باسمه تعالی

امتحان میان‌ترم درس پدیده‌های بحرانی   نام و نام خانوادگی: تاریخ ۹۷/۸/۳۰   ۸۰ نمره

۱- در مدل آیزینگ با حضور میدان مغناطیسی غیرهمگن که با هامیلتونی $\mathrm{H}\{s\} = -J\sum_{\langle ij\rangle} s_i s_j - \sum_i H_i s_i$ داده می‌شود:

الف: انرژی آزاد هلمهولتز و انرژی آزاد گیبس را در تقریب میدان متوسط در d بُعد محاسبه نمایید. **(۱۰ نمره)**

ب: نماهای بحرانی $\beta, \gamma, \delta, \alpha$ را محاسبه نمایید. **(۱۰ نمره)**

ج: در خصوص مفهوم فیزیکی گسستگی در مقدار ظرفیت گرمایی در نقطه بحرانی توضیح دهید. **(۱۰ نمره)**

۲- با استفاده از محک گینزبرگ می‌توان به سه سئوال پاسخ داد: الف: نتایج حاصل از تقریب میدان متوسط تا چه بُعدی معتبر است؟ ب: اگر بخواهیم سهم جمله‌ای که در هامیلتونی دور ریختیم را در نظر بگیریم مرتبه تصحیحات به هامیلتونی چقدر است؟ ج: چه ارتباطی بین مقدار دمای بحرانی واقعی و دمایی که در نظریه میدان میانگین محاسبه می‌شود، وجود دارد؟ اکنون برای هامیلتونی آیزینگ مشخص کنید که نسبت سهم اصلاحات به انرژی میدان متوسط چگونه رفتار می‌کند؟**(۱۰ نمره)**

۳- با استفاده از تعریف آنتروپی گیبس، متوسط مغناطش را برای مدل آیزینگ محاسبه نمایید و آنرا با نتایج به دست آمده برای این مدل در تقریب میدان میانگین مقایسه کنید. هامیلتونی این مدل به صورت $\mathrm{H}\{s\} = -J\sum_{\langle ij\rangle} s_i s_j - H\sum_i s_i$ است. (**راهنمایی: آنتروپی گیبس به صورت** $S = -k_B \sum_c P(c)\ln P(c)$ **و انرژی آزاد به صورت** $F = \langle \mathrm{H}\{s\}\rangle - TS$ **است.** $P(c)$ **احتمال انتخاب پیکربندی** c **است. فرض کنید که احتمال اینکه در یک سایت اسپین مقدار مشخصی اختیار کند مستقل از همسایه‌اش باشد به بیانی دیگر احتمال اضافی خوشگی صفر است. اکنون تلاش کنید که** $P(c)$ **را بر حسب متوسط مغناطش در هر سایت بنویسید.) (۱۰ نمره)**

۴- در نظریه لاندائو رابطه بین انرژی آزاد لاندائو و انرژی آزاد هلمهولتز را استخراج کنید. **(۱۰ نمره)**

۵- الف: معادله حاکم بر تابع همبستگی پارامتر نظم (Φ) را در مدل لاندائو که چگالی انرژی آزاد آن به صورت $L[\Phi] = \int d^d x \left[\mu(\nabla\Phi)^2 + a\Phi^2 + b\Phi^3 + c\Phi^4 - H\Phi\right]$ داده شده است، استخراج کنید. **(۱۰ نمره)**

ب: با توجه به ویژگی‌های فیزیکی که انتظار داریم در نزدیکی نقطه بحرانی رخ دهد در مورد علامت و وابستگی ضرایب به دمای کاهش یافته بحث کنید. **(۵ نمره)**

ج: چه ارتباطی بین تابع پاسخ و تابع همبستگی وجود دارد؟**(۵ نمره)**

موفق باشید

موحد

ابداً خویشتن را محدود به آنچه قدمای ما بدان پرداخته اند نکنیم و سعی نماییم آنچه را که می توان تکمیل کرد تکمیل کنیم (ابوریحان بیرونی قرن ۴ هجری)

	Microcanonical $\Omega(E,V,N)$	Canonical $Z(T,V,N)$	Grand canonical $\mathcal{Z}(T,V,\mu)$
$\frac{S}{k}$	$\ln\Omega$	$\left(\frac{\partial(T\ln Z)}{\partial T}\right)_{V,N}$	$\left(\frac{\partial(T\ln \mathcal{Z})}{\partial T}\right)_{V,\mu}$
F	$E-kT\ln\Omega$	$-kT\ln Z$	$kT\mu^2\left(\frac{\partial(\mu^{-1}\ln \mathcal{Z})}{\partial \mu}\right)_{T,V}$
U	Fixed $(=E)$	$kT^2\left(\frac{\partial(\ln Z)}{\partial T}\right)_{V,N}$	$-\left(\frac{\partial(\ln \mathcal{Z})}{\partial \beta}\right)_{\beta\mu,V}$
N	Fixed	Fixed	$kT\left(\frac{\partial(\ln \mathcal{Z})}{\partial \mu}\right)_{T,V}$
kT	$\left(\frac{\partial(\ln\Omega)}{\partial E}\right)^{-1}_{V,N}$	Fixed	Fixed
$\frac{\mu}{kT}$	$-\left(\frac{\partial(\ln\Omega)}{\partial N}\right)_{E,V}$	$-\left(\frac{\partial(\ln Z)}{\partial N}\right)_{T,V}$	Fixed
P	$kT\left(\frac{\partial(\ln\Omega)}{\partial V}\right)_{E,N}$	$kT\left(\frac{\partial(\ln Z)}{\partial V}\right)_{T,N}$	$\frac{kT}{V}\ln\mathcal{Z}$
$\frac{C_V}{k}$	$-\beta^2\left(\frac{\partial^2(\ln\Omega)}{\partial E^2}\right)^{-1}_{V,N}$	$\beta^2\left(\frac{\partial^2(\ln Z)}{\partial \beta^2}\right)_{V,N}$	$T\left(\frac{\partial^2(T\ln \mathcal{Z})}{\partial T^2}\right)_{V,\mu}$
$(\Delta N)^2$	0	0	$\left(\frac{\partial^2(\ln \mathcal{Z})}{\partial (\beta\mu)^2}\right)_{\beta,V}$
$(\Delta E)^2$	0	$\left(\frac{\partial^2(\ln Z)}{\partial \beta^2}\right)_{V,N}$	$\left(\frac{\partial^2(\ln \mathcal{Z})}{\partial \beta^2}\right)_{\beta\mu,V}$

$\int_0^\infty dx\; x^n\; e^{-\alpha x} = \frac{n!}{\alpha^{n+1}}$ $\qquad$ $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$

$\int_{-\infty}^{\infty} dx \exp\left[-ikx - \frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2}\exp\left[-\frac{\sigma^2 k^2}{2}\right]$ $\qquad$ $\lim_{N\to\infty}\ln N! = N\ln N - N$

$\left\langle e^{-ikx}\right\rangle = \sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\langle x^n\rangle$ $\qquad$ $\ln\left\langle e^{-ikx}\right\rangle = \sum_{n=1}^{\infty}\frac{(-ik)^n}{n!}\langle x^n\rangle_c$

$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$ $\qquad$ $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

Surface area of a unit sphere in d dimensions $\qquad$ $S_d = \frac{2\pi^{d/2}}{(d/2-1)!}$

$$\sum_k \to \frac{l^d}{(2\pi)^d}\int d^dk \to \int g(\varepsilon)d\varepsilon$$

$$\int_{-\infty}^{+\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

$$g(\varepsilon)d\varepsilon = \frac{S_d}{\left(\frac{2\pi}{l}\right)^d}dk \qquad S_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$$

$$\Gamma(t) = \int_0^{+\infty} dx x^{t-1}e^{-x} = (t-1)\Gamma(t-1)$$

$$\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin\pi t} \quad ,\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$V_d = \frac{R^d\pi^{d/2}}{\frac{d}{2}\Gamma\left(\frac{d}{2}\right)}$$

Provided by ***A. Kargaran*** *under supervision of* ***Dr. Movahed***

Answer to Midterm Exam Critical Phenomena

1-(part a)

Hamiltonian for Ising model in in-homogeneous magnetic field is:

$$\mathcal{H}_\Omega\{S\} = -J\sum_{\langle ij\rangle} S_i S_j - \sum_i H_i S_i$$

Suppose for the moment that the spins are independent: $J = 0$. Then

$$\mathcal{Z}_\Omega\{0, H\} = \prod_{i=1}^{N}\left[e^{-\beta H_i} + e^{\beta H_i}\right] = 2^N \prod_{i=1}^{N} \cosh\left(H_i/k_B T\right)$$

Free energy is:

$$\mathcal{F} = -k_B T \log\left[\mathcal{Z}_\Omega\{0, H\}\right] = -k_B T \sum_{i=1}^{N} \log\left[\cosh\left(H_i/k_B T\right)\right]$$

We know that in mean field approximation we can write:

$$-J\sum_{\langle ij\rangle} S_i S_j \approx 2dM \sum_{i=1}^{N} S_i$$

So it changes Hamiltonian and free energy:

$$\mathcal{F} = -k_B T \log\left[\mathcal{Z}_\Omega\{J, H\}\right] = -k_B T \sum_{i=1}^{N} \log\left[\cosh\left(2dM/k_B T + H_i/k_B T\right)\right]$$

We should consider Gibbs free energy here because of average magnetization and magnetic field, so we have:

$$\mathcal{G} = -k_B T \sum_{i=1}^{N} \log\left[\cosh\left(2dM/k_B T + H_i/k_B T\right)\right] - M\sum_{i=1}^{N} H_i$$

part (b)

In here we consider homogeneous magnetic field so we can write:

$$\mathcal{F} = -k_B T \log\left[\mathcal{Z}_\Omega\{J, H\}\right] = -k_B T N \log\left[\cosh\left(\frac{2dM + H}{k_B T}\right)\right]$$

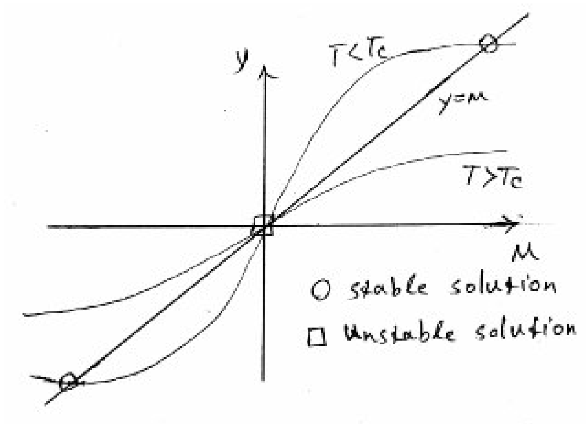
and Magnetization:

$$M = -\frac{1}{N}\frac{\partial \mathcal{F}}{\partial H} = k_B T \tanh\left[\frac{2dM + H}{k_B T}\right]$$

If we use self-consistent method (like Ising model) we can find Ising like results. If we put $H = 0$ and expand above equation right hand side for small M we have:

$$M \approx \frac{2dJM}{k_B T_c} \Rightarrow T_c = \frac{2dJ}{k_B}$$

If $T > T_c$ the only intersection is $M = 0$ and if $T < T_c$ we have three intersections, two of them are stable solution and one of them is unstable. The stable solutions are parallel or anti-parallel with mean magnetization of all spins $\boldsymbol{M}$. Solution $M = 0$ is stable when $T > T_c$ and unstable when $T < T_c$.



We can find:

$$M = \tanh\left(H/k_B T + M\tau\right) = \frac{\tanh H/k_B T + \tanh M\tau}{1 + \tanh H/k_B T \tanh M\tau}$$

Finally:

$$\tanh H/k_B T = \frac{M - \tanh M\tau}{1 + M \tanh M\tau}$$

We expand above for small H and M, we have:

$$\frac{H}{k_B T} \approx M(1-\tau) + M^3\left(\tau - \tau^2 + \frac{\tau^3}{3} + \ldots\right) + \ldots \tag{1}$$

For $H = 0$ and $T \to T_c^-$ we have:

$$M^2 \approx 3\frac{(T_c - T)}{T} + \ldots$$

where the dots indicate corrections to this leading order formula. We can read off the critical exponent $\beta : \beta = 1/2$. The critical isotherm is the curve in the $H - M$ plane corresponding to $T = Tc$. Its shape near the critical point is described by the critical exponent δ:

$$H \sim M^\delta$$

Setting $\tau = 1$ in the equation of state 1 , we find:

$$\frac{H}{k_B T} \sim M^3$$

Showing that the mean field value of δ is 3. The isothermal magnetic susceptibility χ_T also diverges near T_c:

$$\chi_T \equiv \left.\frac{\partial M}{\partial H}\right|_T$$

Differentiating the equation of state 1, gives:

$$\frac{1}{k_B T} \approx \chi_T (1 - \tau) + 3M^2 \chi_T \left(\tau - \tau^2 + \frac{1}{3}\tau^3\right) \tag{2}$$

For $T > Tc$, $M = 0$ and $\tau \to 1$ we have:

$$\chi_T = \frac{1}{k_B} \frac{1}{T - T_c} + \dots$$

Comparing with the definition of the critical exponent γ:

$$\chi_T \sim |T - T_c|^\gamma$$

we conclude that $\gamma = 1$. For $T < T_c$, we have:

$$M \approx \sqrt{3} \left(\frac{T_c - T}{T}\right)^{1/2} + \dots$$

Substituting into 2 gives:

$$\begin{aligned} \frac{1}{k_B T} &\approx \chi_T \left(\frac{T - T_c}{T_c}\right) + 3\chi_T \left(\frac{T_c - T}{T}\right) \\ &= 2\chi_T \left(\frac{T_c - T}{T}\right) \\ \chi_T &= \frac{1}{2k_B} \frac{1}{T_c - T} \dots \end{aligned}$$

which shows that the divergence of the susceptibility below the transition temperature is governed by the critical exponent $\gamma' = \gamma = 1$.

For α we have we put $H = 0$ and we have:

$$C_v = -T \frac{\partial^2 \mathcal{F}}{\partial T^2} = \frac{N(2dJM)^2}{k_B T^2} \text{sech}^2 \left(\frac{2dJM}{k_B T}\right)$$

Now we expand above for small M:

$$C_v = \frac{4d^2 J^2 M^2 N}{k_B T^2} - \frac{16 M^4 \left(d^4 J^4 N\right)}{k_B^3 T^4} + O\left(M^6\right)$$

We Now from above that magnetization behave:

$$M = \begin{cases} 0 & T > T_c \\ \sqrt{3} \left(\frac{T_c - T}{T}\right)^{1/2} + \dots & T < T_c \end{cases}$$

Heat capacity behave like:

$$C_v = \begin{cases} 0 & T > T_c \\ At + Bt^2 + Ct^4 + \dots & T < T_c \end{cases}$$

A, B and C are constants.

part (c)

In Landau theory we can find that heat capacity behave like:

$$C_v = \begin{cases} 0 & T > T_c \\ a^2/(bT_c^2) & T < T_c \end{cases}$$

This function has a discontinuity in critical temperature so we can NOT fit this function with a power law except power zero so we have $\alpha = 0$.

(2)

In Landau theory for Ising model we write Landau free energy as follows:

$$H_\Omega\{S\} = -\sum_i S_i H_i$$

Where:

$$H_i = H + \sum_j J_{ij}\langle S_j\rangle + \sum_j J_{ij}\left(S_j - \langle S_j\rangle\right)$$

Here, the first term is the external field, the second is the mean field, and the final term is the fluctuation, which we ignore in mean field approximation. We put last term of above in Hamiltonian and we have:

$$\begin{aligned}
\left\langle \sum_i S_i \sum_j J_{ij}\left(S_j - \langle S_j\rangle\right) \right\rangle &= \left\langle \sum_{ij} J_{ij}\left(S_i S_j - S_i\langle S_j\rangle\right) \right\rangle \\
&= \sum_{ij} J_{ij}\left(\langle S_i S_j\rangle - \langle S_i\rangle\langle S_j\rangle\right) \\
&= \sum_{ij} J_{ij}(\boldsymbol{r}_i - \boldsymbol{r}_j) G(\boldsymbol{r}_i - \boldsymbol{r}_j) \\
&\approx \frac{J}{a^d}\int_\Omega d^d\boldsymbol{r} G(\boldsymbol{r}) \\
&\approx \int_\Omega d^d\boldsymbol{r} G(\boldsymbol{r})
\end{aligned}$$

J is average of $J_{ij}(\boldsymbol{r}_i - \boldsymbol{r}_j)$ over interaction region. Now we want to find energy of mean field which is:

$$\begin{aligned}
\left\langle \sum_i S_i \sum_j J_{ij}\langle S_j\rangle \right\rangle &= \sum_{ij} J_{ij}\langle S_i\rangle\langle S_j\rangle \\
&= \sum_{ij} J_{ij}(\boldsymbol{r}_i - \boldsymbol{r}_j)\langle S_i\rangle\langle S_j\rangle \\
&\approx J\sum_{ij}\langle S_i\rangle\langle S_j\rangle \\
&= J\sum_i \langle S_i\rangle^2 \\
&= \frac{J}{a^d}\int_\Omega d^d\boldsymbol{r}\phi(\boldsymbol{r})^2
\end{aligned}$$

We can find how this approximation is valid by Ginsburg criterion which is:

$$E_{LG} = \frac{\left|\int_V d^d\boldsymbol{r}\ G(\boldsymbol{r})\right|}{\int_V d^d\boldsymbol{r}\ \phi(\boldsymbol{r})^2}$$

The region must not be larger than ξ (correlation length), otherwise the fluctuations will become uncorrelated, and we will not obtain a true estimate of their(numerator and denominator) strength. In above V is taken to be the correlation volume:$V = \xi(T)^d$. We know that correlation function at critical temperature behave like:

$$\int_V d^d\boldsymbol{r}\ G(\boldsymbol{r}) \sim k_B T_c \chi_T \sim t^{-\gamma}$$

and:

$$\int_V d^d\boldsymbol{r}\ \phi(\boldsymbol{r})^2 \sim \xi^d |t|^{2\beta} \sim t^{2\beta-\nu d}$$

So we have:

$$E_{LG} \sim \frac{t^{-\gamma}}{t^{2\beta-\nu d}} = t^{-\gamma-2\beta+\nu d}$$

This criterion diverge when:

$$d > \frac{\gamma+2\beta}{\nu} = d_c$$

d_c is upper critical dimension which for mean field theory is $d_c = 4$.

(3)

Probability of choosing configuration c is $e^{-\beta H\{c\}}/\mathcal{Z}$, where $\mathcal{Z}$ is partition function. Helmholtz free energy with Gibbs entropy is:

$$
\begin{aligned}
\mathcal{F} &= \langle H\{s\}\rangle - TS \\
&= \sum_c P(c)H\{c\} + k_B T \sum_c P(c)\log P(c) \\
&= \sum_c P(c)\left[H\{c\} + k_B T \log P(c)\right] \\
&= \sum_c \frac{e^{-\beta H\{c\}}}{\mathcal{Z}}\left[H\{c\} - H\{c\} - k_B T \log \mathcal{Z}\right] \\
&= \frac{k_B T \log \mathcal{Z}}{\mathcal{Z}} \sum_c e^{-\beta H\{c\}} = -k_B T \log \mathcal{Z}
\end{aligned}
$$

We approximate the Hamiltonian with:

$$
-J\sum_{\langle ij\rangle} S_i S_j \approx 2dM \sum_{i=1}^{N} S_i
$$

So magnetization become:

$$
M = -\frac{1}{N}\frac{\partial \mathcal{F}}{\partial H} = k_B T \tanh\left[\frac{2dM + H}{k_B T}\right]
$$

Above result is identical with mean field method.

(4)

The Helmholtz free energy given by:

$$e^{-\beta\mathcal{F}} = \int \mathcal{D}\eta \; e^{-\beta\mathcal{H}\{\eta(r)\}}$$

where the integral $\int \mathcal{D}\eta$ is a functional integral over all degrees of freedom associated with η, instead of an integral over all microstates. Landau's assumption is that we can replace the entire partition function by the following:

$$e^{-\beta\mathcal{F}} \approx \int \mathcal{D}\eta \; e^{-\beta\mathcal{L}\{\eta(r)\}} \tag{3}$$

For example, if η is the mean magnetization, a given value for the magnetization can be determined by many different microstates. It is assumed that all of this information is contained in $\mathcal{L}\{\eta(r)\}$. This is a non-trivial assumption which can nonetheless be proven for certain systems. The conversion of the degree of freedom from $S$to η is known as *coarse-graining*, and is at the heart of the relationship between statistical mechanics and thermodynamics. The next step is to minimize $\mathcal{L}\{\eta(r)\}$(to maximize integrated), performing a saddle point approximation (or steepest descent) to the functional integral in 3, giving:

$$e^{-\beta\mathcal{F}} \approx e^{-\beta\mathcal{L}_{\min}\{\eta(r)\}}$$

this is relation between Helmholtz free energy and Landau free energy.

(5)

part (a)

We aim in this section to calculate $G(\boldsymbol{r})$. We do this in two steps:

1) Find the equation satisfied by ϕ by differentiating the Landau free energy and demanding stationarity.

2) Differentiate with respect to H to get an equation for $\chi_T(\boldsymbol{r}-\boldsymbol{r'}) >$ i.e. an equation for $G(\boldsymbol{r}-\boldsymbol{r'})$.

Step 1

$$\mathcal{L} = \mu\left(\nabla\phi\right)^2 + a\phi^2 + b\phi^3 + c\phi^4 - H\phi$$
$$\frac{\partial\mathcal{L}}{\partial\phi} = 0 \Rightarrow -2\mu\nabla^2\phi + 2a\phi + 3b\phi^2 + 4c\phi^3 - H = 0$$

For above relation I use functional integration like:

$$\frac{\delta}{\delta\phi}\int d^d\boldsymbol{r'}\mu\left(\nabla\phi(\boldsymbol{r'})\right)^2 = -2\mu\nabla^2\phi(\boldsymbol{r'})$$

Step 2

$$\frac{\delta}{\delta H(\boldsymbol{r'})}\left[-2\mu\nabla^2\phi + 2a\phi + 3b\phi^2 + 4c\phi^3 - H\right] = 0$$
$$\left[-2\mu\nabla^2 + 2a + 6b\phi + 12c\phi^2\right]\chi_T(\boldsymbol{r}-\boldsymbol{r'}) = \frac{\delta H(\boldsymbol{r})}{\delta H(\boldsymbol{r'})} = \delta(\boldsymbol{r}-\boldsymbol{r'})$$

Thus the two-point correlation function is actually a ***Green function***. For translationally invariant systems, ϕ is just given by the equilibrium value from Landau theory.

For $a > 0$, $\phi = 0$ we have:

$$\left[-\nabla^2 + \xi_>^{-2}\right]G(\boldsymbol{r}-\boldsymbol{r'}) = \frac{k_BT}{2\mu}\delta(\boldsymbol{r}-\boldsymbol{r'})$$

Where $\xi_>^{-2} = a/\mu$.

For $a < 0$ we have $\phi \neq 0$ we should calculate other roots so we take $H = 0$ and we have:

$$a\phi^2 + b\phi^3 + c\phi^4 = \phi^2\left(a + b\phi + c\phi^2\right) = 0 \Rightarrow \phi = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$$

We can find similar equation for correlation function:

$$\left[-\nabla^2 + \xi_<^{-2}\right]G(\boldsymbol{r}-\boldsymbol{r'}) = \frac{k_BT}{2\mu}\delta(\boldsymbol{r}-\boldsymbol{r'})$$

Where $\xi_<^{-2} = (1/2\mu)\left[2a + 6b\left(\frac{-b\pm\sqrt{b^2-4ac}}{2c}\right) + 12c\left(\frac{-b\pm\sqrt{b^2-4ac}}{2c}\right)^3\right]$.

part (b)

Roots for Landau free energy are:

$$a\phi^2 + b\phi^3 + c\phi^4 = \phi^2\left(a + b\phi + c\phi^2\right) = 0 \Rightarrow \phi = 0, \quad \phi = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$$

We should have:

$$\left(\frac{b}{2c}\right)^2 - \frac{2a}{c} > 0 \Rightarrow b^2 > 8ac$$

Above put a constrain on sign of free parameters. Near critical temperature a should change sign (linearly dependent to t) and c should have positive sign (otherwise we don't have finite minimum for ϕ). We don't have any physical constrain for sign of b and it can have either positive and negative signs (b is independent of t).

part (c)

First, we define the two-point function:

$$G(\boldsymbol{r}_i - \boldsymbol{r}_j) = \langle S_i S_j\rangle - \langle S_i\rangle\langle S_j\rangle$$

Partition function is:

$$\mathcal{Z}_\Omega = \mathrm{Tr}\ e^{-\beta\mathcal{H}_\Omega} = \mathrm{Tr}\ \exp\left[\beta J\sum_{\langle ij\rangle} S_i S_j + \beta H\sum_i S_i\right]$$

We can obtain averages by differentiating from partition function:

$$\sum_i\langle S_i\rangle = \frac{1}{\mathcal{Z}_\Omega}\mathrm{Tr}\ \left[\sum_i S_i\right] e^{-\beta\mathcal{H}_\Omega} = \frac{1}{\beta\mathcal{Z}_\Omega}\frac{\partial\mathcal{Z}_\Omega}{\partial H}$$

$$\sum_{ij}\langle S_i S_j\rangle = \frac{1}{\mathcal{Z}_\Omega}\mathrm{Tr}\ \left[\sum_{ij} S_i S_j\right] e^{-\beta\mathcal{H}_\Omega} = \frac{1}{\beta^2\mathcal{Z}_\Omega}\frac{\partial^2\mathcal{Z}_\Omega}{\partial H^2}$$

Now we want to calculate susceptibility:

$$\begin{aligned}
\chi_T &= \frac{\partial M}{\partial H} = -\frac{\partial^2\mathcal{F}}{\partial H^2} = \frac{1}{N\beta}\frac{\partial^2\log\mathcal{Z}_\Omega}{\partial H^2} \\
&= \frac{1}{N\beta}\ \frac{\partial}{\partial H}\ \left[\frac{1}{\mathcal{Z}_\Omega}\frac{\partial\mathcal{Z}_\Omega}{\partial H}\right] \\
&= \frac{1}{N\beta}\ \left[\frac{1}{\mathcal{Z}_\Omega}\frac{\partial^2\mathcal{Z}_\Omega}{\partial H^2} - \left(\frac{1}{\mathcal{Z}_\Omega}\frac{\partial\mathcal{Z}_\Omega}{\partial H}\right)^2\right] \\
&= \frac{\beta}{N}\ \left[\sum_{ij}\langle S_i S_j\rangle - \left(\sum_i\langle S_i\rangle\right)^2\right] \\
&= \frac{\beta}{N}\ \sum_{ij} G(\boldsymbol{r}_i - \boldsymbol{r}_j) = \beta\ \sum_i G(\boldsymbol{x}_i) = \frac{\beta}{a^d}\int_\Omega d^d r\ G(r)
\end{aligned}$$

In last step we take our space to be continuous and summation become integral as follows:

$$\sum_i \longrightarrow \frac{1}{a^d}\int_\Omega$$