بسم الله الرحمن الرحیم

Summary on Data Modeling

① $$\mathcal{L}_{Rel} = e^{-\frac{\Delta\chi^2}{2}} \qquad \Delta\chi^2 \equiv \Delta^T . C^{-1} . \Delta - \chi^2_{min}$$

$$\Delta^T = (y_{obs} - y_{theory})_{1\times N}$$

<u>C = Covariance Matrix for observation</u>

for diagonal <u>C</u> we have.

$$\rightarrow \Delta\chi^2 = \sum_{i=1}^{N} \frac{[y_{obs}(x_i) - y_{theory}(x_i;\{\theta\})]^2}{\sigma_i^2} - \chi^2_{min}$$

عناصر ماتریس کوواریانس

$$\chi^2_{min} = \chi^2(\{\theta\}_{Best})$$

② $$\mathcal{L}_{Rel} = \sqrt{\frac{Det(F)}{(2\pi)^M}}\, e^{-\frac{\Delta\theta^T . F . \Delta\theta}{2}}$$

$$\Delta\theta^T = (\theta - \theta_{best})_{1\times M} ,$$

$$F = \begin{bmatrix} & & \end{bmatrix}_{M \times M} \longrightarrow C_\theta = F^{-1}$$

$\uparrow$

The Goal for Data modeling here.

$\rightarrow \theta_{Best}$

$\rightarrow \boxed{\sigma_{\theta_{Best}}}$

$\rightarrow$ Goodness of fit

① and ② $\rightarrow$

$$\mathcal{L}_{Rel} = e^{-\frac{\Delta\chi^2}{2}} = \sqrt{\frac{Det(F)}{(2\pi)^M}}\; e^{-\frac{\Delta\theta^T \cdot F \cdot \Delta\theta}{2}}$$

$$\rightarrow \ln \mathcal{L}_{Rel} = -\frac{\Delta\chi^2}{2} = -\frac{\Delta\theta^T \cdot F \cdot \Delta\theta}{2} + \ln\left(\quad\right)$$

③ $$F_{ij} = -\left\langle \left. \frac{\partial^2 \ln \mathcal{L}_{Rel}}{\partial\theta_i \, \partial\theta_j} \right|_{\{\theta\}=\{\theta\}_{Best}} \right\rangle$$

$$= \frac{1}{2}\left\langle \left. \frac{\partial^2 \Delta\chi^2}{\partial\theta_i \, \partial\theta_j} \right|_{\{\theta\}=\{\theta\}_{Best}} \right\rangle$$

④ $$f \equiv \underbrace{-\ln \mathcal{L}_{Rel}}$$ $\to \Delta\chi^2$, $\to F$

$$\to f = \frac{1}{2}\,\mathrm{Tr}\left(\underbrace{\Delta^T \cdot C^{-1} \cdot \Delta}_{\text{عدد}}\right) + cte$$

Recall
$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

⑤ $$\frac{\partial f}{\partial \theta_i} = \mathrm{Tr}\left(C^{-1}\frac{\partial}{\partial \theta_i}(\Delta^T \Delta)\right)$$

مشاهده‌ها داده‌ی تجربی

پارامتر مدل

y_{Theory}

$$\left\langle \frac{\partial f}{\partial \theta_i} \right\rangle = \mathrm{Tr}\left(C^{-1}\left\langle (\Delta^T \Delta)_{,i} \right\rangle\right)$$

0

$\{\theta\} \to \{\theta\}_{best}$

Since according to our condition the min value of χ^2 is associated to the best value of parameter therefore χ^2_{min} is extremum

C = Diagonal $\quad f = \sum \frac{[y_{obs} - y_{the}(x_i, \theta)]^2}{\sigma_i^2}$

$$0 = \left\langle \frac{\partial f}{\partial \theta_i} \right\rangle = \sum \frac{2}{\sigma_i^2}\frac{\partial y_{the}}{\partial \theta_i}\left\langle [y_{obs} - y_{the}] \right\rangle \Big|_{\theta = \theta_{Best}}$$

0

⑥ $$\frac{\partial^2 f}{\partial\theta_i\,\partial\theta_j} = \mathrm{Tr}\left[C^{-1} \cdot (\Delta^T \Delta)_{;ij} \right]$$

$$= \mathrm{Tr}\left[\Delta^T_{;i} \cdot C^{-1} \cdot \Delta_{;j} \right]$$

$$F_{ij} = \left\langle \frac{\partial^2 f}{\partial\theta_i\,\partial\theta_j} \Big|_{\{\theta\}=\{\theta\}_{best}} \right\rangle = \mathrm{Tr}\left[\left\langle \Delta^T_{;i} \cdot C^{-1} \cdot \Delta_{;j} \right\rangle \right]\Big|_{\{\theta\}=\{\theta\}_{best}}$$

$$\boxed{F_{ij} = \mathrm{Tr}\left[\left\langle \frac{\partial y_{th}}{\partial\theta_i} \cdot C^{-1} \cdot \frac{\partial y_{th}}{\partial\theta_j} \right\rangle \right]}$$

Diagonal C so.

$$\Delta\chi^2 = \sum_{i=1}^{N} \frac{\left[y_{obs} - y_{th}(x_i, \{\theta\}) \right]^2}{\sigma_i^2}$$

$$F_{ij} = \frac{1}{2} \left\langle \frac{\partial^2 \Delta\chi^2}{\partial\theta_i\,\partial\theta_j} \right\rangle = ?$$

$$\frac{\partial \Delta\chi^2}{\partial\theta_i} = \sum_{i=1}^{N} 2 \frac{\partial y_{th}}{\partial\theta_i} \frac{\left[y_{obs} - y_{th} \right]}{\sigma_i^2}$$

$$\frac{\partial^2 \Delta\chi^2}{\partial\theta_i \partial\theta_j} = \sum_{i=1}^{N} \left\{ 2 \frac{\partial^2 y_{th}}{\partial\theta_i \partial\theta_j} \frac{[\quad]}{\sigma_i^2} \right.$$

$$\left. + \underbrace{2 \frac{\partial y_{th}}{\partial\theta_i} \frac{\partial y_{th}}{\partial\theta_j} \frac{1}{\sigma_i^2}}_{} \right\}$$

$$\boxed{F_{ij} = \frac{1}{2} \left\langle \frac{\partial^2 \Delta\chi^2}{\partial\theta_i \partial\theta_j} \Big|_{\{\theta\}=\{\theta\}_{best}} \right\rangle = \underbrace{\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial y_{th}}{\partial\theta_i} \frac{\partial y_{th}}{\partial\theta_j}}_{}}$$

$\{\theta\}=\{\theta\}_{be}$

σ وزن ها بزرگتر از شبیه، آزمایش.

⑦ F_{ij}'s ✓ $\longrightarrow$ $F =$ ✓

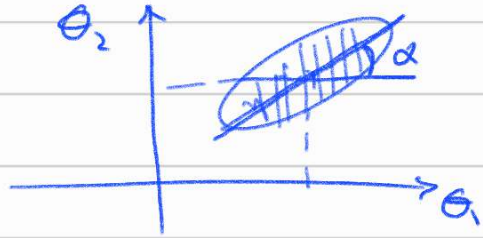
$$\sigma_{\theta_i} = (F^{-1})_{ii}$$

⑧ Determining the Contour Plot

$M=2$ $\boxed{\delta\theta^T . F . \delta\theta = \Delta\chi^2}$

$$\frac{\delta\theta_1^2}{\sigma_{11}^2(1-\rho^2)} + \frac{\delta\theta_2^2}{\sigma_{22}^2(1-\rho^2)} - \frac{2\rho}{(1-\rho^2)} \frac{\delta\theta_1}{\sigma_{11}} \frac{\delta\theta_2}{\sigma_{22}} = \Delta\chi^2$$

$$\tan 2\alpha = \frac{2\rho\,\sigma_{11}\sigma_{22}}{\sigma_{22}^2 - \sigma_{11}^2}$$



(9) for a given Confidence interval.

P% = ✓

$\cancel{\boxed{n = N - M}}$

$$P\% = \int_0^{\Delta\chi^2} d\Delta\chi^2 \, P_n(\Delta\chi^2)$$

$n=2$

$$P_n(\Delta\chi^2) = \frac{1}{2^{n/2}\,\Gamma(n/2)} (\Delta\chi^2)^{n/2} e^{-\frac{\Delta\chi^2}{2}}$$

1σ , 2σ 3σ

(10) Goodness of fit $\longleftrightarrow$ خوبی برازش

$$P_{<P_{obs}} \simeq 2\,P_{\chi^2>\chi^2_{obs}} = 2\int_{\chi^2>\chi^2_{obs}}^{+\infty} d\chi^2 \; P(\chi^2,\nu)$$

$\nu = N - M$

for $\nu \gg 1$ $\quad P(\chi^2, \nu) \rightarrow$ Gaussian funct.

$$\sim e^{-\frac{(\chi^2-\nu)^2}{2(2\nu)}}$$

$$\chi^2 = \nu \pm \sqrt{2\nu}$$

$$\chi^2_\nu = \frac{\nu}{\nu} \pm \frac{\sqrt{2\nu}}{\nu} = 1 \pm \sqrt{\frac{2}{\nu}}$$

(11) How to Compute (Determine) the minimum Value of χ^2 ?

Ⓐ Fully Numerical approach $\rightarrow$ $\left\{ \begin{array}{l} \overbrace{\text{MCMC}}^{\text{HMC}} : \underline{\text{Random}} \text{ Searching} \\ \underline{\text{Deterministic}} \text{ Searching} \end{array} \right.$

Ⓑ Normal Equation $\rightarrow$ Singular Value Decompos.

$\{\theta\}_{best}$ ✓

Ⓒ Gradient Method $\qquad$ روش مبتنی بر گرادیان $\underline{\chi^2}$

$$\vec{\nabla}\chi^2 = \sum_{k=1}^{M} \frac{\partial \chi^2}{\partial \theta_k} \hat{\theta}_k$$

$$(\nabla \chi^2)_k = \frac{\partial \chi^2}{\partial \theta_k} = \frac{\chi^2(\theta_k + \Delta\theta_k) - \chi^2(\theta_k)}{\Delta\theta_k}$$

$$b_k \equiv \frac{\theta_k}{\Delta\theta_k}$$

$$\frac{\partial \chi^2}{\partial b_k} = \frac{\partial \chi^2}{\partial \theta_k} \Delta\theta_k$$

$$\gamma_i = \frac{\frac{\partial \chi^2}{\partial b_i}}{\sqrt{\sum_{k=1}^{M} \left(\left(\frac{\partial \chi^2}{\partial b_k} \right) \right)^2}}$$

Unit Vector for Gradient

$$\boxed{\theta_i(t+1) = \theta_i(t) - \gamma_i \, \Delta\theta_i}$$

تصمیم را بگیریم به سمتی حرکت کنیم
که Chi² کم شود.

according to this approach we essentially go to the value for which the $\chi^2 \rightarrow \chi^2_{min}$