Date: 1403/08/17

In the name of God
Department of Physics, Shahid Beheshti University

STATISTICAL FIELD THEORY AND CRITICAL PHENOMENA

First midterm exam

(Time allowed: 3:00 hours)

NOTE: All question must be answered. **Write** the answer of each question in **separate sheet**.

1. The impossibility of taking place phase transition for Ising model in absence of external magnetic field.
A: For Ising model prove that $Z(H,T,J) = Z(-H,T,J)$. (Hint: Show that $\mathcal{H}(H,J,\{s\}) = \mathcal{H}(-H,J,\{-s\})$. (5 points)
B: Prove that $M(H) = -M(-H)$. Explain the physical consequence of this result for phase transition. (5 points)
C: Show that

$$\lim_{N(\Omega)\to\infty} \lim_{H\to 0} \left(\frac{1}{N(\Omega)} \frac{\partial F}{\partial H}\right) \neq \lim_{H\to 0} \lim_{N(\Omega)\to\infty} \left(\frac{1}{N(\Omega)} \frac{\partial F}{\partial H}\right)$$

Explain the meaning of this inequality. Which one is not equal to zero. (5 points)
D: Why is the thermodynamic limit important for phase transition?. Give an example which possessing no thermodynamic limit? (5 points)
E: From thermodynamic potential point of view, the necessary condition to have phase transition is singularity in $F = -k_B T \ln Z$, in which the partition function is generally $Z = Tr(e^{-\beta\mathcal{H}})$. The partition function is generally considered to be an analytical function of temperature. Therefore, we do not expect to find singularities in the free energy. This implies that phase transitions should not exist in nature. Deduce what wrong is with above statement. (5 points)

2. Ising Model on the network: Suppose that the Hamiltonian is given by: $\mathcal{H} = -\sum_{i<j} J_{ij} a_{ij} s_i s_j - \sum_i H_i s_i$. Here $i,j = 1,2,..,N$ and a_{ij} is an element of the adjacency matrix. The topology of network is encoded via the adjacency matrix.
A: According to the the Mean-field theory, determine the $\langle m\rangle$. (5 points)
B: Determine the scaling exponents as $\langle m\rangle \sim |-t|^{\beta}$ with $\beta = 1/2$ and $\chi = d\langle m\rangle/dH \sim |t|^{-\gamma}$ with $\gamma = 1$. (5 points)

3. Landau-Ginzburg-Wilson theory
A: According to the coarse-graining on the discrete partition function, clarify the relation between coarse-grained partition function with original one. (5 points)
B: Explain the main properties that we should consider to write the Landau free energy. (5 points)
C: According to the $Z = \int D\Psi D\mathcal{A}\, e^{-\beta\mathcal{H}}$, and following effective Hamiltonian as:

$$\beta\mathcal{H} = \int d^d r \left[m^2|\Psi|^2 + b|\Psi|^4 + |(\nabla - ie^*\mathcal{A})\Psi|^2 + \frac{1}{2\mu}(\nabla\times\mathcal{A})^2\right]$$

where the Ψ and $\mathcal{A}$ are scalar and gauge fields, respectively. Derive the dimensional analysis of $[\Psi]$, $[\mathcal{A}]$, $[m^2]$, $[e]$. Suppose that μ is a constant. If we are interested in keeping the long wavelength limit for Ψ, modify the above effective Hamiltonian. (10 points)
D: To model the First-order phase transition with respect to Ψ field, what is the least modification on the $\mathcal{H}$? and why? (5 points)

Good luck, Movahed

بسم الله الرحمن الرحیم

Answer Key for First midterm of Statistical Field Theory and Critical phenomena

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1. A: $Z(H,J,T) \overset{?}{=} Z(-H,J,T)$

For $\mathcal{H} = -J\sum S_i S_j - H\sum S_i$

If ☆ $\begin{cases} H \longrightarrow -H \\ \{S\} \longrightarrow \{-S\} \end{cases}$ therefore $\mathcal{H}(-H,J,\{-S\}) = \mathcal{H}(H,J,\{S\})$

☆ $Z(H,J,T) = \sum_{\{S\}} e^{-\beta\mathcal{H}(H,J,\{S\})}$

$Z(-H,J,T) = \sum_{\{S\}} e^{-\beta\mathcal{H}(-H,J,\{S\})}$

Also, Since we Consider all configuration, therefore we have

$$Z(-H,J,T) = \sum_{\{S\}} e^{-\beta \underbrace{\mathcal{H}(-H,J,\{-S\})}_{=\mathcal{H}(H,J,\{S\})}}$$

$$= \sum_{\{S\}} e^{-\beta\mathcal{H}(H,J,\{S\})} = Z(H,J,T)$$

So $$\boxed{Z(-H, J, T) = Z(H, J, T)}$$ 5

(B) $M(H) \stackrel{?}{=} -M(-H)$

we know $F = -K_B T \ln Z(H, J, T)$ and from above

part we have $F = -K_B T \ln Z(H, J, T) = -K_B T \ln Z(-H, J, T)$

and $$M(H) = -\frac{\partial F(H, J, T)}{\partial H} = -\frac{\partial F(-H, J, T)}{\partial H}$$

$$= -\frac{\partial F(-H, J, T)}{\partial(-H)} \frac{\partial(-H)}{\partial H}$$

$$\boxed{M(H) = -M(-H)}$$

The physical meaning of above is that

$$\lim_{H \to 0} M(H) = -M(-H) \Rightarrow M(0) = -M(0) \Rightarrow M = 0$$

So, for $H = 0$, we don't anticipate to have

pure magnetization. 5

(C) We define $f \equiv \frac{F}{N(\Omega)}$ and for small H

we have

$$f(H) = f(H=0) + H \frac{\partial F}{\partial H} + \mathcal{O}(H^2)$$

$$= f(0) - M|H| + \mathcal{O}(H^2)$$

therefore
$$\lim_{H \to 0^-} -\frac{\partial f}{\partial H} = -M$$

$$\lim_{H \to 0^+} -\frac{\partial f}{\partial H} = +M$$

So at thermodynamic Limit ($N(\Omega) \to \infty$)

f is not analytical therefore

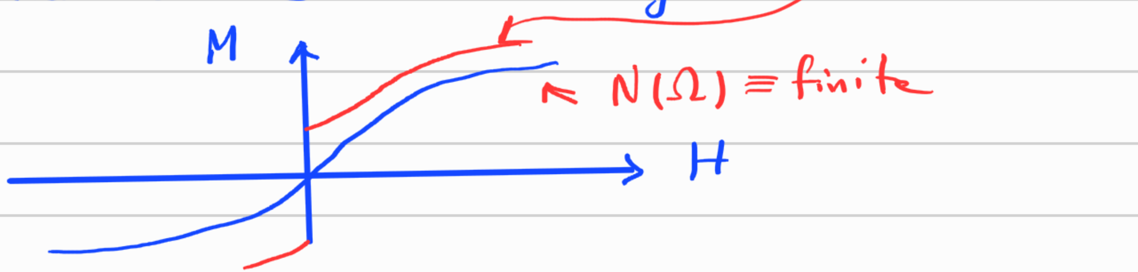
one can not write

$$\lim_{N(\Omega) \to \infty} \lim_{H \to 0} \neq \lim_{H \to 0} \lim_{N(\Omega) \to \infty}$$

therefore
$$\lim_{H \to 0} \lim_{N(\Omega)} -\frac{1}{N(\Omega)} \frac{\partial F}{\partial H} \neq 0$$

while
$$\lim_{N(\Omega)} \underbrace{\left(\lim_{H \to 0} -\frac{1}{N(\Omega)} \frac{\partial F}{\partial H} \right)}_{=0} = 0$$

Such behavior can be illustrated by



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D As we obtained that $\lim_{N(\Omega)\to\infty} \lim_{H\to 0} \neq \lim_{H\to 0} \lim_{N(\Omega)\to\infty}$

therefore, we expect that, the singularity takes place at $\left.\begin{array}{l} N(\Omega)\to\infty \\ V(\Omega)\to\infty \end{array}\right\}$ thermodynamic limit

Consequently, at first we should check the Existence of thermodynamic limit of a typical physical quantity and any divergency of considered measure is a signature of phase transition.

an example that we have no thermodynamic limit is a simple electric charge system including only one kind of charged particles.

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E The Analytical function form is satisfied only for finite Sample Size. and for

$N \to \infty$
$V \to \infty$, the analyticity of F

(or Z) does not make Sense

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Ising on the Network

2) A

Ref. Critical phenomena in Complex Systems 2008

$$\mathcal{H} = -\sum J_{ij} a_{ij} S_i S_j - \sum H_i S_i$$

Adjacency matrix

$$\mathcal{H} = \sum_{i=1}^{N} \mathcal{H}_i = -\sum \bar{H}_i S_i$$

(Effective field at ith spin) $\bar{H}_i \equiv \sum_{j \neq i} J_{ij} a_{ij} S_j + H_i$

Recall that for $\mathcal{H} = -\sum H_i S_i$

$$M = \tanh(\beta H)$$

and for $\mathcal{H} = -\sum J_{ij} S_i S_j - \sum H_i S_i$

$$= -\sum \bar{H}_i \, S_i$$

$$\bar{H}_i \equiv \sum_j J_{ij} S_j + H_i$$

Effective field at S_i $\longleftrightarrow$ $\bar{H}_i = qJM + H_i$

mean value of Neighbors

Now one can use $\bar{H}$ instead of H

therefore $\boxed{M = \tanh(\beta \bar{H}) = \tanh(\beta H + \beta q J M)}$

Now similar to previous statement based on Mean field theory and Graph theory one can deduce that each node has $q \equiv$ mean number of nearest neighbors which its values depend on (a_{ij})

So we again achieve to same result as:

$$M = \tanh(\beta H + \beta q J M)$$

mean degree of links

B One can expand $M = \tanh(\beta H + \beta q J M)$

around $H \to 0$
$M \to 0$

and one can show that

$$M \sim |-t|^{1/2} \longrightarrow \boxed{\beta = 1/2}$$

and $G(r_i, r_j) = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$

$$\chi = \frac{\partial M}{\partial H} = \beta \int d^d R \, G(R) \sim |t|^{-\gamma}$$

$$\boxed{\gamma = 1}$$ 5

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3)

A) $Z = \mathrm{Tr}(e^{-\beta H}) = \sum e^{-\beta \mathcal{H}}$

$$= Z_< Z_>$$

$K \in [0, \Lambda^{-1}]$ $K \in [\Lambda^{-1}, a^{-1}]$

$\underbrace{\phantom{K \in [0, \Lambda^{-1}]}}$

this part has some singularity

$$a \ll \Lambda \lesssim \xi \sim L$$

$$Z = \mathrm{Tr}\, e^{-\beta \mathcal{H}} \xrightarrow{\text{Coarse-graining}} W_\Lambda = \mathrm{Tr}\, e^{-\beta \mathcal{H}_\Lambda}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$F = -K_B T \ln Z \qquad\qquad F_\Lambda = -K_B T \ln W_\Lambda$$

$$Z = \sum_{\{s\}''} W_\Lambda$$

$$Z = \int D\phi \, e^{-\beta L([\phi])} \qquad (5)$$

(B) In general to write an Effective Lagrangian according to $L[\phi] = \int d^d x \, \mathcal{L}(\phi, \partial\phi, \ldots)$

$$\left.\begin{array}{l}\text{Locality}\\ \text{Symmetry}\\ \text{Analyticity}\\ \text{Stability}\end{array}\right\} \text{should be adopted to write } \mathcal{L}(\phi, \partial\phi, \ldots) \qquad (5)$$

(C)

$$\beta \mathcal{H} = \int d^d r \Big[\overset{\text{mass}}{\overset{\downarrow}{m^2}} |\psi|^2 + b |\psi|^4 + |(\nabla - i e^* A)\psi|^2 + \frac{1}{2\mu} (\nabla \times A)^2 \Big]$$

☆ We Know that the Integrand should have

L^{-d} dimension.

☆ We Know that $b = cts$ and $b > 0$

☆ $m^2 \sim t \sim \xi^{-2}$

☆ $[|\nabla \varphi|^2] = [|L^{-1} \varphi|^2] = [L^{-d}]$

$L^{-2} [\varphi^2] = L^{-d} \Rightarrow [\varphi^2] = L^{-d+2}$

$$\boxed{[\varphi] \simeq L^{-\frac{d}{2}+1}}$$

☆ $[m^2 \varphi^2] = L^{-d}$

$[m^2 L^{-d+1}] = L^{-d}$

$[m^2] = L^{-1} \longrightarrow \boxed{[m] \sim L^{-2}}$

☆ $[(\nabla \times A)^2] = L^{-d}$

$[(L^{-1} A)^2] = L^{-d} \longrightarrow [A^2] = L^{-d+2}$

$$\boxed{[A] = L^{-\frac{d+1}{2}}}$$

☆ $\left[(e^* A \psi)^2\right] = L^{-d}$

$$\left[(e^*)^2 L^{-d+1} L^{-d+1}\right] = L^{-d}$$

$$\left[(e^*)^2\right] = L^{-d+2d-2} = L^{d-2}$$

$$\boxed{[e^*] = L^{\frac{d}{2}-1}}$$

Long wavelength of ψ means we should ignore $|\nabla\psi|^2$ and

$$\beta\mathcal{H} = \int d^d r \left[m^2|\psi|^2 + b|\psi|^4 + |(-ie^* A)\psi|^2 + \frac{1}{2\mu}(\nabla\times A)^2 \right]$$

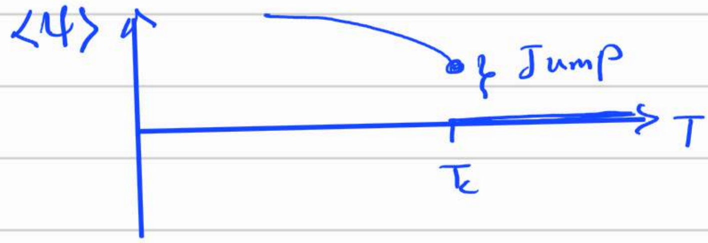
(10)

D To have first order transition, we need to have discontinous transition. In such case at least we need to $c|\psi|^3$ with $c>0$. Or if we would like to hold Z2-symmetry so, we should add

$c|\psi|^6$ as tricritical case.

without $|\psi|^3$ or $|\psi|^6$, we can not expect to

have



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