Date: 1397/01/19

In the name of God

Department of Physics
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STOCHASTIC PROCESSES

Mid-Term exam

(Time allowed: 2 hours)

NOTE: All question must be answered. Legibility, good hand-writing and penmanship have 5 additional marks. Please write the answer of each question in separate sheet.

1. Transformation of variables: suppose that we have stochastic variables in N-Dimension such as $\{x\}$: $x_1, x_2, ..., x_N$ and its **multivariate probability** distribution function is given by $p(\{x\})$, assume that there is a mapping as: $g(\{x\}) : \{x\} \to \{y\}$, Now calculate $p(\{y\})$. (5 points)

2. Weighted and Un-weighted correlation functions:

(a) We have an **isotropic** stochastic field represented by: $\mathbf{x}$: $\{x(\mathbf{r}_1), x(\mathbf{r}_2), ..., x(\mathbf{r}_m)\}$. The weighted n-point correlation function of mentioned random field is written by:

$$C_{\mathbf{x}}^{(n)}(r_{1,2}, ...) \equiv \langle x(\mathbf{r}_1)x(\mathbf{r}_2)x(\mathbf{r}_3)...x(\mathbf{r}_n)\rangle$$

Now write this correlation function by using characteristic function, $Z_{\mathbf{x}}(\lambda) \equiv \langle \mathrm{e}^{\mathbf{i}\lambda.\mathbf{x}}\rangle$. What about connected moment? (10 points)

(b) Kernel contribution: Imagine that we apply a kernel on data in a stochastic process according to $X(t) = \int dt' K(t-t')x(t')$. What is the mathematical form of the weighted TPCF of modified process? (**Hint**: the weighted TPCF can be expressed by power spectrum as $C_x(\tau) = \frac{1}{2\pi}\int d\omega e^{i\omega\tau}S(\omega)$ and $S(\omega) = |\tilde{x}(\omega)|^2$, Use the convolution theorem and try to write your answer in terms of power spectrum). (3 points)

(c) Explain the meaning of bias factor using un-weighted and weighted TPCF. Write a proper mathematical explanation. (5 points)

3. Un-weighted TPCF:

(a) Explain the concept of un-weighted TPCF as an excess probability. (5 points)

(b) A definition for un-weighted TPCF of a typical feature such as local maxima is $\langle \mathcal{N}(r)\rangle_{r,r+dr}^{\text{peak}} = \bar{\mathcal{N}}_{\text{peak}}[1+\Psi_{\text{peak}}(r)]$. Now compute the $\bar{\mathcal{N}}_{\text{peak}}$ corresponding to the number density of peak-pair separated by r and $r+dr$ for 1, 2 and 3-Dimnesion. To this end suppose that we have M local maxima in underlying field. (12 points)

4. Stochastic process with colored-noise: suppose that the evolution of velocity is given by:

$$\frac{dv(t)}{dt} = \mathrm{i}[\omega_0 + \eta(t)]v(t)$$

where $\langle \eta(t)\eta(t')\rangle = \gamma e^{-\gamma|t-t'|}$, $\langle \eta(t)\rangle = 0$ with normal distribution and ω_0 is a constant.

(a) Calculate $\langle v(t)\rangle$. (5 points)

(b) Calculate $\langle v(t_1)v(t_2)\rangle$. (5 points)

(c) Explain your results when $\gamma \to \infty$ and in opposite case, namely $\gamma \to 0$. (5 points)

Good luck, Movahed

بسم الله الرحمن الرحیم

Answer-key Mid-term 97/01/19

① Transformation of variables.

$$\{x\} \xrightarrow{g(\{x\})} \{y\}$$

Conservation of probability shows that

$P(\{x\})d^N_x = P(\{y\})d^N_y$ and we have $P(\{y\}) = \int d^N_x \, \delta_D(\{y\} - g(\{x\}))\, P(\{x\})$

$$P(\{y\}) = P(\{x\}) \left| \frac{d^N_x}{d^N_y} \right|$$

$$J = \left| \frac{d^N_x}{d^N_y} \right| = \left| \frac{dg(\{x\})}{d^N_x} \right|^{-1} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_N} \\ \frac{\partial x_2}{\partial y_1} & \ddots & & \\ \vdots & & \ddots & \\ & & & \frac{\partial x_N}{\partial y_N} \end{vmatrix}$$

Therefore $P(\{y\}) = P(g^{-1}\{y\})\, J^{-1}$

or $P(\{y\}) = \sum_n P(g_n^{-1}(y))\, J^{-1} \Big|_{\{x\} = \{x_n\} = \{g_n^{-1}(\{y\})\}}$

② ⓐ $C_x^{(n)}(r_{12}, r_{13}, \dots) = \langle x(\bar{r}_1)\, x(\bar{r}_2) \dots x(\bar{r}_n) \rangle$

we know that

$$Z_{\{x\}}(\lambda) = \langle e^{i\vec{\lambda}\cdot\vec{x}} \rangle = \langle e^{i\lambda_1 x(r_1) + i\lambda_2 x(r_2) \dots} \rangle$$

Therefore

$$C_x^{(2)}(r_{12}) = \langle x(r_1)\, x(r_2) \rangle$$

$$= (-i)^2 \frac{d^2}{d\lambda_1 d\lambda_2} Z_{\{x\}}(\lambda) \Big|_{\lambda=0}$$

$$= (-i)^2 (i)(i) \langle x(r_1)\, x(r_2) \rangle$$

So

$$C_{\{x\}}^{(n)} = (-i)^n \frac{d^n}{d\lambda_1 d\lambda_2 \dots d\lambda_n} Z_{\{x\}}(\lambda) \Big|_{\lambda=0}$$

for Connected Moments or Cumulants we have

$$\langle x(r_1)\, x(r_2) \dots x(r_n) \rangle_c \Big|_{r_1 = r_2 = r_3 \dots = r_n} = (-i)^n \frac{d^n}{d\lambda_1 \dots d\lambda_n} \ln Z_{\{x\}}(\lambda) \Big|_{\lambda=0}$$

$$\times \underbrace{\delta_{r_1 r_2 \dots r_n}}$$

due to have Cumulant ←

rather than Correlation

② - (b) $$X(t) = \int dt' \, K(t-t') \, x(t')$$

Using Fourier Transformation we have

$$X(t) = \int dt' \left[\int d\omega_1 \, e^{i\omega_1(t-t')} \tilde{K}(\omega_1) \right] \left[\int d\omega_2 \, e^{i\omega_2 t'} \tilde{x}(\omega_2) \right]$$

$$= \underbrace{\int dt' \, e^{it'(\omega_2-\omega_1)}}_{\delta_D(\omega_2-\omega_1)} \int d\omega_1 \, e^{i\omega_1 t} \tilde{K}(\omega_1) \int d\omega_2 \, \tilde{x}(\omega_2)$$

$$X(t) = \int d\omega \, e^{i\omega t} \tilde{K}(\omega) \tilde{x}(\omega)$$

Now according to definition of weighted TPCF for $X(t)$, we have

$$C_X(\tau) = \langle X(t) X(t+\tau) \rangle_t$$

$$= \left\langle \left[\int d\omega_1 \, e^{i\omega_1 t} \tilde{K}(\omega_1) \tilde{x}(\omega_1) \right] \left[\int d\omega_2 \, e^{i\omega_2(t+\tau)} \tilde{K}(\omega_2) \tilde{x}(\omega_2) \right] \right\rangle_t$$

$$= \int dt \int d\omega_1 d\omega_2 \, e^{it(\omega_1+\omega_2)} \, e^{i\omega_2\tau} \tilde{K}(\omega_1) \tilde{x}(\omega_1) \tilde{K}(\omega_2) \tilde{x}(\omega_2)$$

$$= \frac{\delta_D(\omega_1+\omega_2)}{2\pi} \int d\omega_2 \, e^{i\omega_2\tau} \tilde{K}(-\omega_2) \tilde{K}(\omega_2) \tilde{x}(-\omega_2) \tilde{x}(\omega_2)$$

$$= \frac{1}{2\pi} \int d\omega \, e^{i\omega\tau} |\tilde{K}(\omega)|^2 |\tilde{x}(\omega)|^2$$

Where $|\tilde{x}(\omega)|^2 = S(\omega)$ is so-called Power spectrum

② - ⓒ $\xi(r) = b^2 C(r)$ or

$$\xi(r) = \frac{P_{12}}{P_1 P_2} - 1$$ and for

pixel above threshold we have

$$P_{12} = \int_{\nu}^{+\infty} d\alpha_1 \int_{\nu}^{+\infty} d\alpha_2 \; P_{12}(\alpha_1, \alpha_2)$$

$$P_1 = \int_{\nu}^{+\infty} d\alpha_1 \; P_1(\alpha_1) \quad , \ldots$$

for Gaussian PDF we showed that

$$b^2 \sim \nu^2 \quad \text{for } r \to \infty, \; \nu \gg 1$$

The Physical meaning of bias factor is that e.g. a typical feature such as local maxima don't follow the stochastic field itself. Instead it follows by a bias factor such that $\delta_p = b\,\delta$

③ $P_{12} = P_1 P_2 [1 + \xi(r)]$

$\langle N \rangle_{Pair} = \bar{n}_{Pair} \Delta r [1 + \xi(r)]$ for 1D

ⓑ 1D $\bar{N}_{Peak} = \dfrac{\binom{M}{2}}{\Delta r} = \dfrac{M(M-1)}{2\Delta r}$

2D $= \dfrac{\binom{M}{2}}{2\pi r \Delta r}$

3D $= \dfrac{\binom{M}{2}}{4\pi r^2 \Delta r}$

④ $\dot{v} = i[\omega_0 + \eta(t)]v(t)$

$$v(t) = v_0 \exp\left[i\omega_0 t + i\int_0^t \eta(t')dt' \right]$$

$$\langle v(t) \rangle = v_0 e^{i\omega_0 t} \underbrace{\left\langle e^{i\int_0^t \eta(t')dt'} \right\rangle}_{?}$$

$$e^{i\int_0^t \eta(t')dt'} = 1 + i\int_0^t \eta(t')dt' - \frac{1}{2}\int_0^t dt_1 \int_0^t dt_2\, \eta(t_1)\eta(t_2) + \cdots$$

$$\left\langle e^{i\int_0^t \eta(t')dt'} \right\rangle = 1 + i\int^t \langle \eta(t') \rangle dt' - 1/2 \int_0^t dt_1 \int_0^t dt_2 \langle \eta(t_1)\eta(t_2) \rangle$$

$$+ \cdots$$

$$= 1 + 0 - \frac{1}{2}\, 2\int_0^t dt_1 \int_0^{t_1} dt_2\, \gamma e^{-\gamma(t_1 - t_2)}$$

$$+ \cdots$$

$$= 1 - t + \frac{1}{\gamma}(1 - e^{-\gamma t}) + \cdots$$

$$= e^{-t + \frac{(1 - e^{-\gamma t})}{\gamma}}$$

and

$$\langle v(t) \rangle = v_0 e^{+i\omega_0 t - \left(t - \frac{(1 - e^{-\gamma t})}{\gamma}\right)}$$

for $\gamma \to \infty$ $\quad \langle \eta(t)\eta(t') \rangle \longrightarrow \delta_D$: Dirac Delta function

for $\gamma \to 0$ $\quad \langle v(t) \rangle = v_0 e^{i\omega_0 t}$ No Noise and behaves as determinstic for $\gamma \to \infty$ $\quad \langle v(t) \rangle = v_0 e^{+i\omega_0 t - t}$

Since $\lim_{\gamma \to \infty} \frac{(1-e^{-\gamma t})}{\gamma} = 0$

And $\lim_{\gamma \to \infty} \langle \eta(t)\eta(t') \rangle \simeq \delta_D(t-t')$

for $\gamma \to 0$ $\quad \lim_{\gamma \to 0} \frac{(1-e^{-\gamma t})}{\gamma} = t e^{-\gamma t} = t$

$\lim_{\gamma \to 0} \langle \eta(t')\eta(t) \rangle = 0$ $\longrightarrow$ No Noise

$\lim_{\gamma \to 0} \langle v(t) \rangle = v_0 e^{i\omega_0 t}$ $\quad$ Deterministic Process

for $\langle v(t_1) v(t_2) \rangle$ Do the Same.

Notice that we have Used

Gaussian Property in which

$\langle \eta(t_1)\eta(t_2)\eta(t_3) \rangle = 0$