**Date: 1400/02/09**

In the name of God

Department of Physics, Shahid Beheshti University

# STOCHASTIC PROCESSES

## Mid-Term exam

## (Time allowed: 3 hours)

**NOTE:** All questions must be answered. Please write the answer of each question in separate sheet.

**1.** Moments and Cumulants: Consider that we have two statistically independent random variables $x$ and $y$. Now we define $z$ according to $z = x + y$. Calculate the 4th moment and 4th cumulant of $z$ and based on your results explain the advantages of cumulant compared to moment. (10 points)

**2.** PDF transformation: Suppose the for $\{\xi\} : \{\xi_1, \xi_2, ..., \xi_n\}$ we have following PDF:

$$p_\xi(x) \begin{cases} \frac{2x}{\pi^2} & |x| \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

if we have a transformation as $\{\xi\} \to \{\eta\}$ according to $y_\eta = \sin(x_\xi)$, compute the $p_\eta(y)$. (5 points)

**3.** For a typical stochastic field, $f$, in $d$-dimension, with $\langle f \rangle = 0$, $\sigma_0^2 = \langle f^2 \rangle$, $\alpha \equiv \frac{f}{\sigma_0}$, $\eta_i \equiv \partial\alpha/\partial x_i$ and $\sigma_1^2 \equiv -\langle f \nabla^2 f \rangle$, show that:

$$\left\langle \left(\frac{\partial}{\partial\alpha}\right)^k \left(\frac{\partial}{\partial\eta_1}\right)^\ell \delta_D(\alpha - \vartheta)|\eta_1| \right\rangle = \frac{h_{\ell-2}}{\pi} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0}\right)^{1-\ell} e^{-\vartheta^2/2} H_k(\vartheta)$$

where

$$\begin{aligned} H_n(0) &= \begin{cases} 0, & (n : \text{odd}) \\ (-1)^{n/2}(n-1)!! & (n : \text{even}) \end{cases} \\ &\equiv h_n. \end{aligned}$$

and

$$H_n(\vartheta) = e^{\vartheta^2/2} \left(-\frac{\partial}{\partial\vartheta}\right)^n e^{-\vartheta^2/2},$$

mentioned relation is necessary to compute crossing statistics. (10 points)

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**4.** Multivariate Gaussian Distribution: A multivariate Gaussian distribution for $\vec{X} \equiv [X]_{N\times 1}$ can be given by $P(X) = \frac{e^{-\frac{1}{2}X^T.C^{-1}.X}}{\sqrt{(2\pi)^N Det(C)}}$, where covariance matrix is $C = \langle X^T X \rangle$.

(a) Show that this distribution function is normal. (5 points)

(b) Show that after an affine transformation in the form of $X \to Y = RX$ which $R^T = R^{-1}$, $P(Y)$ stays normal as well. (**Hint**: $R$ can be considered as a typical rotation) (10 points)

**5.** Weighted and Un-weighted correlation functions:

(a) We have an **isotropic** stochastic field represented by: $\mathbf{x} : \{x(\mathbf{r}_1), x(\mathbf{r}_2), ...., x(\mathbf{r}_m)\}$. The weighted $n$-point correlation function of mentioned random field is written by:

$$C_{\mathbf{x}}^{(n)}(r_{1,2}, ...) \equiv \langle x(\mathbf{r}_1)x(\mathbf{r}_2)x(\mathbf{r}_3)...x(\mathbf{r}_n)\rangle$$

Now write this correlation function by using characteristic function, $Z_{\mathbf{x}}(\lambda) \equiv \langle \mathrm{e}^{\mathbf{i}\lambda.\mathbf{x}}\rangle$. What about connected moment? (10 points)

(b) Kernel contribution: Imagine that we apply a kernel on data in a stochastic process according to $X(t) = \int dt' K(t-t')x(t')$. What is the mathematical form of the weighted TPCF of modified process? ( 5 points)

(c) Explain the meaning of bias factor using un-weighted and weighted TPCF. Write a proper mathematical explanation. (5 points)

Good luck, Movahed

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