Stochastic000226

بسم الله الرحمن الرحیم — جلسه — ۱۴۰۰/۲/۲۶

Ornestein-Ohlenbeck Process

linear Langevin Equation (N-Variable)

$$\dot{x}_i + \xi_{ij}\, x_j = \eta_i \qquad i=1,N$$

if $i=1$ $\rightarrow$ $\boxed{\dot{x} = -\xi x + \eta}$

Ex1 if $\xi=0$ $\rightarrow$ Wiener Process $\boxed{\dot{x} = \eta}$

$D^{(1)}=0$, $D^{(2)}=\sigma^2 \leftarrow$ cts شدت نوفه است

$D^{(n)}=0$ for $n \geqslant 3$ $\rightarrow$ Fokker-Planck

$$\boxed{\frac{\partial p(x,t)}{\partial t} = \underset{\text{cts}}{D^{(2)}} \frac{\partial^2 p(x,t)}{\partial x^2}}$$

for Conditional PDF

$$\boxed{\frac{\partial p(x,t|x',t')}{\partial t} = D^{(2)} \frac{\partial^2}{\partial x^2} p(x,t|x',t')}$$

Initial Condition $p(x,t'|x',t') = \delta_D(x-x')$

$t_i = t'$

for $t>t'$ in General

$$- \frac{\left[x - x' - D^{(1)}(x',t)\tau\right]^2}{4 D^{(2)}(x',t)\tau}$$

$$p(x,t\,|\,x',t) = \frac{1}{2\sqrt{\pi D^{(2)}(x',t)\,\tau}}\, e^{\,\cdots} \qquad \tau \rightarrow (t-t')$$

→ for $D^{(1)} = 0$ and $D^{(2)} = \text{cts} = D$

(A)

$$p_W(x,t\,|\,x',t') = \frac{1}{2\sqrt{\pi D (t-t')}}\, e^{-\frac{(x-x')^2}{4D(t-t')}}$$

$p(x,t;x',t')$

$$p_W(x,t) = \int dx'\, \underbrace{p_W(x,t\,|\,x',t')}_{\text{Kernel (Green's function)}}\, p_W(x',t')$$

Ex2: $D^{(1)} \neq 0$, $\boxed{D^{(1)} = -\xi x}$ and $\boxed{D^{(2)} = D = \text{cts}}$

$D^{(n)} = 0 \quad n \geq 3$

$$\frac{\partial p(x,t\,|\,x',t')}{\partial t} = \left[\left(-\frac{\partial}{\partial x}\right) D^{(1)} + \left(\frac{\partial}{\partial x}\right)^2 D^{(2)}\right] p(x,t\,|\,x',t')$$

$$\boxed{\frac{\partial p(x,t\,|\,x',t')}{\partial t} = \xi \frac{\partial}{\partial x}\big(x\, p(x,t\,|\,x',t')\big) + D \frac{\partial^2}{\partial x^2} p(x,t\,|\,x',t')}$$

$\frac{\partial}{\partial t}$; $\frac{\partial^2}{\partial x^2}$, $\frac{\partial}{\partial x}$ → PDE حل این

Fourier Transformation method

$$p(x,t\,|\,x',t') = \frac{1}{2\pi}\int dk\, e^{ikx}\, \tilde{p}(k,t\,|\,x',t')$$

$x \rightarrow k$

$$\left\{ \begin{array}{l} \frac{\partial}{\partial x} \longrightarrow ik \\ x = -i\frac{\partial}{\partial k} \end{array} \right\}$$

$$\frac{\partial}{\partial t}\tilde{p}(k,t|x',t') = \xi(ik)\left[-i\frac{\partial}{\partial k}\tilde{p}(k,t|x',t')\right] + D(ik)^2\tilde{p}(k,t|x',t')$$

$$\boxed{\frac{\partial}{\partial t}\tilde{p} = \xi k\frac{\partial}{\partial k}\tilde{p} - Dk^2\tilde{p}}$$

$\frac{\partial}{\partial t}$, $\frac{\partial}{\partial k}$ → Ordinary Differential Equation (ODE) → **Characteristic Method**

$\boxed{t_i = t'}$

$$p_{\text{initial}}(x,t'|x',t') = \delta_D(x-x')$$

$$\tilde{p}_{\text{initial}} = ? \;\longrightarrow\; p_{\text{initial}}(x,t'|x',t') = \frac{1}{2\pi}\int dk\, e^{ikx}\,\tilde{p}(k,t'|x',t')$$

$$\delta_D(x-x') = \frac{1}{2\pi}\int dk\, e^{ik(x-x')}$$

$$p_{\text{initial}}(x,t'|x',t') = \text{''} = \frac{1}{2\pi}\int dk\, e^{ikx}\, e^{-ikx'}$$

$$= \frac{1}{2\pi}\int dk\, e^{ikx}\,\tilde{p}(k,t'|x',t')$$

$$\delta_D(x-x') = p_{\text{initial}}(x,t'|x',t') \xrightarrow{\text{F.T.}} \boxed{\tilde{p}(k,t'|x',t') = e^{-ikx'}}$$

$\frac{\partial}{\partial t}, \frac{\partial}{\partial k}$ → ODE $\frac{d}{ds}$ ← روش مشخصه

پارامتری

$$\frac{\partial}{\partial t}\tilde{p}(k,t|x',t') = \xi k\frac{\partial}{\partial k}\tilde{p}(k,t|x',t') - Dk^2\tilde{p}(k,t|x',t')$$

$$\frac{d\tilde{p}}{ds} = \frac{\partial \tilde{p}}{\partial t}\frac{dt}{ds} + \frac{\partial \tilde{p}}{\partial k}\frac{dk}{ds}$$

$$\frac{\partial}{\partial t}\tilde{p} - \xi K \frac{\partial}{\partial K}\tilde{p} = -DK^2\tilde{p}$$

$$\frac{d\tilde{p}}{ds} = -DK^2\tilde{p} \Rightarrow \left\{ \frac{\partial \tilde{p}}{\partial t}\frac{dt}{ds} + \frac{\partial \tilde{p}}{\partial k}\frac{dk}{ds} = -DK^2\tilde{p} \right\}$$

$$\frac{\partial \tilde{p}}{\partial t}1 - \xi K\frac{\partial}{\partial K}\tilde{p} = -DK^2\tilde{p}$$

یعنی

$$\frac{dt}{ds} = 1 \rightarrow t = S + cts_1 \;—\; t(s,r) = S + cts_1$$

t' شروع — شرط اولیه $t(S=0,r) = t'$ برای $\boxed{cts_1 = t'}$

$$\frac{dk}{ds} = -\xi K \rightarrow K = K_0 e^{-\xi S} \rightarrow K(S=0,r) = K \rightarrow \boxed{K(s,r) = Ke^{-\xi S}}$$

$$t(s,r) = S + t' \rightarrow S = t(s,r) - t'$$

$$K(s,r) = Ke^{-\xi S} \rightarrow \boxed{K(s,r) = Ke^{-\xi(t(s,r)-t')}}$$

$$\frac{d\tilde{p}}{ds} = -DK^2(s,r)\tilde{p} = -DK^2e^{-2\xi(t-t')} \rightarrow \tilde{p}(s,r) = \tilde{p}_0\, e^{\frac{DK^2(s,r)}{2\xi}(e^{-2\xi S}-1)}$$

$$\tilde{p}_0 = \tilde{p}(s=0,r) = e^{-iKx'}$$

پس نهایتا

$$\boxed{\tilde{p}(s,r) = e^{-iK(s,r)x'}\, e^{\frac{DK^2(s,r)}{2\xi}(e^{-2\xi S}-1)}}$$

$(t-t')$

برحسب (K, t) بجای (r, s) داریم

$$\tilde{p}(k,t|x',t') = e^{-ikx e^{...}} \; e^{+\frac{Dk^2}{2\xi} e^{...}(e^{...}-1)}$$

$$\tilde{p}(k,t|x',t) = e^{-ikx' e^{-\xi(t-t')}} \times e^{-\frac{Dk^2 e^{2\xi(t-t')}}{2\xi}\left(1-e^{-2\xi(t-t')}\right)}$$

$\frac{d}{ds} \longrightarrow \frac{\partial}{\partial t}, \frac{\partial}{\partial k}$

Inverse F. T.

$\dot{x} = -\xi x + \eta$

$x = x_h + x_p = e^{-\xi(t-t')} x_0 + \cdots$

$x_h = G(t-t')\, x(t')$

$$p(x,t|x',t') = \frac{1}{2\pi}\int dk\, e^{ikx}\, \tilde{p}(k,t|x',t')$$

$$p(x,t|x',t') = \sqrt{\frac{\xi}{2\pi D\left[1-e^{-2\xi(t-t')}\right]}}\; e^{-\frac{\xi\left(x-e^{-\xi(t-t')}x'\right)^2}{2D\left(1-e^{-2\xi(t-t')}\right)}}$$

تغییر نسبتی به آنچه شرح داده شد

① if $\xi \to 0 \longrightarrow D^{(1)} \to 0 \longrightarrow$ Weiner process

$$p(x,t|x',t') = \frac{1}{\sqrt{4\pi D(t-t')}}\; e^{-\frac{(x-x')^2}{2D(t-t')}}$$

② $\xi(t-t') \gg 1 \longrightarrow$ Stationary Case

$$p(x,t|x',t') \longrightarrow p_{st}(x,t|x',t') = p(x,t) = \sqrt{\frac{\xi}{2\pi D}}\; e^{-\frac{\xi x^2}{2D}}$$

حافظه خود را از دست می‌دهد

Gaussian

$\bar{x} = 0 \longrightarrow \langle x \rangle = 0$ $\longleftarrow N(\bar{x}, \sigma_x)$

$\sigma_x^2 = D/\xi$

Ex3 :

$$p(x,t\,;\,x',t') = p(x,t\,|\,x',t')\,p(x',t')$$

$$\downarrow$$

$$\underbrace{p_{st}(x,t\,;\,x',t')}_{?} = \underbrace{\overset{Ex2}{\widetilde{p(x,t\,|\,x',t')}}}_{\text{propagator (General)}}\;\underbrace{p_{st}(x',t')}_{\text{Ex2}}$$

$$p_{st}(x,t\,;\,x',t') = \underbrace{\sqrt{\frac{\xi}{2\pi D\left(1-e^{-2\xi(t-t')}\right)}}\; e^{-\frac{\xi\left(x-e^{-\xi(t-t')}x'\right)^2}{2D\left(1-e^{-2\xi(t-t')}\right)}}}_{\text{propagator}} \times \sqrt{\frac{\xi}{2\pi D}}\; e^{-\xi x'^2/2D}$$

$t'<t$

$$p_{st}(x,t\,;\,x',t') = \frac{\xi}{2\pi D\sqrt{1-e^{-2\xi|t-t'|}}}\; e^{-\xi\frac{x^2+x'^2-2xx'e^{-\xi|t-t'|}}{2D\left(1-e^{-2\xi|t-t'|}\right)}}$$

$\leftarrow \xi|t-t'|\gg 1$ برای

$$\lim_{\xi|t-t'|\gg 1} p_{st}(x,t\,;\,x',t') = p_{st}(x,t)\,p_{st}(x',t') \longleftarrow \checkmark$$

$$= p_{st}(x,t\,|\,x',t')\,p_{st}(x',t')$$

$$\lim_{\xi|t-t'|\gg 1} p(x,t\,|\,x',t') = p_{st}(x,t)$$ [illegible] ✓

Green's function method

to solve O-h

$$\dot{x}_i + \xi_{ij} x_j = \eta_i \qquad i = 1, \dots, N$$

$x_i(t) = ?$ → Green's function

$$\dot{x}_i + \sum_j \xi_{ij} x_j = \eta_i$$

initial condition $\quad x_i(t=0) = x_i^{(0)}$ که مقدار مشخص از زمان است

$$x_i(t) = x_i^{(h)}(t) + x_i^{(p)}(t)$$

$$x_i(t) = G_{ij}(t)\, x_j^{(0)} + \int dt'\, G_{ij}(t,t')\, \eta_j(t')$$

N-variable → $$x_i(t) = \sum_j G_{ij}(t)\, x_j^{(0)} + \sum_j \int dt'\, G_{ij}(t,t')\, \eta_j(t')$$

یادآوری $\quad i=1$

$$x(t) = \underbrace{G(t)}_{e^{-\xi t}} x(0) + \int_0^t dt'\, G(t-t')\, \eta(t')$$

قبلاً دیده بودیم:

$$\boxed{x(t) = e^{-\xi t} x_0 + \int_0^t dt'\, e^{-\xi(t-t')}\, \eta(t)}$$

$$\hat{\mathcal{L}}_i = \frac{d}{dt} + \xi_{ij}$$

$$\hat{\mathcal{L}}\, G(t,t') = \delta_D(t-t')$$

→ $$\boxed{\dot{G}_{ij} + \xi_{ik} G_{kj} = 0}$$

تمرین: نشان دهید:

$$* \quad \bar{G}(t) = e^{-\bar{\xi} t} \quad *$$

مارکی

حالت اولیه for $\bar{\xi} t < 1$ → $\bar{G}(t) = 1 - \bar{\xi} t + \frac{1}{2} \bar{\xi}^2 t^2 + \dots$

حالت نهایی → $\boxed{\xi t \gg 1}$

① $$M_{1i} = \langle x_i(t) \rangle = \langle x_i^{(h)}(t) + x_i^{(p)}(t) \rangle$$

$$= \langle x_i^{h}(t) \rangle + \langle x_i^{(p)}(t) \rangle$$

$\boxed{\langle \eta \rangle = 0}$

$$M_{1i} = \langle x_i^{h}(t) \rangle = \langle G_{ij}(t)\, x_j^{(0)} \rangle = G_{ij}(t)\, x_j^{(0)}$$

G_{ij}: Propagator; $x_j^{(0)}$: مقدار اولیه

② $$K_{2ij} = \langle [x_i(t) - M_{1i}][x_j(t) - M_{1j}] \rangle$$

$$= \int_0^t \int_0^t dt_1\, dt_2\, G_{ik}(t-t_1)\, G_{js}(t-t_2)\, g_{ks}\, \delta_D(t_1 - t_2)$$

g_{ks}: شدت

$g_{ks}\, \delta_D(t_1 - t_2)$: $\langle \eta_k \eta_s \rangle$

$$\boxed{K_{2ij} = \int_0^t dt'\, G_{ik}(t-t')\, G_{js}(t-t')\, g_{ks}}$$

③ $$\frac{d}{dt} K_{2ij} = \dot{K}_{2ij} = G_{ik}(t-t')\, G_{js}(t-t')\, g_{ks}$$

$- \quad dt \quad {}_{2ij} \quad {}_{ij}$

$$\frac{d^2}{dt^2} K_{2ij} = \ddot{K}_{2ij} = \dot{G}_{ik} G_{js} g_{ks} + G_{ik} \dot{G}_{js} g_{ks}$$

$$= -\xi_{il} G_{lk} G_{js} g_{ks} + G_{ik} (-\xi_{jl} G_{ls}) g_{ks}$$

$$\ddot{K}_{2ij} = -\xi_{il} \dot{K}_{2lj} - \xi_{jl} \dot{K}_{2li}$$