بسم الله الرحمن الرحیم

جلسه ۳۰/۹/۱۴۰۰

Exercise set 7

9 ① **Fisher forecast :** $y_{theory} = a x^{H}$

$\{\theta\} = \{a, H\}$. $j = 1, 2$ $\boxed{M = 2}$

$$\boxed{\{D\} : \{(x_i, y_i)\} \, , \quad i = 1, \quad N}$$

"if available"

We are going to evaluat errors for free Parameter (Not Best Value)

فرض می‌کنیم وجود دارد.

According to Q1 (Set 6) $\longrightarrow$ $\{a_{best}, H_{best}\}$.

این عبارت را داریم.

Recall that :

عدد

$$\mathcal{L}_{Rel} = e^{-\frac{\Delta\chi^2}{2}} \quad , \quad \Delta\chi^2 \equiv \Delta^T . C^{-1} . \Delta - \chi^2_{min}$$

$$\Delta^T \equiv \left(y_{obs} - y_{theory} \right)_{1 \times N}$$

$\longrightarrow$ $C \equiv$ Covariance for observation (systematic) if C is diagonal

$$\Delta\chi^2 = \sum_{i=1}^{N} \frac{\left[y_{obs}(i) - y_{theory}(x_i, \{\theta\})\right]^2}{\sigma_i^2} - \chi^2_{min}$$

$\to \sigma_i^2$ Diagonal Element of C

$$\chi^2_{min} = \chi^2(\{\theta\}_{Best})$$

$$\mathcal{L}_{Rel} = \sqrt{\frac{Det(F)}{(2\pi)^2}}\, e^{-\frac{\Delta\theta^T \cdot F \cdot \Delta\theta}{2}}$$

$$\Delta\theta^T \equiv (\theta - \theta_{best})_{1\times M = 1\times 2}$$

$$F: \begin{bmatrix} & \\ & \end{bmatrix}_{2\times 2} \longrightarrow \boxed{C_\theta = F^{-1}}$$

$$F_{ij} = \frac{1}{2}\left\langle \frac{\partial^2 \Delta\chi^2}{\partial\theta_i \partial\theta_j} \right\rangle$$

$$\boxed{F_{ij} = \left\langle \left(\frac{\partial y_{theor}}{\partial\theta_i}\right)^T_{best} \cdot C^{-1} \cdot \left.\frac{\partial y_{the}}{\partial\theta_j}\right|_{best} \right\rangle}$$

if $N=2$, $C = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 \end{bmatrix}$

$$y_{the} = a x_i^{H}$$

$\theta_2 \rightarrow H$, $a \rightarrow \theta_1$

$$\left.\frac{\partial y_{th}}{\partial a}\right|_{best} = x_i^{H} = x_i^{H_{best}}$$

$$\left.\frac{\partial y_{th}}{\partial H}\right|_{best} = \left(a_{best}\, x^{H_{best}}\right) \ln x$$

$$F_{11} = \left(\frac{\partial y_{th}(x_1)}{\partial a} , \frac{\partial y_{th}(x_2)}{\partial a} \right) \left\{ \frac{1}{\sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}\sigma_{21}} \begin{bmatrix} \sigma_{22}^2 & -\sigma_{12}^2 \\ -\sigma_{21}^2 & \sigma_{11}^2 \end{bmatrix} \right\}$$

$$\begin{pmatrix} \frac{\partial y_{th}(x_1)}{\partial a} \\ \frac{\partial y_{th}(x_2)}{\partial a} \end{pmatrix}$$

$$F_{11} = \frac{\left(x_1^{H} , x_2^{H} \right)}{\sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}\sigma_{21}} \begin{bmatrix} \sigma_{22}^2 x_1^{H} - \sigma_{12}^2 x_2^{H} \\ -\sigma_{21}^2 x_1^{H} + \sigma_{22}^2 x_2^{H} \end{bmatrix} =$$

$$F_{11} = \frac{1}{\sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}\sigma_{21}} \left(\sigma_{22}^2 (x_1^{H})^2 - \sigma_{12}^2 x_1^{H} x_2^{H} - \sigma_{21}^2 x_2^{H} x_1^{H} + \sigma_{22}^2 (x_2^{H})^2 \right)$$

$$F_{12} = \left\langle \overbrace{\frac{\partial y_{th}}{\partial a} \cdot C^{-1}} \cdot \frac{\partial y_{th}}{\partial H} \right\rangle_{ens} = \mathrm{Tr}\left[\bigcirc \right]$$

$$= \left(x_1^H ,\ x_2^H \right) \frac{1}{\sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}^2 \sigma_{21}^2} \begin{pmatrix} \sigma_{22}^2 & -\sigma_{12}^2 \\ -\sigma_{21}^2 & \sigma_{11}^2 \end{pmatrix} \begin{pmatrix} a x_1^H \ln x_1 \\ a x_2^H \ln x_2 \end{pmatrix}$$

$$= \frac{\left(x_1^H , x_2^H \right)}{\sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}^2 \sigma_{21}^2} \begin{pmatrix} \sigma_{22}^2 a x_1^H \ln x_1 - \sigma_{12}^2 a x_2^H \ln x_2 \\ -\sigma_{21}^2 a x_1^H \ln x_1 + \sigma_{11}^2 a x_2^H \ln x_2 \end{pmatrix}$$

$$F_{12} = \frac{1}{\sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}^2 \sigma_{21}^2} \Big[x_1^H \sigma_{22}^2 a x_1^H \ln x_1 - x_1^H \sigma_{12}^2 a x_2^H \ln x_2 - x_2^H \sigma_{21}^2 a x_1^H \ln x_1 + x_2^H \sigma_{11}^2 a x_2^H \ln x_2 \Big]$$

$$F_{22} = \left(\frac{\partial y_{th}(x_1)}{\partial H} ,\ \frac{\partial y_{th}(x_2)}{\partial H} \right) \cdot C^{-1} \cdot \begin{pmatrix} \dfrac{\partial y_{th}(x_1)}{\partial H} \\ \dfrac{\partial y_{th}(x_2)}{\partial H} \end{pmatrix}$$

$$= \left(a x_1^H \ln x_1 ,\ a x_2^H \ln x_2 \right) \frac{1}{\sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}^2 \sigma_{21}^2} \begin{pmatrix} \sigma_{22}^2 & -\sigma_{12}^2 \\ -\sigma_{21}^2 & \sigma_{11}^2 \end{pmatrix} \begin{pmatrix} a x_1^H \ln x_1 \\ a x_2^H \ln x_2 \end{pmatrix}$$

$$= \frac{\left(a x_1^H \ln x_1 , a x_2^H \ln x_2 \right)}{\sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}^2 \sigma_{21}^2} \begin{pmatrix} \sigma_{22}^2 a x_1^H \ln x_1 - \sigma_{12}^2 a x_2^H \ln x_2 \\ -\sigma_{21}^2 a x_1^H \ln x_1 + \sigma_{22}^2 a x_2^H \ln x_2 \end{pmatrix}$$

$$F_{22} = \frac{1}{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2\sigma_{21}^2}\Big[ax_1^H \ln x_1\, \sigma_{22}^2\, ax_1^H \ln x_1 - ax_1^H \ln x_1\, \sigma_{12}^2\, ax_2^H \ln x_2$$

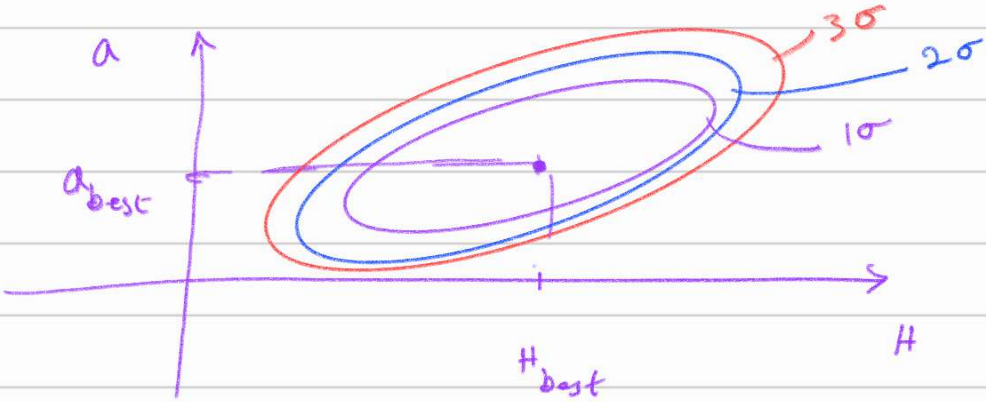
$$- ax_2^H \ln x_2\, \sigma_{21}^2\, ax_1^H \ln x_1 + ax_2^H \ln x_2\, \sigma_{22}^2\, ax_2^H \ln x_2 \Big]$$

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{bmatrix} \longrightarrow C_\theta = \frac{1}{F_{11}F_{22} - F_{12}F_{21}} \begin{bmatrix} F_{22} & -F_{12} \\ -F_{12} & F_{11} \end{bmatrix}$$

$$C_\theta = \begin{bmatrix} \sigma_{aa}^2 & \sigma_{aH}^2 \\ \sigma_{Ha}^2 & \sigma_{HH}^2 \end{bmatrix} = \begin{bmatrix} \dfrac{F_{22}}{F_{11}F_{22} - F_{12}F_{21}} & \dfrac{-F_{12}}{F_{11}F_{22} - F_{12}F_{21}} \\[2ex] -\dfrac{F_{21}}{F_{11}F_{22} - F_{12}F_{21}} & \dfrac{F_{11}}{F_{11}F_{22} - F_{21}F_{12}} \end{bmatrix}$$

$$\frac{(a - a_{best})^2}{\sigma_{aa}^2(1-\rho^2)} + \frac{(H - H_{best})^2}{\sigma_{HH}^2(1-\rho^2)} - \frac{2\rho}{(1-\rho^2)} \frac{(a - a_{best})(H - H_{best})}{\sigma_{aa}\,\sigma_{HH}} = \Delta\chi^2$$

$$\rho = \frac{\sigma_{aH}^2}{\sigma_{aa}\,\sigma_{HH}} \Rightarrow \boxed{\langle \delta a\, \delta H \rangle = \rho\, \sigma_{aa}\, \sigma_{HH}}$$



for M=2 (a, H)

68.3% $\leftarrow$ $\Delta\chi^2 = 2.3$ $\longrightarrow$ 1σ Confidence Interval.

95.4% $\leftarrow$ $\Delta\chi^2 = 6.17$ $\longrightarrow$ 2σ

99.73% $\leftarrow$ $\Delta\chi^2 = 11.8$ $\longrightarrow$ 3σ

Q4 (Set 7)

$$\mathcal{H} = \sum_{i=1}^{3N} \left\{ \frac{p_i^2}{2m} + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} u_{ij} \right\}$$

$$U = E = \langle \mathcal{H} \rangle = -\frac{1}{\beta} \frac{\partial}{\partial \beta} \ln Z(T, V, N)$$

$$Z(T, V, N) = \int d\Gamma \, e^{-\beta \mathcal{H}}$$

Independent par

$$y_{the} \rightarrow E_{Th}(T) = \frac{3 k_B T}{3} + \frac{n}{2} \int u(r) \, g(r) \, d^3 r$$

g of r $\rightarrow$ Clustering

Un-Weighted TPCF

$$U(r) = 4\epsilon\left[\left(\frac{\alpha}{r}\right)^{a} - \left(\frac{\sigma}{r}\right)^{b}\right]$$

$$g(r) = e^{-\beta u(r) + \beta n F(r)}$$

$$F = \rho\beta,$$

$$\{\theta\} = \{\bar{a} = 12,\ \bar{b} = 6,\ \bar{\epsilon} = 0.997,\ \bar{\alpha} = 3.40,\ \bar{\sigma} = 3.45,\ \bar{\rho} = 0.001\}$$

$M = 6 \Rightarrow$ # of free Parameters.

$$[F]_{6\times 6} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} & F_{26} \\ F_{31} & F_{32} & F_{33} & F_{34} & F_{35} & F_{36} \\ F_{41} & F_{42} & F_{43} & F_{44} & F_{45} & F_{46} \\ F_{51} & F_{52} & F_{53} & F_{54} & F_{55} & F_{56} \\ F_{61} & F_{62} & F_{63} & F_{64} & F_{65} & F_{66} \end{bmatrix}_{6\times 6}$$

$$F_{ij} = \left\langle \frac{\partial E(T,\{\theta\})}{\partial \theta_i} \cdot C^{-1} \cdot \frac{\partial E(T,\{\theta\})}{\partial \theta_j} \right\rangle_{ensem}$$

$$F_{aa} = \left\langle \frac{\partial E(T,\{\theta\})}{\partial a} \cdot C^{-1} \cdot \frac{\partial E(T,\{\theta\})}{\partial a} \right\rangle_{en}$$

$$\frac{\partial E(\vec{T}, \{\theta\})}{\partial a} = \frac{\partial}{\partial a}\left[\frac{3 k_B T}{3} + \frac{1}{2}\int_0^\infty 4\epsilon\left[\left(\frac{\alpha}{r}\right)^a - \left(\frac{\sigma}{r}\right)^b\right] \cdot \times e^{-\beta\left[4\epsilon\left\{\left(\frac{\alpha}{r}\right)^a - \left(\frac{\sigma}{r}\right)^b\right\}\right] + \beta^2 q} \, 4\pi r^2 dr \right]$$

$$'' = \frac{1}{2}\int dr\, 4\pi r^2\, 4\epsilon\, \frac{\partial}{\partial a}\left\{ \left[\left(\frac{\alpha}{r}\right)^a - \left(\frac{\sigma}{r}\right)^b\right] e^{-\beta 4\epsilon\left[\left(\frac{\alpha}{r}\right)^a - \left(\frac{\sigma}{r}\right)^b\right] + \beta^2 q} \right\}$$

$$I = \int_0^\infty d^3\vec{r}\; u(\vec{r})\, g(\vec{r})$$

$$I(\{\theta\}, T) = \int d^3 r\; u(r, \{\theta\})\, g(r, \{\theta\})$$

$$\frac{\partial I}{\partial \theta_i} \simeq \frac{I(\theta_i + \Delta\vec{\theta}_i, \theta_2, \theta \ldots) - I(\theta_i - \Delta\theta_i, \ldots)}{2\Delta\theta_i}$$

Q3 (Set 7)

$G(x, \{\theta\}) = y_{the}(x)\{\theta\}$

$\{\theta\} = \{\Omega_m, \Omega_\lambda, H_0, \omega_0, \omega_1\}$.

$$F_{ij} = \left\langle \frac{\partial G(x, \{\theta\})}{\partial \theta_i} \cdot C^{-1} \cdot \frac{\partial G(x, \{\theta\})}{\partial \theta_j} \right\rangle$$

$$\frac{\partial G(x, \{\theta\})}{\partial \Omega_m} = ?$$

$y \equiv \ln x$

$$\left\{ \frac{d^2 G}{dy^2} + \frac{3}{2}\left(\frac{1}{3} + \frac{\Omega_k}{2} - \omega \Omega_\lambda\right) \frac{dG}{dy} - \frac{3}{2} \Omega G = 0 \right\}$$

$$A = \frac{dG}{dy}$$

$$\frac{d^2 G}{dy^2} = \frac{dA}{dy} = -\frac{3}{2}\left(\frac{1}{3} + \frac{\Omega_k}{2} - \omega \Omega_\lambda\right) A + \frac{3}{2} \Omega G$$

$$\frac{dG}{dy} = A$$

$$\left\{ \begin{array}{l} A(\underbrace{0.001}_{x}) = 1 \\ G(0.001) = 0 \end{array} \right\}$$

(G, A) دو معادله جفت شده مرتبه اول ←

$G(x)$ — Numerical Solution

$G(\Omega_m, x, \ldots) = \ldots$

$G(\Omega_m + \Delta\Omega_m, x) \ldots = \ldots$

$G(\Omega_m - \Delta\Omega_m, x \ldots) = \ldots$

$$\Rightarrow \frac{\partial G}{\partial \Omega_m} = \frac{G(\Omega_m + \Delta\Omega_m) - G(\Omega_m - \Delta\Omega_m)}{2\Delta\Omega_m}$$