Stochastic000131

بسم الله الرحمن الرحیم — جلسه — ۱۴۰۰/۱/۲۴

* Power spectrum (spectral analysis)

* Spectral Index $\rightarrow$ $\langle f \rangle = \langle$ Conditions correspond to feature $\rangle$

Power Spectrum — طیف توان

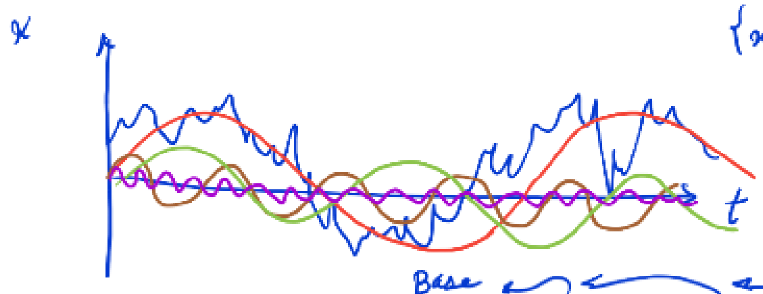
عناصر سازنده یک میدان تصادفی با در نظر گرفتن یک مجموعه پایه متعامد و کامل

Complete orthonormal Base.

کدام پایه؟

for 1+1-Dimension $\rightarrow$ Time Series $\{x(t)\}$ — پیوسته

$\{x\} : \{x_1, x_2, \dots x_N\}$ — گسسته



Base

$$x(t) = A_1 F_1(t) + A_2 F_2 + A_3 F_3(t) + A_4 F_4(t) + \dots$$

$$x(t) = \sum_{i=1}^{\infty} A_i F_i(t) \quad — \quad \{A_i\}$$

ضرایب بسط که باید تعیین شوند

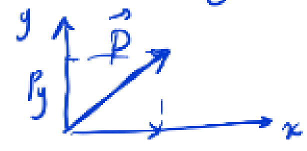
{مشخص می‌شود که هر کدام از پایه‌ها چه نقشی در تولید $x(t)$ دارند}

Recall that $|\psi\rangle = \sum_n a_n |\phi_n\rangle$ $\qquad \langle \phi_n | \phi_m \rangle = \delta_{nm}$

$$\langle F_i | F_j \rangle = \delta_{ij}$$

$$\langle F_j | x(t) \rangle = \sum_i A_i \langle F_j | F_i \rangle \qquad \int dt\, F_i F_j = \delta_{ij}$$

$$\langle F_j | x(t) \rangle = A_j \longrightarrow$$



P_x

$$\vec{P} = P_x \hat{i} + P_y \hat{j}$$

$$\{A\} \qquad \{F\}$$

$\underline{\underline{F}}$ که توابع متعامد و کامل هستند

امواج تخت

Fourier Base $\leftarrow$ $e^{i\omega t}$, e^{ikx}

spherical Harmonic $\leftarrow$ $Y_{lm}(\theta,\varphi)$

توابع خاص $\leftarrow$

سوال است چه پایه انتخابی ساده و شبیه‌تر باشد به داده‌ها $\leftarrow$ تعداد پایه کمتری می‌توان سیستم را توصیف کرد

$$x(t) = \sin(\omega_1 t)$$

$$\underline{\sin(\omega_1 t)} = \sum_{i=1}^{\infty} A_i F_i$$

Fourier Transformation تبدیل فوریه

$$\{x(t)\} \xrightarrow{F.T.} \{X(\omega)\} \xrightarrow{I.F.T} \{x(t)\}$$

الزاماً این دو با هم برابر نیستند

$$x(t) = \int d\omega \, \underbrace{e^{+i\omega t}}_{F} \, \underbrace{X(\omega)}_{A} \longrightarrow \underline{X(\omega)} = \underline{\int dt \, \underline{\underline{e}}^{-i\omega t} x(t)}$$

محدود باشد

پایه‌های فضای فوریه

دامنه آن پایه

$$\text{(1)} \quad \int_{-\infty}^{+\infty} |x(t)| \, dt < \infty$$

شرایط لازم (نه کافی) برای وجود داشتن تبدیل فوریه

② x مشتق پذیر باشد $\longrightarrow$

$$x \longrightarrow X \longrightarrow x$$

③ x شانسی باشد

Weighted Two-Point Correlation $\quad \underline{C_x(\tau)} = \langle \underline{x(t)}\, \underline{x(t+\tau)} \rangle$

①, ②, ③

① $\int d\tau\, |C_x(\tau)| < \infty$

② هموار است

③ فرض کردیم

$$S(\omega) = \int d\tau\, e^{-i\omega\tau}\, C_x(\tau)$$

Power spectrum

$$\underline{C_x(\tau)} = \int d\omega\, \underline{e^{i\omega\tau}}\, \underline{S(\omega)}$$

$$S(\omega) \overset{?}{\underset{?}{\longleftrightarrow}} X(\omega)$$

تبدیل فوریه تابع همبستگی دونقطه‌ای است

تبدیل فوریه سیگنال است

چند ویژگی طیف توان

$\langle x \rangle = 0$

① $C_x(\tau) = \int d\omega\, e^{i\omega\tau}\, S(\omega)$

if $\tau = 0 \quad C_x(0) = \langle x(t)\, x(t) \rangle = \sigma_x^2 = \int d\omega\, S(\omega)$

سطح زیر منحنی طیف توان

سارا ی. داریوش زاده ،

② $$S(\omega) = A(\omega) + i\,B(\omega)$$

$$S(\omega) = \int d\tau\, e^{-i\omega\tau}\, C_x(\tau)$$

$$A(\omega) + i\,B(\omega) = \int d\tau \left[\cos\omega\tau - i\sin\omega\tau\right] C_x(\tau)$$

$$\rightarrow A(\omega) = \int d\tau\, \cos(\omega\tau)\, C_x(\tau)$$

$$\rightarrow B(\omega) = -\int d\tau\, \sin(\omega\tau)\, C_x(\tau)$$

اگر $C_x(\tau)$ $\leftarrow$ $C_x(\tau) = C_x(-\tau)$ $\leftarrow$ جفت ها وجود دارد

$$B(\omega) = -\int_{-\infty}^{+\infty} d\tau\, \underbrace{\sin(\omega\tau)}_{\text{فرد}}\, \underbrace{C_x(\tau)}_{\text{زوج}} = 0$$

در حالت ما، بخش موهومی طیف توان صفر است ⟵ طیف توان حقیقی است.

③ $$S(\omega) \xrightarrow{?} X(\omega)$$

$$S(\omega) = \int d\tau\, e^{-i\omega\tau}\, C_x(\tau)$$

$$= \int d\tau\, e^{-i\omega\tau} \left\langle x(t)\, x(t+\tau) \right\rangle_t$$

$$= \int d\tau\, e^{-i\omega\tau} \int dt\, x(t)\, x(t+\tau)$$

$$= \int d\tau\, e^{-i\omega\tau} \int dt \int d\omega_1\, e^{i\omega_1 t} X(\omega_1) \int d\omega_2\, e^{i\omega_2 (t+\tau)} X(\omega_2)$$

$$= \int d\omega_2 \underbrace{\int d\tau\, e^{-i\tau(\omega-\omega_2)}}_{\delta_D(\omega-\omega_2)} \int dt \int d\omega_1\, e^{i\omega_1 t}\, X(\omega_1)\, e^{i\omega_2 t}\, X(\omega_2)$$

$$= \delta_D(\omega-\omega_2) \int d\omega_1 \underbrace{\int dt\; e^{i\omega_1 t}\, e^{i\omega t}}\; X(\omega_1)\; X(\omega)$$

$$= \delta_D(\omega-\omega_2)\, \delta_D(\omega_1+\omega)\quad X(-\omega)\; X(\omega)$$

$$S(\omega) = X(-\omega)\, X(\omega) = |X(\omega)|^2$$ [illegible]

④

$$\begin{array}{ccc} \{x(t)\} & \longrightarrow & \{\dot{x}(t)\} \\ \downarrow & & \downarrow \\ S_x(\omega) = \checkmark & \xrightarrow{?} & S_{\dot{x}}(\omega) = ? \end{array}$$

$$C_x(\tau) = \langle x(t)\, x(t+\tau) \rangle_t$$

$$\frac{dC_x(\tau)}{d\tau} = \frac{d}{d\tau} \langle x(t)\, x(t+\tau)\rangle_t = \left\langle x(t) \frac{d}{d\tau} x(t+\tau) \right\rangle_t$$

$$= \left\langle x(t) \frac{d}{d(\tau+t)} x(t+\tau) \frac{d(t+\tau)}{d\tau} \right\rangle_t$$

$$= \langle x(t)\, \dot{x}(t+\tau) \rangle_t$$

[illegible] $t \to t-\tau$ [illegible]

$$\frac{d}{d\tau} C_x(\tau) = \left\langle x(t-\tau)\, \dot{x}(t) \right\rangle_t$$

$$\frac{d^2}{d\tau^2} C_x(\tau) = -\left\langle \dot{x}(t-\tau)\, \dot{x}(t) \right\rangle_t \qquad t-\tau \to t$$

$$= -\underbrace{\left\langle \dot{x}(t)\, \dot{x}(t+\tau) \right\rangle_t}$$

$$\boxed{\frac{d^2}{d\tau} C_x(\tau) = -C_{\dot{x}}(\tau)}$$

ترانهادن

$$\boxed{\ddot{C}_x(\tau) = -C_{\dot{x}}(\tau)}$$

$$C_x(\tau) = \int d\omega\, e^{i\omega\tau} S(\omega)$$

$$\frac{d}{d\tau} C_x(\tau) = \int d\omega\, i\omega\, e^{i\omega\tau} S_x(\omega)$$

$$\frac{d^2}{d\tau^2} C_x(\tau) = \int d\omega\, -\omega^2\, e^{i\omega\tau} S_x(\omega)$$

$$\ddot{C}_x(\tau) = \int d\omega\, e^{i\omega\tau}\, -\omega^2 S_x(\omega) = -C_{\dot{x}}(\tau)$$

مستقل

$$\text{〃} \qquad = -\int d\omega\, e^{i\omega\tau} S_{\dot{x}}(\omega)$$

$$-\omega^2 S_x(\omega) = -S_{\dot{x}}(\omega)$$

$$\boxed{S_{\dot{x}}(\omega) = \omega^2 S_x(\omega)}$$

$$\boxed{S_{x^{(n)}}(\omega) = (-1)\ \omega\ S_x(\omega)}$$

$$x^{(n)} \equiv \frac{d^n x}{dt^n}$$

(5) $\langle x(t)\, \dot{x}(t+\tau) \rangle = ?$ زرا شیریان

$$C_x(\tau) = \langle x(t)\, x(t+\tau) \rangle_t$$

$$\frac{d}{d\tau} C_x(\tau) = \langle x(t)\, \dot{x}(t+\tau) \rangle = \frac{d}{d\tau}\left[\int d\omega\, e^{i\omega\tau} S_x(\omega) \right]$$

$$= \int d\omega\ i\omega\ e^{i\omega\tau} S_x(\omega)$$

if $\tau = 0 \Rightarrow \underbrace{\langle x(t)\, \dot{x}(t) \rangle_t}_{\text{حقیقی}} = \underbrace{\int d\omega \overbrace{i\omega}^{\text{موهومی}} \underbrace{S_x(\omega)}_{\text{حقیقی (زوج)}}}_{\text{موهومی}}$

$$\langle x(t)\, \dot{x}(t) \rangle_t = 0$$

$$\boxed{\langle x(t)\, \dot{x}(t+\tau) \rangle_t \neq}$$

$$\begin{cases} \sigma_0 = \langle x^2 \rangle \\ \eta = \dot{x} \end{cases}$$

Recall That:

(1+1)-Dim

$$\langle n_{cross} \rangle = \langle \delta_D(\alpha - \nu\sigma_0)\, |\eta| \rangle =$$

$$= \int d\alpha\, d\eta\ \delta_D(\alpha - \nu\sigma_0)\, |\eta|\, \underbrace{P(\alpha, \eta)}_{\text{[illegible]}}$$

$$P(\alpha,\eta) = \frac{1}{2\pi\sqrt{\mathrm{Det}(\mathrm{Cov})}}\, e^{-(\alpha,\eta)\cdot \mathrm{Cov}^{-1}\cdot \binom{\alpha}{\eta}}$$

$$\mathrm{Cov} = \begin{bmatrix} \langle \alpha\alpha\rangle & \langle \alpha\eta\rangle \\ \langle \eta\alpha\rangle & \langle \eta\eta\rangle \end{bmatrix} = \begin{bmatrix} \langle \alpha\alpha\rangle & \overset{0}{\langle \alpha\dot\alpha\rangle} \\ \cancel{\langle \dot\alpha\alpha\rangle} & \underbrace{\langle \dot\alpha\dot\alpha\rangle}_{\sigma_1^2} \end{bmatrix}$$

$$P(\alpha,\eta) = P(\alpha)\,P(\eta)$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\, e^{-\frac{\alpha^2}{2\sigma_0^2}}\right)\left(\frac{1}{\sqrt{2\pi\sigma_1^2}}\, e^{-\frac{\eta^2}{2\sigma_1^2}}\right)$$

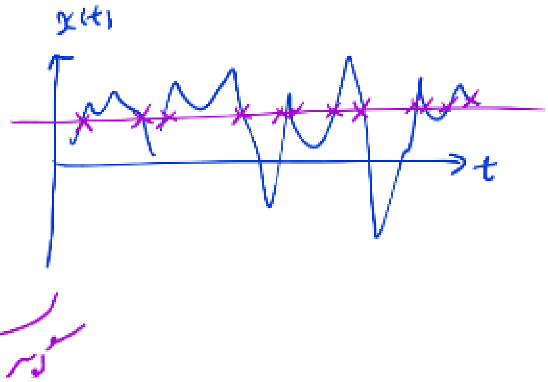
$$\langle n_{\mathrm{cross}}\rangle = \int d\alpha\, \delta_D(\alpha - \nu\sigma_0)\, \frac{1}{\sqrt{2\pi\sigma_0^2}}\, e^{-\alpha^2/2\sigma_0^2}$$

$$\times \int d\eta\, |\eta|\, \frac{1}{\sqrt{2\pi\sigma_1^2}}\, e^{-\eta^2/2\sigma_1^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma_0^2}}\, e^{-\nu^2/2} \times \int_{-\infty}^{+\infty} d\eta\, |\eta|\, \frac{1}{\sqrt{2\pi\sigma_1^2}}\, e^{-\eta^2/2\sigma_1^2}$$

$$= \frac{1}{\sqrt{\cancel{2}\pi\sigma_0^2}}\, e^{-\nu^2/2}\, \sqrt{\frac{2}{\pi}}\,\sigma_1$$

$$\boxed{\langle n_{\mathrm{cross}}(\nu)\rangle = \frac{1}{\pi}\,\frac{\sigma_1}{\sigma_2}\, e^{-\nu^2/2}}$$



(ω)

انرژی های طیفی

$$\longrightarrow \sigma_1^2 = \langle \dot{x}(t)\,\dot{x}(t) \rangle = \langle \eta(t)\,\eta(t) \rangle = \int d\omega\, \omega^2 S_2(\omega)$$

$$\longrightarrow \sigma_0^2 = \langle x(t)\,x(t) \rangle = \int d\omega\, S_2(\omega) \qquad = \int d\omega\, S_2(\omega)$$

جمع بندی انرژی های طیفی براساس طیف توان

$\{\mathcal{H}(\vec{r})\}$ – $\underline{(1+D)\text{-Dimension}}$

$\langle \mathcal{H}(\vec{r}) \rangle = 0$ فرض

$\langle \mathcal{H}(r)\,\mathcal{H}(r) \rangle = \sigma_0^2$

$\langle \mathcal{H}(r)\,\mathcal{H}_{,i}(r) \rangle = 0$ Isotropic field $\qquad \mathcal{H}_{,i} = \dfrac{\partial \mathcal{H}}{\partial r_i}$

$\langle \mathcal{H}(\vec{r})\,\mathcal{H}_{,ij}(\vec{r}) \rangle \overset{?}{=} -\dfrac{1}{D}\,\sigma_1^2\,\delta_{ij} \qquad \mathcal{H}_{,ij} = \dfrac{\partial^2 \mathcal{H}}{\partial r_i\,\partial r_j}$

$1 = i = j \rightarrow \mathcal{H}_{,11} = \dfrac{\partial^2 \mathcal{H}}{\partial x^2}$

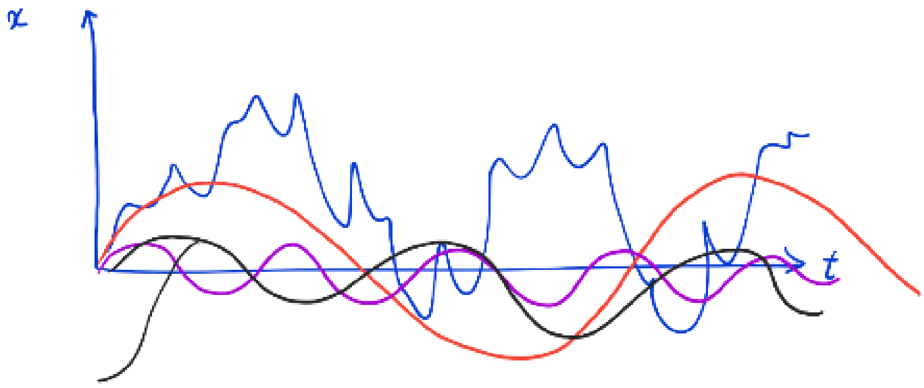
$\langle \mathcal{H}_{,i}(\vec{r})\,\mathcal{H}_{,j}(\vec{r}) \rangle \overset{?}{=} \dfrac{\sigma_1^2}{D}\,\delta_{ij}$

$$\boxed{\sigma_n^2 = \int d^D K\, K^m\, S(K)} \longleftarrow \boxed{\text{Spectral Index}}$$

for $D=3 \qquad m = 2n+2$

for $D=2 \qquad m = 2n+1$

for $D=1 \qquad m = 2n \longrightarrow \sigma_1^2 = \int d\omega\, \omega^2 S(\omega)$



$$\lambda_2 \to \omega_2 = \frac{2\pi}{\lambda_2} \qquad\qquad \omega = 0 \to \lambda = \infty$$

$$\lambda_1 \to \omega_1 = \frac{2\pi}{\lambda_1}$$

$$\lambda_3 \to \omega_3 = \frac{2\pi}{\lambda_3} \qquad\qquad \omega = \infty \to \lambda = 0$$

$$\underset{O(1)}{x(t)} = \int_{-\infty}^{+\infty} \underset{[\frac{1}{t}]}{d\omega}\, e^{i\omega t}\, X(\omega)$$

$$= \int X(\omega)\, e^{i\omega t}\, d\omega$$

$$= \sum_{j=0}^{\infty} X(\omega_j)\, e^{i\omega_j t}$$

$$\underset{O(1)}{x(t)} = X(\omega_0)\, e^{i\omega_0 t} + X(\omega_1)\, e^{i\omega_1 t} + \dots$$

$$X(\omega_n)\, e^{i\omega_n t} + \dots$$

Superposition

$$H(x) = \int dk\, e^{ikx}\, S(k) \qquad\qquad K = \frac{2\pi}{\lambda},$$

مكان

$\frac{1}{[\text{مكان}]}$

Wavenumber

عدد موجي