Stochastic000219

بسم الله الرحمن الرحیم   جلسه ۱۴۰۰/۲/۱۹

ادامه بحث بسط کرامرز-مویال را خواهیم داشت: **Kramers-Moyal Expansion**

برای یادآوری

$$\frac{\partial p(x,t)}{\partial t} = \hat{\mathcal{L}}_{KM}(x,t)\, p(x,t)$$

K.M عملگر

$$\hat{\mathcal{L}}_{KM} \equiv \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n D^{(n)}(x,t) \rightarrow$$

در حالت کلی شامل بی‌نهایت جمله است

تعاریف معادل

$$D^{(n)}(x,t) \equiv \lim_{\tau \to 0} \frac{M_n(x,t,\tau)}{n!\,\tau}$$

$$M_n(x,t,\tau) = \int dx'' \,(x''-x)^n\, p(x'',t+\tau \,|\, x,t)$$

$$= \left\langle \left[\xi(t+\tau)-\xi(t)\right]^n \right\rangle \Big|_{\xi(t)=x}$$

If $\hat{\mathcal{L}}_{KM}$ is independent from "t" ← مستقل از زمان

(A)

$$\rightarrow \frac{\partial p(x,t)}{\partial t} = \hat{\mathcal{L}}_{KM}(x)\, p(x,t) \rightarrow \boxed{p(x,t) = e^{\hat{\mathcal{L}}_{KM}(x)(t-t')}\, p_{\text{initial condition}}}$$

شرط اولیه

$$\rightarrow \begin{cases} \dfrac{\partial p(x,t|x',t')}{\partial t} = \hat{\mathcal{L}}_{KM}\, p(x,t|x',t') \\ \text{Initial Cond: } p(x,t'|x',t') = \delta_D(x-x') \end{cases}$$

$$p(x,t|x',t') = e^{\hat{\mathcal{L}}_{KM}(x)(t-t')}\, \delta_D(x-x')$$

$$\text{if } \boxed{t-t' = \tau \ll 1} \Rightarrow \boxed{p(x,t|x',t') = \left[1+\hat{\mathcal{L}}_{KM}(x)\,\tau + \mathcal{O}(\tau^2)\right] \times \delta_D(x-x')}$$

(B)

$$Ⓐ\,Ⓑ \Longrightarrow \boxed{p(x,t|x',t') \simeq \left[1+\hat{\mathcal{L}}_{KM}\,\tau\right]\delta_D(x-x')}$$

$$\underset{\text{زمان}}{\frac{\partial p(x,t)}{\partial t}} = \hat{\mathcal{L}}_{KM}(x,t)\, p(x,t) \qquad \{x(t)\}$$

متغیر فضایی

Time Series

※ Dyson Series Solution (Dyson Expansion method)

$t'<t \qquad x' = \xi(t')$ Forward approach

$x = \xi(t>t')$

$$\frac{\partial p(x,t|x',t')}{\partial t} = \hat{\mathcal{L}}_{KM}(x,t)\, p(x,t|x',t') \longrightarrow \boxed{p(x,t|x',t') = ?}$$

Initial condition $\quad p(x,t'|x',t') = \delta_D(x-x')$

$$p(x,t|x',t') = \delta_D(x-x') + \underbrace{\int_{t'}^{t} dt_1\, \hat{\mathcal{L}}_{KM}(x,t_1)\,\delta_D(x-x')}_{①}$$

$$+ \underbrace{\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\, \hat{\mathcal{L}}_{KM}(x,t_1)\,\hat{\mathcal{L}}_{KM}(x,t_2)\,\delta_D(x-x')}_{②}$$

$$+ \underbrace{\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \int_{t'}^{t_2} dt_3\, \hat{\mathcal{L}}_{KM}(x,t_1)\,\hat{\mathcal{L}}_{KM}(x,t_2)\,\hat{\mathcal{L}}_{KM}(x,t_3)\,\delta_D(x-x')}_{③}$$

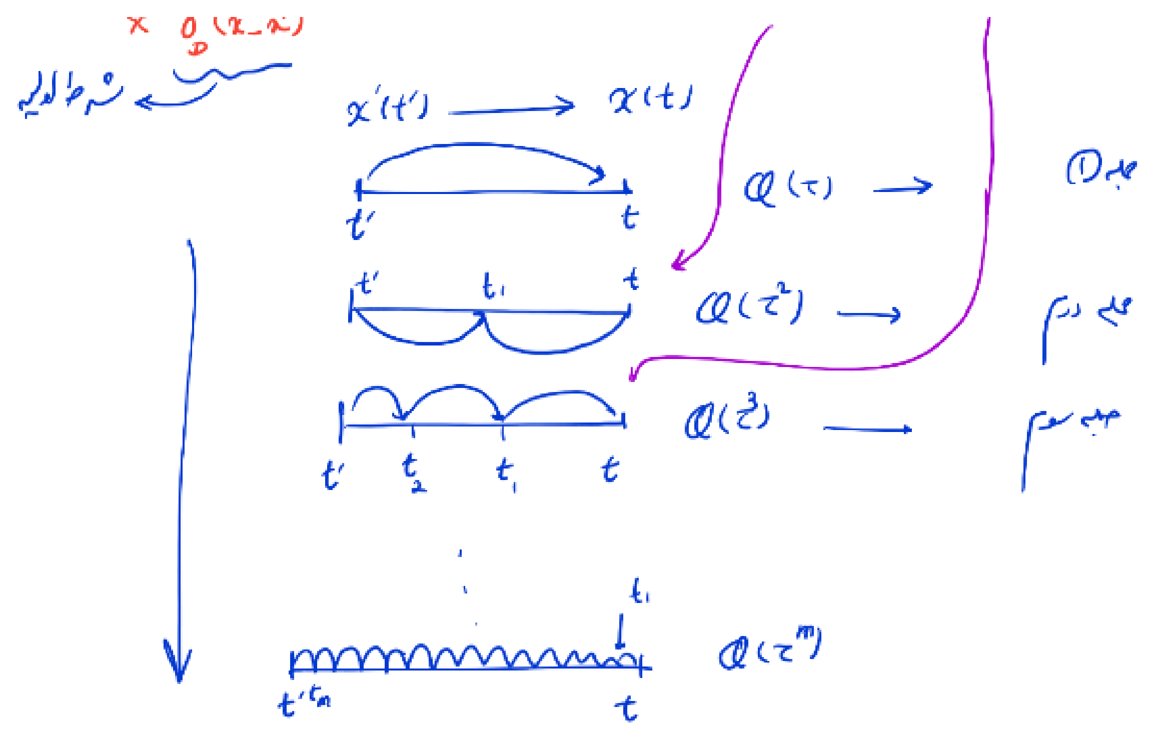
$$+ \cdots + \cdots$$

خاتمه　　شروع

$$p(x,t|x',t') = \left[1 + \sum_{m=1}^{\infty} \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{m-1}} dt_m\, \overbrace{\hat{\mathcal{L}}_{KM}(x,t_1)\,\hat{\mathcal{L}}_{KM}(x,t_2)\cdots\hat{\mathcal{L}}_{KM}(x,t_m)}^{m}\right]$$

مقدار اولیه است

$t'<t_1<t$ or $t'<t_2<t_1<t$



Dyson Expansion based on $\hat{T}$ (time ordering operator) is:

$$p(x,t|x',t') = \overleftarrow{T}\left[1+\sum_{m=1}^{\infty}\frac{1}{m!}\int_{t'}^{t}dt_1\int_{t'}^{t}dt_2\cdots\int_{t'}^{t}dt_m\,\hat{\mathcal{L}}_{KM}(x,t_1)\,\hat{\mathcal{L}}_{KM}(x,t_2)\cdots\mathcal{L}_{KM}(x,t_m)\right]\delta_D(x-x')$$

Forward Expansion ← زمان‌های بزرگتر را به سمت چپ می‌برد $\overleftarrow{T}$

در همین حالت می‌توان ملاحظه کرد که اگر $t-t'=\tau\ll 1$ در آن صورت

$$p(x,t|x',t') = \left[1+\int_{t'}^{t}dt_1\,\hat{\mathcal{L}}_{KM}(x,t_1)+Q(\tau^2)\right]\delta_D(x-x')$$

$$\simeq\left[1+\hat{\mathcal{L}}_{KM}(x,t)\,\tau\right]\delta_D(x-x')$$

تو تابع‌ها بگذاریم

* Backward KM Expansion

* Backward KM expansion

$$t > \underbrace{t'+\tau}_{x''} > t'$$
$$\quad x \qquad\qquad\qquad x'$$

مسیر تحول

تحول زمانی نسبت به x', t' است

(در روش FKM تحول نسبت به x و t را داشتیم)

اول ، مرحله میانی ، بعد t'

فرآیند مارکوف

(A) $$p(x,t|x',t') = \int dx''\, p(x,t|x'',t'+\tau)\, p(x'',t'+\tau|x',t') \leftarrow \text{Markov Process}$$

$$\boxed{\cancel{=\int} dx'\, p(x,t|x'',t'+\tau, x',t')\, p(x'',t'+\tau|x',t')} \rightarrow \text{تحول کلی}$$

(B) $$p(x'',t'+\tau|x',t') = \int dy\, \delta_D(y-x'')\, p(y,t'+\tau|x',t')$$

$$\delta_D(y-x'') = \delta_D(y-x''+x'-x') = \delta_D(x'-x''+\underbrace{y-x'}_{\epsilon})$$

$$\delta_D(y-x'') = \sum_{n=0}^{\infty} \frac{(y-x')^n}{n!}\left(\frac{\partial}{\partial x'}\right)^n \delta_D(x'-x'')$$

(C) $$p(x'',t'+\tau|x',t') = \sum_{n=0}^{\infty} \frac{1}{n!}\int dy\, (y-x')^n\, p(y,t'+\tau|x',t')\left(\frac{\partial}{\partial x'}\right)^n \delta_D(x'-x'')$$

(A) $\leftarrow$ $$p(x'',t'+\tau|x',t') = 1 + \sum_{n=1}^{\infty}\frac{1}{n!} M_n(x',t',\tau)\left(\frac{\partial}{\partial x'}\right)^n \delta_D(x'-x'')$$

(D) $$p(x,t|x',t') - p(x,t|x',t'+\tau) = \overset{\text{Backward}}{-}\frac{\partial p(x,t|x',t')}{\partial t'}\tau + O(\tau^2)$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!} M_n(x',t',\tau)\left(\frac{\partial}{\partial x'}\right)^n p(x,t|x',t'+\tau)$$

$$= \tau\sum_{n=1}^{\infty} D^{(n)}(x',t')\left(\frac{\partial}{\partial x'}\right)^n p(x,t|x',t')$$

Back_ward

$$\begin{cases} \dfrac{\partial p(x,t|x',t')}{\partial t'} = -\hat{\mathcal{L}}^{\dagger}_{KM}(x',t')\, p(x,t|x',t') \\[2ex] \hat{\mathcal{L}}^{\dagger}_{KM} \equiv \sum_{n=1}^{\infty} D^{(n)}(x',t')\left(\dfrac{\partial}{\partial x'}\right)^{n} \end{cases}$$

Forward

$$\begin{cases} \dfrac{\partial p(x,t|x',t')}{\partial t} = \hat{\mathcal{L}}_{KM}(x,t)\, p(x,t|x',t') \\[2ex] \hat{\mathcal{L}}_{KM} = \sum_{n=1}^{\infty} \left(-\dfrac{\partial}{\partial x}\right)^{n} D^{(n)}(x,t) \end{cases}$$

Exercise show that Forward and Back word Solution are Equivalence

Dyson expansion for back_word

$$p(x,t|x',t') = \overrightarrow{T}\left[1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \ldots \int_{t'}^{t} dt_m\, \mathcal{L}^{\dagger}_{KM}(x',t_1) \ldots\right]$$

زمان‌های عقب‌تر به سمت چپ می‌روند

$$\delta_D(x-x')$$

for $t-t'=\tau \ll 1$

$$\boxed{p(x,t|x',t') = \left[1 + \mathcal{L}^{\dagger}_{KM}(x',t)\,\tau + O(\tau^2)\right] \delta_D(x-x')}$$

Forward $p(x,t|x',t') = p(x,t'+\tau|x',t') \simeq \left[1 + \hat{\mathcal{L}}_{KM}(x,t)\,\tau\right] \delta_D(x-x')$

$$\text{Back word} \quad p(x,t|x',t') = p(x,t'+\tau|x',t') \cong \left[1 + \mathcal{L}^{+}_{KM}(x',t)\,\tau\right] \delta_D(x-x')$$

Forward
$$p(x,t+\tau|x',t) = \hat{A}(x)\, \delta_D(x-x')$$
$$\hat{A}(x) \equiv \overleftarrow{T} \exp\left[\int_{t'}^{t} dt''\, \mathcal{L}_{KM}(x,t'')\right]$$

هم‌ارز هستند

Back word
$$p(x,t+\tau|x',t) = \hat{A}^{+}(x)\, \delta_D(x-x')$$
$$\hat{A}^{+}(x) = \overrightarrow{T} \exp\left[\int_{t'}^{t} dt''\, \hat{\mathcal{L}}^{+}_{KM}(x,t'')\right]$$

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* Pawula theorem قضیه پاولا

$$\hat{\mathcal{L}}_{KM} \longrightarrow \hat{\mathcal{L}}_{\text{Fokker-Planck}}$$

$$\frac{\partial p(x,t)}{\partial t} \longrightarrow \text{Master Equation}$$

$\downarrow$

$$\frac{\partial p(x,t)}{\partial t} \longrightarrow \text{Kramers\_Moyal Expansion}$$

$\downarrow$? $\longrightarrow$ $\boxed{\text{Pawula theorem}}$

$$\partial p(x,t) \longrightarrow \boxed{\text{Fokker - Planck}} \text{ ? Expansion}$$

∂t $\longrightarrow$.

$$\frac{\partial p(x,t)}{\partial t} = \hat{\mathcal{L}}_{KM}(x,t)\, p(x,t) \qquad , \quad \hat{\mathcal{L}}_{KM} = \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial x}\right)^{n} D^{(n)}(x,t)$$

if $\underline{D^{(4)} = 0} \Rightarrow D^{(n)} = 0$ for $n \geq 3$

$D^{(1)}, D^{(2)} \neq 0$

$$\boxed{\frac{\partial P(x,t)}{\partial t} = \left[\left(-\frac{\partial}{\partial x}\right) D^{(1)}(x,t) + \left(\frac{\partial}{\partial x}\right)^{2} D^{(2)}(x,t)\right] p(x,t)}$$

Fokker – Planck Equation

According to schwartz in-equality

$$\left[\int f(x)\, g(x)\, p(x)\, dx\right]^{2} \leq \left[\int f^{2}(x)\, p(x)\, dx\right]\left[\int g^{2}(x)\, p(x)\, dx\right]$$

توابع دلخواه

حاصل‌ضرب توزیع

$$\iint \left[f(x)\, g(y) - f(y)\, g(x)\right]^{2} p(x)\, p(y)\, dx\, dy \geq 0$$

$\rightarrow f(x) \equiv (x - x')^{n}$ $\qquad g(x) \equiv (x - x')^{n+m}$

$p(x) \equiv p(x, t+\tau \,|\, x', t)$

$$\left[\int (x-x')^{n} (x-x')^{n+m}\, p(x, t+\tau \,|\, x', t)\, dx\right]^{2} \leq \int (x-x')^{2n}\, p(x, t+\tau \,|\, x', t)\, dx \int (x-x')^{2n+2m}\, p(x, t+\tau \,|\, x', t)\, dx$$

$$\left(M_{2n+m}\right)^2 \leq M_{2n}\, M_{2n+2m}$$

$$\boxed{\left[(2n+m)!\, D^{(2n+m)}\right]^2 \leq (2n)!\, D^{(2n)}\, (2n+2m)!\, D^{(2n+2m)}}$$

شرط مکمل

$$\begin{cases} (2n+m) \geq 0 \\ 2n \geq 0 \\ 2n+2m \geq 0 \\ D^{(n)} = \lim\limits_{\tau\to 0} \dfrac{M_n}{n!\,\tau} \end{cases} \longrightarrow D^{(0)} = \lim\limits_{\tau\to 0} \dfrac{1}{\tau} \to \text{واگراست}$$

شرط لازم

$$\begin{cases} 2n+m \geq 0 \\ 2n \geq 0 \longrightarrow n \geq 0 \to \boxed{n \geq 1} \\ 2n+2m \geq 0 \longrightarrow n+m \geq 0 \to \boxed{n+m \geq 1} \end{cases}$$

$$\text{if } D^{(2n)} = 0 \longrightarrow D^{(2n+m)} = 0$$

$$\uparrow \qquad\qquad \downarrow$$

$$n \geq 1 \qquad 2n+m \geq 0 \to m \geq -2n \to m \geq -2n+1$$

$$n \geq 1 \qquad \boxed{m \geq -1}$$

البته حزان

$$\boxed{m \geq 0} \quad \text{اشتراک} \quad m \geq -1$$

$$m \geq 0 \leftarrow \begin{cases} m \geq 1-n \\ n \geq 1 \end{cases}$$

$2n+m$ $\quad 2n+0 \quad 2n+1 \quad 2n+2 \quad 2n+\infty$

$$\boxed{D^{(2n)}=0 \longrightarrow D^{\ldots}=0 \longrightarrow D^{\ldots}=D^{\ldots}=D^{\ldots}=\ldots D^{\ldots}=0}$$

اگر $D^{(2n)}=0$ آنگاه D^{2n+1} و بقیه همگی صفرهستند

$$\text{if } D^{(2n+2m)} = D^{2r} = 0 \longrightarrow D^{2n+m} = 0 \qquad 2n+m = r+n$$

$(r \equiv n+m)$ تعریف شود

$$D^{2r}=0 \longrightarrow D^{r+n}=0 \longrightarrow \underbrace{D^{r+1}=D^{r+2}=D^{r+3}=\ldots}_{\text{همه صفرهستند}}=D^{2r}=0$$

بالاترین عضو صفر

$2r=4$

$D^4=0$

$r=2$

$$D^5=D^6=\ldots=0$$

$$D^{2+1}=0=D^{2+2}=0$$

$D^{(3)}$

$$\boxed{\text{اگر ربع طی } D^4=0 \longleftarrow D^n=0 \text{ به ازای } n\geq 3}$$

$$\boxed{\text{البته دقت کنید } D^{(2n)}=0 \text{ و } 2n\geq 4 \text{ در آن صورت } D^n=0 \text{ به ازای } n\geq 3}$$

Ex: $r=5 \longrightarrow D^{2r}=D^{10}=0 \longrightarrow \begin{cases} D^{11}=D^{12}=\ldots=0 \\ D^6=D^7=D^8=D^9=0 \end{cases}$

$$\boxed{\text{if } D^{10}=0 \longrightarrow D^6=D^7=\ldots=0}$$

$r=3 \quad D^6=0 \longrightarrow D^7=D^8=\ldots \infty =0$

$D^4=D^5=0$

$r=2 \quad D^4=0 \longrightarrow D^5=D^6=\ldots \infty =0$

✓ $D^3 = D^4$ 50