In the name of God

# Department of Physics
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## ADVANCED STATISTICAL MECHANICS I

### Exercise Set 1

### (Due Date: 1402/07/29)

**1.** Different Probability density functions:

**A** : For A binomial distribution $P_{binomial}(k) = \frac{N!}{k!(N-k)!}p^k q^{(N-k)}$, compute $\langle k\rangle$, $\langle (k-\langle k\rangle)^2\rangle$, $\langle (k-\langle k\rangle)^3\rangle$ and show $P(k)$ for binomial is normalized. (Hint: thr normalization is given by $1 = \sum_{k=0}^{N} P(k)$, and $\langle f(k)\rangle = \sum_{k=0}^{N} P(k)\ f(k)$)
**B** : For A Poisson distribution $P_{poisson}(k) = \frac{\lambda^k}{k!}e^{-\lambda}$ and $\lambda \equiv Np$, compute $\langle k\rangle$, $\langle (k-\langle k\rangle)^2\rangle$, $\langle (k-\langle k\rangle)^3\rangle$ and show $P_{binomial}(k)$ for binomial is normalized. Also show:

$$P_{poisson}(k) = \lim_{N\to\infty} P_{binomial}(k)$$

**C** : Show that $P_{Gaussian}(k) = \lim_{\lambda\to\infty} P_{poisson}(k)$
where a Gaussian distribution is $P_{Gaussian} = \frac{1}{\sqrt{2\pi\lambda}}e^{-\frac{(k-\lambda)^2}{2\lambda}}$, and show that $P_{Gaussian}(k)$ is normalized.

**2.** Give some examples of the phenomena in the nature for which the governing distribution for the relevant parameters is given by Poisson distribution. Give examples for Gaussian and Binomial distributions as well.

**3.** Fluctuations: As discussed in the class, the fluctuation evaluation is a feasible way to show which macrostate to be occurred. Suppose that we have a room divided into two equivalent parts.
**A** : At first, suppose that a room contains $N = 100$ particles. Compute the ratio of probability to have $N_{left} = 99$ to the probability to have $N_{left} = 50$. For each case compute the relative fluctuation defined by $\langle (n-\langle n\rangle)^2\rangle/\langle n\rangle^2$.
**B** : Do the same for $N = 1000$ and compute the the ratio of probability to have $N_{left} = 999$ to the probability to have $N_{left} = 500$.
**C** Compare your results for above cases and deduce the behavior of fluctuations for thermodynamical limit.
Hint: Use the Stirling's approximation to compute $N!$.

**4.** : Central Limit Theorem: Show that the probability of $x$ which is computed by the mean value of various random variables as:

$$x = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} y_i$$

is given by Gaussian Distribution, namely:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}e^{-\frac{x^2}{2\sigma_x^2}}$$

Suppose that the mean value of $\langle y_i\rangle = 0$ for $i = 1, ..., N$ and $\langle y_i^2\rangle = \sigma_y^2$ for all $y_i$ and it is finite. Show that $\sigma_x^2 = \sigma_y^2/N$

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**5.** Characteristic Function: In Probabilistic approach, the characteristic function is defined by:

$$Z_x(\lambda) \equiv \langle e^{i\lambda x} \rangle = \int dx\ P(x) e^{i\lambda x}$$

which is somehow the Fourier transformation of $P(x)$. Also $P(x)$ is probability density function satisfies in normalization condition as $1 = \int dx\ P(x)$.

**A** : Show that the moment of $x$ which is written by $M_n \equiv \langle x^n \rangle = \int dx\ P(x)\ x^n$ is:

$$M_n = \langle x^n \rangle = \left( \frac{d}{d(i\lambda)} \right)^n Z_x(\lambda)|_{\lambda=0}$$

**B** : Show that the Cumulant of $x$ which is written by $K_n \equiv \langle x^n \rangle_c$ is:

$$K_n = \left( \frac{d}{d(i\lambda)} \right)^n \ln(Z_x(\lambda))|_{\lambda=0}$$

and the relation between some moment and cumulant is:

$$K_1 = M_1$$

$$K_2 = M_2 - M_1^2 = \langle x^2 \rangle - \langle x \rangle^2$$

as indicated by e.g. $K_2$, we find that $K_2$ is identical to variance. Therefore, this quantity reveals a measure for fluctuations. Another interesting thing is that if $M_1 = \langle x \rangle = 0$, for a Gaussian process, Only $K_2$ survives as we expect, and therefore, the checking whether the $K_n = 0$ for $n > 3$ if we have $M_1 = 0$ is a proper measure to show the Gaussianity of underlying process. (Gaussianity means that the probability of variable is given by Gaussian function. )

**6.** : Solve question 1.16 of "Fundamentals of Statistical and Thermal Physics 56946th Edition by Frederick Reif".

Good luck, Movahed