Date: 1403/09/29

Name:

In the name of God
Department of Physics, Shahid Beheshti University

STATISTICAL FIELD THEORY AND CRITICAL PHENOMENA

Second midterm exam

(Time allowed: 3:00 hours)

NOTE: All question must be answered. **Write** the answer of each question in **separate sheet**.

1. **Ising model:** According to the RG approach in coordinate space and without extraction the exact form of recursive relation, show that for the 1-D Ising model, we have no non-trivial fixed point. On the contrary, show that for the 2-D Ising model, we expect to have $T_c \neq 0$ as an repulsive fixed point. What about the 3-D Ising model? (Hint: suppose that $h = 0$ and consider the regular substrate for the Ising model.) (10 points)

2. **Recursive relation and β-function:**
   (a) Derive the relation between β-function and $\mathcal{R}_\ell$. (5 points)
   (b) What is the meaning of hyper-critical surface and its relation with Universality class? (5 points)
   (c) Classify the relevant and irrelevant coupling coefficients by means of linearized $\mathcal{R}_\ell$ and β-function? (5 points)

3. **Dynamical critical exponent:** According to dissipation theorem and time-dependent Landau-Ginzburg theory, one can write $\frac{\partial \eta(r,t)}{\partial t} = -\Gamma \frac{\delta \mathcal{L}([K],\eta)}{\delta \eta(r,t)} + \zeta(r,t)$, where t is time, $[K]$'s are coupling and $\zeta(r,t)$ is noise term with Gaussian PDF and $\langle \zeta(r,t)\zeta(r',t')\rangle = D\delta_{Dirac}(r-r')\delta_{tt'}$, Γ is a rate of interaction and

$$\mathcal{L}([K],\eta) = \int d^d r \left[\frac{1}{2}\gamma(\nabla\eta(r,t))^2 + a\eta^2(r,t) + \frac{1}{2}b\eta^4(r,t) - h\eta(r,t)\right]$$

   Also the Linear response function is defined by $\chi(r,t) \equiv \lim_{h\to 0} \frac{\delta\langle \eta(r,t)\rangle}{\delta h}$.

   (a) Now according to the Linear response function defined by $\chi(r,t) \equiv \lim_{h\to 0} \frac{\delta\langle \eta(r,t)\rangle}{\delta h}$, determine the evolution equation for response function. (5 points)
   (b) Determine the static and stationary response function. Explain the physical meaning of your results. (5 points)
   (c) **Bonus question:** Show that the evolution equation of probability for order parameter according to:

$$\partial_t P(\eta(r),t) = \langle \partial_t \delta_{Dirac}(\eta - \eta(\{\bar{\zeta}\}))\rangle_\zeta = \int d^d r' \frac{\delta}{\delta\eta(r')}\left[\Gamma P(\eta(r'))\frac{\delta\mathcal{L}}{\delta\eta(r')} - \langle \zeta\delta_{Dirac}(\eta-\bar{\eta})\rangle_\zeta\right]$$

   would be

$$\partial_t P(\eta(r),t) = \int d^d r' \frac{\delta}{\delta\eta(r')}\left[\Gamma\frac{\delta\mathcal{L}}{\delta\eta(r')}P_\eta + \frac{D}{2}\frac{\delta P_\eta}{\delta\eta(r')}\right]$$

   What is the meaning of your result when $\Gamma = 0$? (10 points)

4. **Widom Hypothesis:** Considering $t_\ell = \ell^{x_t} t$ and $h_\ell = \ell^{x_h} h$, derive the following scaling exponents of scaling relations in terms of x_t and x_h: $M \sim t^\beta$, $M \sim h^{1/\delta}$, $\chi \sim t^\gamma$, $C \sim t^{-\alpha}$, $\xi \sim t^{-\nu}$. (10 points)

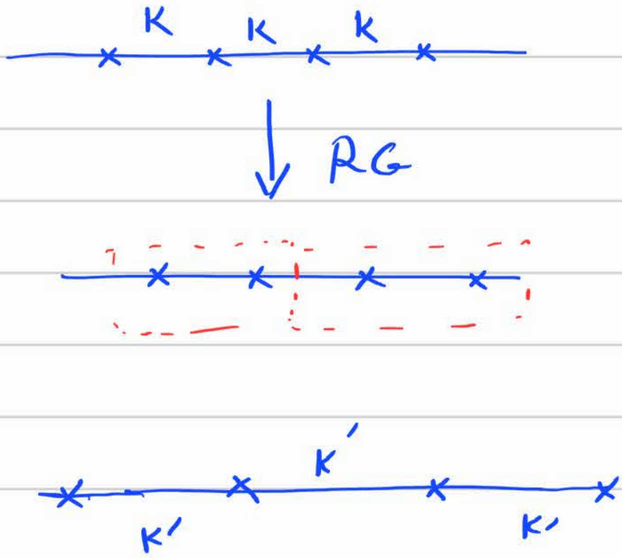
Good luck, Movahed

بسم الله الرحمن الرحيم

Second midterm exam Answer-Key.

① Without deriving Recursive relation we have

1-D Ising model:



$K' = R[K]$

we can deduce that for 1-D Ising

$K' < K$ So by implementing successively The R_ℓ we obtain $K_\ell < K$

Also based on definition of fixed Point, namely

$K^* = R_\ell[K^*]$ $\Rightarrow$ we have two trivial fixed Point

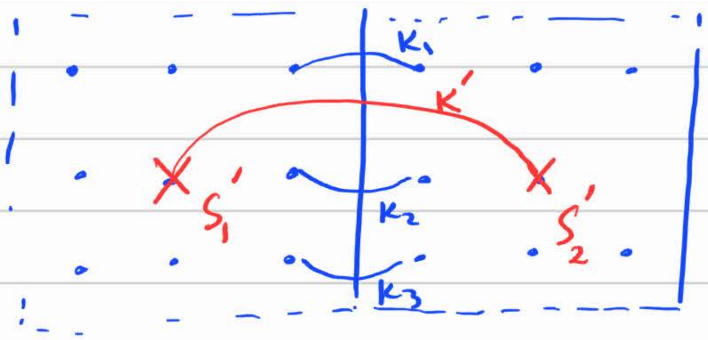
$K^* = 0$ and $K^* = \infty$

Therefore $K^* = 0 \rightarrow T_c = \infty$

$K^* = \infty \rightarrow T_c = 0$ and $T_c = 0$ is repulsive fixed Point.

2-D Ising model. $K_l = R_l[K]$

K



$K' > K$
after applying RG
One find that

$$K^* = R[K^*] \longrightarrow \left\{ \begin{array}{l} K^* = 0 \\ K^* = \infty \end{array} \right\}$$

So $K' > K$ and in this case

$K^* = 0$ and $K^* = \infty$ are both fixed points and therefore we essentially needs to have

$T_c \neq 0$ in somewhere $0 < T_c < \infty$

above consequence is also valid for 3-D Ising model.

(10 - Points)

② -a $[K]_l = R_l[K]$

$$\vec{K}_l = R_l[\vec{K}^*] + (\vec{K} - \vec{K}^*) \left.\frac{\partial R_l}{\partial K}\right|_{\bar{K} = \bar{K}^*} + O(\Delta K^2)$$

$$\vec{K}' - \vec{K}^* \simeq (\vec{K} - \vec{K}^*) \left.\frac{\partial R_\ell}{\partial \vec{K}}\right|_{\vec{K}=\vec{K}^*}$$

$$\vec{k}' \simeq \vec{k} \left.\frac{\partial \vec{K}'}{\partial \vec{K}}\right|_{\vec{K}=\vec{K}^*} \implies \vec{k}' = T_\ell^* \vec{k}$$

$$T\phi = \lambda \phi \qquad \& \qquad \vec{k} = \sum u_i \phi_i$$

$$\vec{k}' = \sum u_i' \phi_i$$

$$u' = \lambda u \quad \& \quad \lambda = \ell^x$$

$$x = \frac{d \ln \lambda}{d \ln \ell}$$

For β-function we have

$$\vec{K}' = K^* + \vec{k} \left.\frac{\partial R_\ell}{\partial \vec{K}}\right|_{\vec{K}=\vec{K}^*}$$

$$\vec{K}' = \vec{K} + \left.\frac{\partial K'}{\partial \ell}\right|_{K'=K} \delta\ell$$

$$= \vec{K} - \frac{\partial \beta_\ell}{\partial \ell} \delta\ell$$

$$\beta_\ell \equiv - \left.\frac{\partial \vec{K}'}{\partial \ell}\right|_{K'=K} \qquad \& \qquad \beta_\ell [K = K^*] = 0$$

$$T\phi_i = \lambda_i \phi_i$$

$$\left(\mathbb{1} - \frac{\partial \vec{\beta}_e}{\partial K}\delta l\right)\phi_i = \lambda_i \phi_i \;\hookrightarrow\; \boxed{T = \left(\mathbb{1} - \frac{\partial \vec{\beta}_e}{\partial K}\delta l\right)}$$

$$\boxed{T = \frac{\partial \vec{R}_e[K]}{\partial K}\bigg|_{K=K^*}}$$

after linearization $\boxed{l = 1 + \delta l}$ we have

$$-\frac{\partial \beta_e}{\partial K}\bigg|_{K=K^*}\phi_i = x_i \phi_i$$

5 Points

(b) The hypercritical surface is a surface that all irrelevant coupling coefficients are living on that and the RG flow of relevant coupling is prependicular on that while the RG flow of irrelevant coupling are tangent on that.

The Universality class includs all models whose

☆ RG flow for irrelevant coupling coefficients are on the same hyper critical surface. and RG flow for relevant coupling coefficients are prependicular on the same hyper critical surface

5 Points

ⓒ $T\phi_i = \lambda_i \phi_i$ $\quad \lambda_i = e^{x_i}$

$-\frac{\partial \beta_e}{\partial k} \phi_i = x_i \phi_i$

$\lambda_i > 1 \ (x_i > 0)$ → Relevant

$\lambda_i < 1 \ (x_i < 0)$ → Irrelevant

$\lambda_i = 1 \ (x_i = 0)$ → Marginal.

5 Point

---

③ a: $$\frac{\partial \eta}{\partial t} = -\Gamma \frac{\delta \mathcal{L}}{\delta \eta} + \zeta(t)$$

also for $\mathcal{L} = \int d^d r \left[\frac{1}{2} \gamma (\nabla \eta)^2 + a\eta^2 + \frac{1}{2} b\eta^4 - h\eta \right]$

$$\left.\frac{\delta \mathcal{L}}{\delta \eta}\right|_{h=0} = -\gamma \nabla^2 \delta\eta + 2a\delta\eta$$

therefore

$$\frac{\partial \delta\eta(t)}{\partial t} = -\Gamma\left[-\gamma\nabla^2\delta\eta + 2a\delta\eta\right] + \zeta(t)$$

F.T.

$$\frac{\partial \delta\eta_k}{\partial t} = -\frac{\delta\eta_k}{\tau_k} + \zeta_k$$

$$\tau_k^{-1} = \tau_0^{-1} + \gamma\Gamma k^2$$ ← Fourier mode

Also

$$\chi = \left.\frac{\delta\langle\eta\rangle}{\delta h}\right|_{h\to 0}$$

$$\frac{\delta \chi}{\delta t} = -\frac{\chi}{\tau_0} + \gamma\Gamma\nabla^2\chi + \Gamma$$

F.T.

$$-i\omega\tilde{\chi}(\omega,k) = -\frac{\tilde{\chi}}{\tau_0} - \gamma\Gamma k^2\tilde{\chi} + \Gamma$$

$$\tilde{\chi}(\omega,k) = \frac{\Gamma}{-i\omega + \tau_k^{-1}}$$

5 points

b: Static Response function $\omega \to 0$

$$\tilde{\chi}(0, K) = \frac{\Gamma}{\tau_K^{-1}}$$

Global Response funct $K \to 0$

$\tilde{\chi}(0,0) \longrightarrow \infty$: this means that we have an infinite response in our system (Thermodynamical limit) corresponds on divergency in Phase Transition.

$\boxed{\text{5 Points}}$

c: We are interested in deriving.

$$\frac{\partial P}{\partial t} = \int d^d r' \frac{\delta}{\delta \eta(r')} \left[\Gamma \frac{\delta \mathcal{L}}{\delta \eta(r)} P + \frac{D}{2} \frac{\delta P}{\delta \eta} \right]$$

in analogous of Fokker-Planck Equation

$$\frac{\partial P}{\partial t} = \left[-\frac{\partial}{\partial \eta} (D^1) + \frac{1}{2} \left(\frac{\partial}{\partial \eta} \right)^2 D^{(2)} \right] P(\eta, t)$$

According to the definition of $P(\eta,t) = \langle \delta_D(\eta-\eta') \rangle_{\eta'}$

we have. $$\frac{\partial}{\partial t} P = \left\langle \frac{\partial}{\partial t} \delta_D(\eta-\eta') \right\rangle_{\eta'} = \left\langle \frac{\partial}{\partial t} \delta_D(\eta-\eta'(\zeta)) \right\rangle_{\zeta}$$

$$= \int d^d r' \left\langle \frac{\partial \eta'}{\partial t} \frac{\partial}{\partial \eta'} \delta_D(\eta-\eta'(\zeta)) \right\rangle_{\zeta}$$

Also $$\left\langle \frac{\partial \eta'}{\partial t} \frac{\partial}{\partial \eta'} \delta_D \right\rangle_{\zeta} = \left\langle \frac{\partial \eta'}{\partial t} \frac{\partial}{\partial \eta'} \delta_D(\eta-\eta') \right\rangle_{\eta'}$$

$$= \int d\eta' \frac{\partial \eta'}{\partial t} \frac{\partial}{\partial \eta'} \delta_D(\eta-\eta') P(\eta')$$

$$d\eta' \frac{\partial}{\partial \eta'} \delta_D(\eta-\eta') = dV \qquad \frac{\partial \eta'}{\partial t} P(\eta') = U$$

$$\int dV\, U = UV - \int d\eta' \delta_D(\eta-\eta') \frac{\partial}{\partial \eta'} \frac{\partial \eta'}{\partial t} P(\eta')$$

Boundary condition

$$= 0 - \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial t} P(\eta)$$

$$= -\int d^d r' \frac{\partial}{\partial \eta(r')} \left\langle \frac{\partial \eta'}{\partial t} \delta_D(\eta-\eta') \right\rangle_{\zeta}$$

Also $$\frac{\partial \eta}{\partial t} = -\Gamma \frac{\delta \mathcal{L}}{\delta \eta} + \zeta$$

Therefore

$$\frac{\partial P}{\partial t} = -\int d^d r' \frac{\partial}{\partial \eta(r')} \left\langle \left(-\Gamma \frac{\delta \mathcal{L}}{\delta \eta(r')} + \xi \right) \delta_D(\eta - \eta') \right\rangle_\xi$$

$$= \int d^d r' \frac{\partial}{\partial \eta(r')} \left[\Gamma \frac{\delta \mathcal{L}}{\delta \eta(r')} P_\eta - \underbrace{\left\langle \xi \, \delta_D(\eta - \eta') \right\rangle_\xi}_{?} \right]$$

Generally $\left\langle f(\xi) \xi \right\rangle_\xi = D \left\langle \frac{\delta f}{\delta \xi} \right\rangle_\xi$ To this end we have

$$D \left\langle \frac{\delta f}{\delta \xi} \right\rangle_\xi = D \int \left(\frac{\delta f}{\delta \xi} \right) P(\xi) \, d\xi$$

$$= D \int \left[\frac{\delta}{\delta \xi} \left(F P(\xi) \right) - f(\xi) \frac{\delta P(\xi)}{\delta \xi} \right] d\xi$$

$$= -D \int f(\xi) \frac{\delta}{\delta \xi} \overbrace{P(\xi)}^{\text{Gaussian form}} \, d\xi$$

$$\boxed{D \left\langle \frac{\delta f}{\delta \xi} \right\rangle_\xi = \int f(\xi) \xi \, P(\xi) \, d\xi = \left\langle f(\xi) \xi \right\rangle_\xi}$$

therefore $\left\langle \xi \, \delta_D(\eta - \eta') \right\rangle_\xi = D \left\langle \frac{\delta}{\delta \xi} \delta_D(\eta - \eta') \right\rangle_\xi$

$$= D \int d^d r'' \left\langle \frac{\delta \eta'}{\delta \xi} \frac{\delta}{\delta \eta'} \delta(\eta - \eta') \right\rangle_\xi$$

(A.8.4) & (A8.5) & (A8.6) $\Rightarrow$ Q.E.D. (10 points)

4: $t = \ell^{x_t} t$

$h = \ell^{x_h} h$

☆ $M \sim t^{\beta}\Big|_{h=0}$, $M = \frac{\partial f}{\partial h}\Big|_{h=0}$

$$f(\ell^{x_t} t, \ell^{x_h} h) = \ell f(t,h)$$

$$\ell^{x_h} M(\ell^{x_t} t, \ell^{x_h} h) = \ell M(t,h)$$

$h=0$ and $\ell = t^{-1/x_t}$

$$\left(t^{-\frac{1}{x_t}}\right)^{x_h} M(1,0) = t^{-\frac{1}{x_t}} M(t,0)$$

$$M(t) = t^{\frac{1-x_h}{x_t}} M(1,0) \sim t^{\beta}$$

$$\boxed{\beta = \frac{1-x_h}{x_t}}$$

☆ $M \sim h^{1/\delta}$ $t=0$, $\ell = h^{-1/x_h}$ $\longrightarrow$ $M(0,h) = h^{\frac{1-x_h}{x_h}} M$

$\sim h^{1/\delta}$

$$\boxed{\delta = \frac{x_h}{1-x_h}}$$

☆ $\chi \sim t^{-\alpha}$ $\qquad \chi = \frac{\partial M}{\partial h}\Big|_{h=0}$

$$l^{2x_h} \chi(l^{x_t} t, l^{x_h} h) = l \chi(t, h)$$

$$l = t^{-1/x_t} \quad , \; h = 0$$

$$t^{-\frac{2x_h}{x_t}} \chi(1,0) = t^{-\frac{1}{x_t}} \chi(t,0)$$

$$\chi(t,0) = t^{\frac{1-2x_h}{x_t}} \chi(1,0) \sim t^{\gamma}$$

$$\boxed{\gamma = \frac{1-2x_h}{x_t}}$$

☆ $C = T \frac{\partial^2 f}{\partial T^2}\Big|_{h=0}$

$$l^{2x_t} C(l^{x_t} t, l^{x_h} h) = l C(t, h)$$

$$l = t^{-1/x_t}, \; h = 0$$

$$t^{-2} C(1,0) = t^{-\frac{1}{x_t}} C(t,0)$$

$$C(t,0) = t^{\frac{1}{x_t} - 2} C(1,0) \sim t^{-\alpha}$$

$$\boxed{\alpha = 2 - \frac{1}{x_t}}$$

$\star \quad \xi \sim t^{-\nu}$

$\xi_\ell = \ell^{-1} \xi$

$\xi(\ell^{x_t} t, \ell^{x_h} h) = \ell^{-1} \xi(t,h)$

$$\xi(1,0) = t^{+1/x_t} \xi(t,0) \longleftarrow \left\{ \begin{array}{l} \ell^{x_t} t = 1 \\ \ell = t^{-1/x_t} \\ h = 0 \end{array} \right\}$$

$$\xi(t,0) = t^{-\frac{1}{x_t}} \xi(1,0) \sim t^{-\nu}$$

$$\boxed{\nu = +\frac{1}{x_t}}$$

10 points

8.5 SUMMARY

The scaling hypothesis is a valuable way to correlate data for systems near the critical point, and for some non-equilibrium systems as they approach equilibrium. These are empirical statements. In the following chapter, we will discuss how the renormalisation group accounts for the success of scaling ideas, for systems in equilibrium. Certain driven non-equilibrium systems, such as models of atomic deposition, exhibit a formal analogy to critical dynamics, and this has proven to be useful in analysing these systems. Finally, there are tantalising suggestions that some systems approaching equilibrium may also be usefully described by renormalisation group ideas. Further discussion of this topic is to be found in chapter 10.

APPENDIX 8 - THE FOKKER-PLANCK EQUATION

In this appendix, we sketch the derivation of the Fokker-Planck equation, starting from the Langevin equation (8.41).

We begin with the definition of the probability distribution for the order parameter $\eta(\mathbf{r})$, and differentiate with respect to time:

$$\begin{aligned}
\partial_t P_\eta(\{\eta(\mathbf{r})\},t) &= \langle \partial_t \delta\left[\eta(\mathbf{r}) - \overline{\eta}(\mathbf{r},t,\{\zeta\})\right]\rangle_\zeta \\
&= \int d^d\mathbf{r}' \left\langle \frac{\partial\overline{\eta}}{\partial t}\frac{\delta}{\delta\overline{\eta}(\mathbf{r}',t)}\delta\left[\eta(\mathbf{r}) - \overline{\eta}(\mathbf{r},t,\{\zeta\})\right]\right\rangle_\zeta \\
&= -\int d^d\mathbf{r}' \frac{\delta}{\delta\eta(\mathbf{r}')}\left\langle \frac{\partial\overline{\eta}}{\partial t}\delta\left[\eta-\overline{\eta}\right]\right\rangle_\zeta \\
&= -\int d^d\mathbf{r}' \frac{\delta}{\delta\eta(\mathbf{r}')}\left\langle \left[-\Gamma\frac{\delta L}{\delta\eta(\mathbf{r}')} + \zeta(\mathbf{r}',t)\right]\delta\left[\eta-\overline{\eta}\right]\right\rangle_\zeta \\
&= \int d^d\mathbf{r}' \frac{\delta}{\delta\eta(\mathbf{r}')}\left[\Gamma P_\eta \frac{\delta L}{\delta\eta(\mathbf{r}')} - \langle\zeta\delta(\eta-\overline{\eta})\rangle_\zeta\right] \qquad \text{(A8.1)}
\end{aligned}$$

The evaluation of $\langle\zeta\delta(\eta-\overline{\eta})\rangle_\zeta$ is accomplished by noting the general result

$$\begin{aligned}
\langle F\{\zeta\}\zeta\rangle_\zeta &= \int D\zeta\ (\zeta F)\ P_\zeta \\
&= D\int D\zeta\,\frac{\delta F}{\delta\zeta}P_\zeta \\
&= D\left\langle\frac{\delta F}{\delta\zeta}\right\rangle_\zeta \qquad \text{(A8.2)}
\end{aligned}$$

where we integrated by parts and used the expression (8.42) for the probability distribution. These operations can be verified, if desired, by writing the functional integral as a multiple integral. In the present case,

$$\begin{aligned}\langle\zeta(\mathbf{r}',t)\delta(\eta-\overline{\eta})\rangle_\zeta &= D\left\langle\frac{\delta}{\delta\zeta(\mathbf{r}',t)}\delta(\eta-\overline{\eta})\right\rangle_\zeta\\ &= D\int d\mathbf{r}''\left\langle\frac{\delta\overline{\eta}(\mathbf{r}'',t)}{\delta\zeta(\mathbf{r}',t)}\frac{\delta}{\delta\overline{\eta}(\mathbf{r}'',t)}\delta(\eta-\overline{\eta})\right\rangle_\zeta\\ &= -D\int d\mathbf{r}''\frac{\delta}{\delta\eta(\mathbf{r}'',t)}\left\langle\frac{\delta\overline{\eta}(\mathbf{r}'',t)}{\delta\zeta(\mathbf{r}',t)}\delta(\eta-\overline{\eta})\right\rangle_\zeta \quad \text{(A8.3)}\end{aligned}$$

The quantity being averaged in the above expression is essentially a response function, and can be evaluated from the formal solution to the Langevin equation (8.41):

$$\overline{\eta}(\mathbf{r},t) = \overline{\eta}(\mathbf{r},0) - \int_0^t dt'\,\Gamma\frac{\delta L}{\delta\eta}\left(\overline{\eta}(\mathbf{r},t')\right) + \int_0^t dt'\,\zeta(\mathbf{r},t'). \qquad \text{(A8.4)}$$

Differentiating with respect to $\zeta(\mathbf{r}',t'')$ and noting that causality implies that $\eta(\mathbf{r},t)$ only depends on $\zeta(\mathbf{r},t')$ when $t > t'$, we find that

$$\frac{\delta\overline{\eta}(\mathbf{r},t)}{\delta\zeta(\mathbf{r}',t'')} = \delta(\mathbf{r}-\mathbf{r}')\left[-\int_{t''}^t dt'\left\{\frac{\delta}{\delta\zeta(\mathbf{r}',t'')}\Gamma\frac{\delta L}{\delta\eta}\left(\overline{\eta}(\mathbf{r},t')\right)\right\} + \theta(t-t'')\right]. \qquad \text{(A8.5)}$$

In eqn. (A8.3), we require the above quantity evaluated at $t = t''$. The value of the Heaviside function $\theta(t)$ at zero is in this case $\theta(0) = 1/2$, as can be seen by repeating the above derivation with the two-point correlation function of ζ being proportional not to $\delta(t-t')$, but to a sharply-peaked even function of $(t-t')$. Thus

$$\frac{\delta\overline{\eta}(\mathbf{r}'',t)}{\delta\zeta(\mathbf{r}',t)} = \frac{1}{2}\delta(\mathbf{r}-\mathbf{r}'). \qquad \text{(A8.6)}$$

Substituting into eqn. (A8.3), and collecting results back through eqn. (A8.1), we finally obtain

$$\partial_t P_\eta(\{\eta(\mathbf{r})\},t) = \int d^d\mathbf{r}'\,\frac{\delta}{\delta\eta(\mathbf{r}')}\left[\Gamma\frac{\delta L}{\delta\eta(\mathbf{r}')}P_\eta + \frac{D}{2}\frac{\delta P_\eta}{\delta\eta(\mathbf{r}')}\right], \qquad \text{(A8.7)}$$

which is our desired result.