Stochastic000221

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Fokker-Planck Equation

Ex 1: Fokker-Planck Equation for one Variable.

if $D^{(4)}=0 \rightarrow D^{(n)}=0$ for $n \geqslant 3$

$KM \longrightarrow FP$

$\hat{\mathcal{L}}_{KM} \longrightarrow \hat{\mathcal{L}}_{FP}$

$$\frac{\partial p(x,t)}{\partial t} = \hat{\mathcal{L}}_{FP}\, p(x,t) = \left[-\frac{\partial}{\partial x} D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2} D^{(2)}(x,t)\right] p(x,t)$$

$$\boxed{PDE\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}\right]}$$

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x} \underbrace{\left[D^{(1)}(x,t) - \frac{\partial}{\partial x} D^{(2)}(x,t)\right] p(x,t)}_{S(x,t)\ \equiv\ \text{Source function}}$$

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x} S(x,t)$$

$$\boxed{\frac{\partial}{\partial t} p(x,t) + \frac{\partial}{\partial x} S(x,t) = 0}$$

For Stationary Regime $\quad \dfrac{\partial p(x,t)}{\partial t} = 0 \rightarrow \boxed{p(x,t) = p(x,t_1) = \cdots}$

$$\frac{\partial}{\partial x} S(x,t) = 0 \longrightarrow S' = cts = 0 \leftarrow$$ دکوله

$$\left[D^{(1)} - \frac{\partial}{\partial x} D^{(2)} \right] p(x) = 0 \longrightarrow \boxed{p(x) = p(x_0)\, e^{\int_0^x \frac{D^{(1)}}{D^{(2)}} dx'}}$$

حل در رژیم ها

چطور بگردیم؟ معادله تحول برای متغیر تصادفی ← لانژون

ریتمک $\longrightarrow \dot{v} = -\xi v + \eta(t)$

تغییر متغیر $\longrightarrow \dot{x} = -\xi x + \eta(t) \longrightarrow \langle \eta(t) \rangle = 0$

$$\langle \eta(t)\, \eta(t') \rangle = 2\sigma^2 \delta_D(t-t')$$

$D^{(n)} = ?$

با دانستن شکل معادله تحول متغیر تصادفی

$$D^{(n)} \equiv \lim_{\tau \to 0} \frac{1}{n!} \frac{\left\langle [x(t+\tau) - x(t)]^n \right\rangle}{\tau} \Bigg|_{x(t)=x}$$

$$D^{(1)} = \lim_{\tau \to 0} \frac{1}{1!} \frac{\left\langle [x(t+\tau) - x(t)]^1 \right\rangle}{\tau} \Bigg|_{x(t)=x}$$

$$\dot{x}(t) = \lim_{\tau \to 0} \frac{x(t+\tau) - x(t)}{\tau} = -\xi x + \eta$$

$$\lim_{\tau \to 0} [x(t+\tau) - x(t)] = -\xi x \tau + \tau \eta$$

$$D^{(1)} = \lim_{\tau \to 0} \frac{\left\langle [-\xi x \tau + \tau \eta] \right\rangle}{\tau} \Bigg|_{x(t)=x}$$

$$= -\xi \langle x(t) \rangle \Big|_{x(t)=x} + \langle \eta \rangle = -\xi x$$

$$\boxed{D^{(1)} = -\xi x} \longrightarrow$$ ضریب تعیین معادلات تتقد

$$\left\langle [x(t+\tau) - x(t)]^2 \right\rangle \Big|_{x(t)=x}$$

$$D^{(2)} = \lim_{\tau \to 0} \frac{1}{2!} \frac{\langle \cdots \rangle|_{x(t)=x}}{\tau}$$

$$= \lim_{\tau \to 0} \frac{1}{2!} \frac{\left\langle \left[-\xi x \tau + \eta \tau\right]^2 \right\rangle_{x(t)=x}}{\tau}$$

$$= \lim_{\tau \to 0} \frac{1}{2!} \frac{\left\langle \xi^2 x^2 \tau^2 - 2\xi x \tau^2 \eta + \tau^2 \eta^2 \right\rangle \Big|_{x(t)=x}}{\tau}$$

$$= \lim_{\tau \to 0} \frac{1}{2!} \frac{\xi^2 \tau^2 \langle x^2 \rangle|_{x(t)=x} - 2\xi \tau^2 \langle x\eta \rangle|_{x(t)=x} + \tau^2 \langle \eta^2 \rangle}{\tau}$$

$$= \lim_{\tau \to 0} \underbrace{\cancel{\frac{\tau \xi^2 \langle x^2 \rangle}{2!}}}_{0} + \lim_{\tau \to 0} \underbrace{\cancel{\frac{-2\xi \tau \langle x\eta \rangle}{2!}}}_{0} + \lim_{\tau \to 0} \frac{\tau \overbrace{\langle \eta^2 \rangle}^{2\sigma^2 \delta_0 (t-t')}}{2!}$$

$$D^{(2)} = \lim_{\tau \to 0} \frac{\cancel{\tau}\, 2\sigma^2 / \cancel{\tau}}{2!} = \frac{\cancel{2}\sigma^2}{\cancel{2!}} = \sigma^2$$

$$\boxed{D^{(2)} = \sigma^2} \qquad \langle \eta(t) \eta(t') \rangle = 2\sigma^2 \delta_0 (t-t')$$

بخش تصادفی معادله لانژوین را نشان می‌دهد.

$$p_{st}(x) = p(x_0)\, e^{+\int_0^{x} \frac{D^1}{D^2} dx'}$$

پس

$$p_{st}(v) = p(v_0)\, e^{+\int_0^{v} \frac{-\xi v'}{\sigma^2} dv} \qquad \sigma^2 = \frac{K_B T \xi}{m}$$

$$= p(v_0)\, e^{-\frac{m v^2}{2KT}} \longleftarrow \text{MB - Distribution}$$

Ex 2: Conditional Probability for $\tau \to 0$

Ex2: Conditional Probability for ...

$$p(x,t+\tau|x',t) = ?$$

نشان دادیم برای $\tau\to 0$

$$p(x,t+\tau|x',t) = \left[1+\hat{\mathcal{L}}_{KM}\,\tau + \mathcal{O}(\tau^2)\right]\delta_D(x-x')$$

اگر $D^{(n)}=0$ for $n\geq 3$

$$p(x,t+\tau|x',t) = \left[1+\hat{\mathcal{L}}_{FP}\,\tau + \mathcal{O}(\tau^2)\right]\delta_D(x-x')$$

$$p(x,t+\tau|x',t) = \left[1-\frac{\partial}{\partial x}D^{(1)}\tau + \frac{\partial^2}{\partial x^2}D^{(2)}\tau\right]\delta_D(x-x')$$

$\delta(x-x')f(x) = \delta(x-x')f(x')$

$$= \exp\left[\left(-\frac{\partial}{\partial x}D^{(1)}(x',t) + \frac{\partial^2}{\partial x^2}D^{(2)}(x',t)\right)\tau\right]\delta_D(x-x')$$

$$p(x,t+\tau|x',t) = e^{-\frac{\partial}{\partial x}D^{(1)}(x',t)\tau + \frac{\partial^2}{\partial x^2}D^{(2)}(x',t)\tau}\int \frac{1}{2\pi}\,d\lambda\, e^{-i\lambda(x-x')}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda\; e^{-i\lambda D^{1}(x',t)\tau - \lambda^2 D^{2}(x',t)\tau - i\lambda(x-x')}$$

(A) $\rightarrow$
$$\boxed{\underbrace{p(x,t+\tau|x',t)} = \frac{1}{2\sqrt{\pi D^{2}(x',t)\tau}}\; e^{-\frac{\left[x-x'-D^{1}(x',t)\tau\right]^2}{4D^{(2)}(x',t)\tau}}}$$

حل تابع چگالی احتمال شرطی در حد $\tau\to 0$

اکنون می‌خواهیم بررسی کنیم که آیا جواب بالا با تعریف D ها سازگار است یا نه؟

$$D^{(n)}(x',t) = \lim_{\tau\to 0}\frac{M_n(x',t,\tau)}{\tau}$$

$$M_n(x',t,\tau) = \int dx\,(x-x')\; p(x,t+\tau\,|\,x',\tau)$$

برای اینکار از تعریف زیر استفاده می‌کنیم

$$\int_{-\infty}^{+\infty} dx\; x^n\, e^{-(x-\beta)^2} = (2i)^{-n}\sqrt{\pi}\, H_n(i\beta)$$

↑ Hermite Polynomials

$H_0 = 1 \quad H_1 = 2x$

$H_2 = 4x^2 - 2$

$$M_n(x',t,\tau) = \int_{-\infty}^{+\infty} dx\,(x-x')^n \frac{1}{2\sqrt{\pi D^{(2)}(x',t)\tau}}\, e^{-\frac{\left[x-x'-D^{(1)}(x',t)\tau\right]^2}{4D^{(2)}(x',t)\tau}}$$

$$M_n(x',t,\tau) = \left[-i\sqrt{D^{(2)}(x',t)\tau}\right]^n H_n\left[\frac{1}{2}\, i\, D^{(1)}(x',t)\sqrt{\frac{\tau}{D^{(2)}(x',t)}}\right]$$

$$\lim_{\tau\to 0}\frac{M_n(x',t,\tau)}{n!\,\tau} = \begin{cases} D^{(1)}(x',t) & \text{for } n=1 \\ D^{(2)}(x',t) & \text{for } n=2 \\ 0 & \text{for } n\geq 3 \end{cases}$$

پس نتیجه اینکه جواب مثبت است سازگار با انتظارمان

Ex 3: $$\hat{\mathcal{L}}_{FP} = -\frac{\partial}{\partial x} D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2} D^{(2)}(x,t)$$

$$\hat{\mathcal{L}}_{FP} = -\frac{\partial D^{(1)}}{\partial x} + \frac{\partial^2}{\partial x^2}D^2 - \left[D^1 - 2\frac{\partial}{\partial x}D^2\right]\frac{\partial}{\partial x} + D^2\frac{\partial^2}{\partial x^2}$$

for $\tau\to 0$ $$p(x,t+\tau\,|\,x',t) = \left[1 + \hat{\mathcal{L}}_{FP}\,\tau + \mathcal{O}(\tau^2)\right]\delta_D(x-x')$$

(B) $$\boxed{p(x,t+\tau\,|\,x',t) = \frac{1}{2\sqrt{\pi D^{(2)}(x,t)\tau}}\; e^{-\frac{\partial D^{(1)}}{\partial x}\tau + \frac{\partial^2 D^{(2)}}{\partial x^2}\tau - \frac{[x-x'-[D^{(1)}-2\frac{\partial}{\partial x}D^{(2)}]\tau]^2}{4D^{(2)}\tau}}}$$

Ex4: Path Integral Solution for Markovian Regim

$\boxed{x \equiv x_n,\ t = t_n}$

$$\begin{cases} p(x,t) = \int p(x,t\,|\,x_{n-1},t_{n-1};\, x_{n-2},t_{n-2};\cdots;\, x_0,t_0) \times \\ \qquad p(x_{n-1},t_{n-1}\,|\,x_{n-2},t_{n-2};\cdots;x_0,t_0) \times \\ \qquad \vdots \\ \qquad p(x_1,t_1\,|\,x_0,t_0)\, p(x_0,t_0)\, dx_{n-1}\, dx_{n-2} \cdots dx_0 \end{cases}$$

for MarKov Process

$$p(x,t) = \int p(x,t\,|\,x_{n-1},t_{n-1})\, p(x_{n-1},t_{n-1}\,|\,x_{n-2},t_{n-2}) \cdots p(x_1,t_1\,|\,x_0,t_0)\, \underline{p(x_0,t_0)} \times dx_{n-1}\, dx_{n-2} \cdots dx_0$$

$t = t_n = t_0 + n\tau$

اگر از (A) استفاده کنیم خوب به تقریب

$$\underbrace{p(x_n,t_n\,|\,x_{n-1},t_{n-1})}_{p(x,t+\tau\,|\,x',t)}$$

قبلاً این فرمول را دارم

(C) $$p(x_n,t_n) = p(x,t) = \lim_{n\to\infty} \underbrace{\int\cdots\int}_{n\ \text{times}} \prod_{i=0}^{n-1} \left\{ \left[4\pi D^{(2)}(x_i,t_i)\tau\right]^{-1/2} dx_i \right\}$$

$$e^{-\sum^{n-1} \frac{[x_{i+1} - x_i - D^{(1)}(x_i,t_i)\tau]^2}{\cdots}}$$

$$\times\, e^{\;\cdots\; \sum_{i=0} \;\cdots\; 4D(x_i,t_i)\tau} \qquad \underbrace{p(x_0,t_0)}$$

درصورتیکه $\lim_{n\to\infty}$ مشتق آن نگاه کنیم

پس $x_{i+1}-x_i = \dot{x}\,\tau$

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{\left[x_{i+1}-x_i - D^{(1)}(x_i,t_i)\tau\right]^2}{4D^{(2)}(x_i,t_i)\tau}$$

$$= \lim \sum_{i=0}^{n-1} \frac{\left[\dot{x}_i\,\tau - D^{(1)}(x_i,t_i)\tau\right]^2}{4D^{(2)}(x_i,t_i)\tau}$$

$$= \int_{t_0}^{t} dt' \, \frac{\left[\dot{x}(t') - D^{(1)}(x(t'),t')\right]^2}{4D^{(2)}(x(t'),t')}$$

Generalized Onsager-Machlup function

Ornstein_Uhlenbeck Process

$$\dot{x}_i + \sum_{j=1}^{N} \varepsilon_{ij}\, x_j = \eta_i$$

کلی‌ترین شکل معادلات لانژوین (خطی) را درنظر بگیرید

N-variable.

$i=1,\ldots N$

$$\langle \eta_i \rangle = 0 \quad , \quad \langle \eta_i(t)\, \eta_j(t') \rangle = 2\,\sigma_{ij}^2\, \delta_D(t-t')$$

$$\sigma_{ij}^2 = \sigma_i^2\, \delta_{ij}$$ ([illegible])

اگر $\varepsilon_{ij}=0$ یعنی بخش تعینی صفر باشد $\longrightarrow$ Wiener Process

Ex1: $\quad i=1 \;,\quad \varepsilon_i = 0$

$$\dot{x} = \eta \equiv \text{Wiener Process}$$

$$D^{(1)} = \lim_{\tau \to 0} \frac{M_1(x,t,\tau)}{\tau} = \lim_{\tau \to 0} \frac{\langle [x(t+\tau) - x(t)]^1 \rangle |_{x(t)=x}}{\tau}$$

$$D^{(1)} = \lim_{\tau \to 0} \frac{\langle \eta(t) \rangle \tau}{\tau} = \langle \eta(t) \rangle = 0$$

$D^{(2)} \leftarrow \xi$ [illegible] $\leftarrow \xi \leftarrow$ [illegible] $\leftarrow \boxed{D^{(1)} = 0}$

$$D^{(2)} = \lim_{\tau \to 0} \frac{M_2(x,t,\tau)}{\tau\, 2!} = \lim_{\tau \to 0} \frac{\langle [\overbrace{x(t+\tau) - x(t)}^{\tau \eta(t)}]^2 \rangle |_{x(t)=x}}{2!\, \tau}$$

$$D^{(2)} = \lim_{\tau \to 0} \frac{\cancel{\tau}^{2}\, 2\sigma^2 / \cancel{\tau}}{2!\, \cancel{\tau}} = \sigma^2 = cts$$

همان انتظاری که داریم