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Answer to Exercise Set #5 of Critical Phenomena

1- (**Kardar's book exercise 4-2**):

part(a)

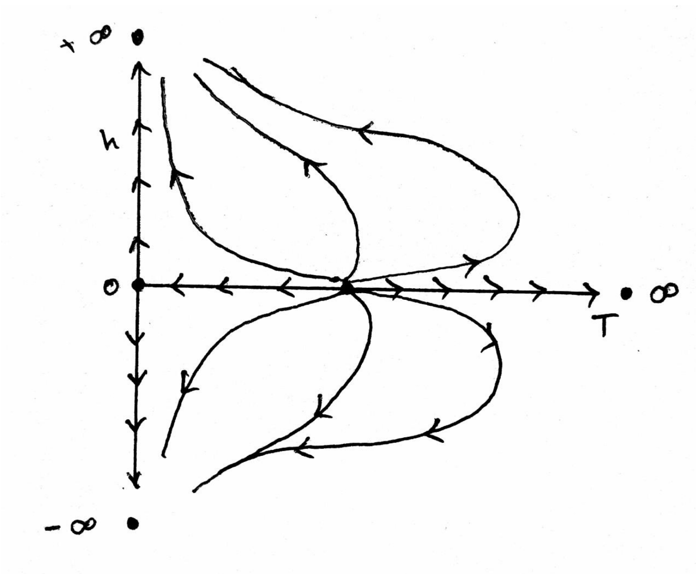
We have to find fixed points from this equation:

$$\begin{cases} \frac{dT}{d\ell} = -\epsilon T + \frac{T^2}{2} \\ \frac{dh}{d\ell} = dh \end{cases}$$

We have:

$$\frac{dT}{d\ell} = -\epsilon T + \frac{T^2}{2} = T\left(-\epsilon + \frac{T}{2}\right) = 0 \Rightarrow T^* = 0 \quad \text{and} \quad T^* = 2\epsilon$$
$$\frac{dh}{d\ell} = dh = 0 \Rightarrow h^* = 0$$

So we have one fixed point on origin and one on coupling constant T axis and three other fixed point on infinities. We know that this is Ising model so we can use our physical senses here, fixed point on infinities should be attractive because in Ising model if we have big external magnetic field, system stable ferromagnetic phase along magnetic field. In high temperature system have stable (attractive) paramagnetic phase. Other fixed point are unstable (repulsive) and RG flow comes out from them.



part(b)

We should linearised RG transformation, we have:

$$\begin{bmatrix} \Delta T' \\ \Delta H' \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} \Delta T \\ \Delta H \end{bmatrix}$$

With

$$\boldsymbol{M} = \begin{bmatrix} \partial R_\ell^T / \partial T & \partial R_\ell^T / \partial H \\ \\ \partial R_\ell^H / \partial T & \partial R_\ell^H / \partial H \end{bmatrix}_{T=T^*, h=h^*}$$

We should find R_ℓ^T and R_ℓ^H to construct matrix $\boldsymbol{M}$. We start from equations:

$$\begin{cases} \frac{dT}{d\ell} = -\epsilon T + \frac{T^2}{2} \\ \frac{dh}{d\ell} = dh \end{cases}$$

We start integrate of first equation, integral equation become:

$$\int_T^{T(\ell)} \frac{dT}{T^2/2 - \epsilon T} = \int_0^\ell d\ell \tag{1}$$

The left hand side of above is:

$$\begin{aligned} \int_T^{T(\ell)} \frac{dT}{T^2/2 - \epsilon T} &= \int_T^{T(\ell)} \frac{a}{T}\, dT + \int_T^{T(\ell)} \frac{b}{T/2 - \epsilon}\, dT \\ &= -\frac{1}{\epsilon} \int_T^{T(\ell)} \frac{dT}{T}\, dT + \frac{1}{\epsilon} \int_T^{T(\ell)} \frac{dT/2}{T/2 - \epsilon} \\ &= \frac{1}{\epsilon} \log\left(\frac{1}{2} - \frac{\epsilon}{T}\right)\Big|_T^{T(\ell)} \\ &= \frac{1}{\epsilon} \log\left(\frac{1}{2} - \frac{\epsilon}{T(\ell)}\right) - \frac{1}{\epsilon} \log\left(\frac{1}{2} - \frac{\epsilon}{T}\right) \\ &= \log\left(\frac{1/2 - \epsilon/T(\ell)}{1/2 - \epsilon/T}\right)^{1/\epsilon} \end{aligned}$$

Equation 1 becomes:

$$\log\left(\frac{1/2 - \epsilon/T(\ell)}{1/2 - \epsilon/T}\right)^{1/\epsilon} = \ell \Rightarrow \frac{1/2 - \epsilon/T(\ell)}{1/2 - \epsilon/T} = b^\epsilon \Rightarrow \frac{1}{T(\ell)} = \frac{1}{2\epsilon}(1 - b^\epsilon) + \frac{b^\epsilon}{T}$$

Finally the transformation that gives new temperature is:

$$R_\ell^T = T(\ell) = \frac{1}{(1 - b^\epsilon)/2\epsilon + b^\epsilon/T}$$

Now we take derivative in none trivial fixed point, we have:

$$\frac{\partial R_\ell^T}{\partial T}\Big|_{T=2\epsilon} = \frac{b^\epsilon}{4\epsilon^2 \left(\frac{b^\epsilon}{2\epsilon} + \frac{1-b^\epsilon}{2\epsilon}\right)^2} = b^\epsilon$$

Finally we can write:

$$\Lambda_\ell^t = b^{y_t} = b^\epsilon \Rightarrow y_t = \epsilon$$

Similarly for second equation we have:

$$\int_h^{h(\ell)} \frac{dh}{h} = \log\left(\frac{h(\ell)}{h}\right)$$

and

$$\log\left(\frac{h(\ell)}{h}\right) = (1+\epsilon)\ell \Rightarrow \frac{h(\ell)}{h} = e^{(1+\epsilon)\ell} = \left(e^\ell\right)^{1+\epsilon} = b^{1+\epsilon}$$

The new magnetic field comes from the equation:

$$R_\ell^H = h(\ell) = h\, b^{1+\epsilon}$$

derivative in fixed point is:

$$\left.\frac{\partial R_\ell^H}{\partial T}\right|_{h=0} = b^{1+\epsilon}$$

Scaling exponent for magnetic field is:

$$\Lambda_\ell^h = b^{y_h} = b^{1+\epsilon} \Rightarrow y_h = 1+\epsilon$$

part(c)

We know that correlation length behave like (in Goldenfeld notation):

$$\xi(t,h) = \ell^n \xi\left(\ell^{ny_t} t,\ \ell^{ny_h} h\right)$$

in Kardar's notation by using $b = e^\ell$ we have:

$$\xi(t,h) = b^n \xi\left(b^{ny_t} t,\ b^{ny_h} h\right) \tag{2}$$

We can write:

$$b^{ny_t} t = 1 \Rightarrow b^n = \left(\frac{1}{t}\right)^{1/y_t}$$

Put above in 2 we have:

$$\xi(t,h) = \left(\frac{1}{t}\right)^{1/y_t} \xi\left(1,\ h\left(\frac{1}{t}\right)^{y_h/y_t}\right)$$

$$\xi(t,h) = t^{-1/y_t} \xi\left(h/|t|^{y_h/y_t}\right)$$

Now we can use Widom hypothesis to write:

$$\nu = 1/y_t = 1/\epsilon, \qquad \Delta = y_h/y_t = (\epsilon+1)/\epsilon = 1 + \frac{1}{\epsilon}$$

part (d)

We know that total free energy before and after of RG transformation should be identical but scale of our system is different (after RG transformation) and this difference show itself in free energy density by:

$$f_s(t,h) = b^{-d} f_s(tb^{y_t}, hb^{y_h})$$

Note that we have not specified b, and thus we are at liberty to choose b as we please. Thus, we will choose:

$$tb^{y_t} = 1 \Rightarrow b = |t|^{-1/y_t}$$

Now we can write:

$$f_s(t,h) = |t|^{d/y_t} f_s(1, h|t|^{-y_h/y_t}) \Rightarrow \Delta = \frac{y_h}{y_t}, \quad 2-\alpha = \frac{d}{y_t}$$

finally we have:

$$\alpha = 2 - \frac{d}{y_t} = 2 - \frac{d}{\epsilon} = 2 - \frac{1+\epsilon}{\epsilon} = 1 - \frac{1}{\epsilon}$$

part(e)
The magnetization is obtained by differentiation:

$$M = -\frac{1}{k_B T}\frac{\partial f_s}{\partial h} \sim |t|^{(d-y_h)/y_t} f_s'(h/t^{\Delta}) \Rightarrow \beta = \frac{d-y_h}{y_t} = \frac{1+\epsilon-1-\epsilon}{\epsilon} = 0$$

The isothermal susceptibility is obtained by one more differentiation:

$$\chi_T = \frac{\partial M}{\partial h} \sim |t|^{(d-2y_h)/y_t} f_s''(h/t^{\Delta}) \Rightarrow \gamma = \frac{2y_h - d}{y_t} = \frac{2+2\epsilon-1-\epsilon}{\epsilon} = \frac{1}{\epsilon} + 1$$

part(f)
Hamiltonian is given by:

$$\begin{aligned}\beta H_{\Omega} &= \beta J \sum_{\langle ij \rangle} S_i S_j - \beta H \sum_i S_i \\ &\equiv -K \sum_{\langle ij \rangle} S_i S_j - h \sum_i S_i\end{aligned}$$

Spin transformation
We define the block spin S_I in block I by:

$$S_I \equiv \frac{1}{|\bar{m}_\ell|}\frac{1}{\ell^d}\sum_{i\in I} S_i$$

where the average magnetization of the block I is:

$$\bar{m}_\ell \equiv \frac{1}{\ell^d}\sum_{i\in I}\langle S_i\rangle$$

With this normalization, the block spins S_I have the same magnitude as the original spins:

$$\langle S_I \rangle = \pm 1$$

Assumption 1: ***Coupling constant changes with RG transformation.***
According to assumption 1, the effective Hamiltonian H_ℓ for the block spins is given by:

$$\beta H_\ell = -K_\ell \sum_{\langle ij \rangle} S_i S_j - h_\ell \sum_i S_i$$

Correlation length changes like:

$$\xi_\ell = \frac{\xi_1}{\ell}$$

Assumption 2: ***Coupling constant changes as power law.***
Since, we seek to understand the power-law and scaling behavior in the critical region, we assume that:

$$t_\ell = t\ell^{y_t} \quad y_t > 0;$$
$$h_\ell = t\ell^{y_h} \quad y_h > 0.$$

We can find similar to correlation length re-scaled effective field h_ℓ, we have:

$$H\sum_i S_i \cong h\bar{m}_\ell \ell^d \sum_I S_I \equiv h_\ell \sum_I S_I \Rightarrow h_\ell = h\bar{m}_\ell \ell^d \Rightarrow \bar{m}_\ell = h_\ell \ell^{-d}/h = \ell^{y_h - d}$$

Now consider the correlation function for the block spin Hamiltonian

$$G(\boldsymbol{r}_\ell, t_\ell) \equiv \langle S_I S_J \rangle - \langle S_I \rangle \langle S_J \rangle$$

where r_ℓ is the displacement between the centers of blocks I and J in units of $a\ell$; if denotes the displacement between the centers of blocks I and J in units of a(lattice constant), then $\boldsymbol{r}_\ell = \boldsymbol{r}/\ell$. In order that the notion of a block spin correlation function be well-defined, we require that the separation between the blocks be much larger than the block size itself: thus, we are concerned only with the long-wavelength limit $r \gg a$. How is $G(r_\ell, t_\ell)$ related to $G(r, t)$! Using the definition S_I:

$$\bar{m}_\ell = h_\ell \ell^{-d}/h = \ell^{y_h - d}$$

Thus the correlation function transforms as:

$$\begin{aligned} G(\boldsymbol{r}_\ell, t_\ell) &= \frac{1}{\ell^{2(y_h - d)} \cdot \ell^{2d}} \sum_{i \in I} \sum_{j \in} \left[\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \right] \\ &= \frac{1}{\ell^{2(y_h - d)}} \cdot \frac{1}{\ell^{2d}} \cdot \ell^d \cdot \ell^d \left[\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \right] \\ &= \ell^{2(d - y_h)} G(r, t) \end{aligned}$$

Including the dependence on h, we have:

$$G\left(\frac{\boldsymbol{r}}{\ell}, t\ell^{y_t}, h\ell^{y_h}\right) = \ell^{2(d - y_h)} G(\boldsymbol{r}, t, h)$$

Again, we can choose ℓ as we please, and so we set $\ell = t^{-1/y_t}$ by $t\ell^{y_t} = 1$ as before, to obtain:

$$G(\boldsymbol{r}, t, h) = t^{2(d - y_h)/y_t} G\left(\boldsymbol{r} t^{1/y_t}, 1, h t^{-y_t/y_h}\right) \tag{3}$$

The prefactor $t^{2(d-y_h)/y_t}$ unfamiliar, but we can easily re-arrange this expression by writing:

$$G\left(\boldsymbol{r} t^{1/y_t}, 1, h t^{-y_t/y_h}\right) \equiv (rt)^{-2(d - y_h)/y_t} F_G\left(\boldsymbol{r} t^{1/y_t}, h t^{-y_t/y_h}\right)$$

which defines the scaling function F_G Substituting into 3 we obtain our final result:

$$G(\boldsymbol{r}, t, h) = \frac{1}{r^{2(d - y_h)}} F_G\left(\boldsymbol{r} t^{1/y_t}, h t^{-y_t/y_h}\right) \sim r^{-2(d - y_h)}$$

We know that correlation function behave like (Goldenfeld Eq(8.30)):

$$G(\boldsymbol{r}, t, h) \sim r^{2-d-\eta}$$

Finally we have:

$$2 - d - \eta = -2d + 2y_h \Rightarrow \eta = 2 + d - 2y_h \Rightarrow \eta = 2 - 1 - \epsilon = 1 - \epsilon$$

part(g)
We know that if $d = 1$ then $\epsilon = 0$ so:

$$\begin{cases} \frac{dT}{d\ell} = \frac{T^2}{2} \\ \frac{dh}{d\ell} = h \end{cases}$$

We can integrate above to find $T(\ell)$, we have:

$$\int_T^{T(\ell)} \frac{2dT}{T^2} = \int_1^{\ell} d\ell \Rightarrow \left[-\frac{2}{T}\right]_T^{T(\ell)} = \ell - 1 \Rightarrow \left[\frac{2}{T} - \frac{2}{T(\ell)}\right] = \ell - 1$$

Then:

$$T(\ell)^{-1} = T^{-1} - \left(\frac{\ell - 1}{2}\right)$$

For other equation we have:

$$\frac{dh}{d\ell} = h \Rightarrow \int_h^{h(\ell)} \frac{dh}{h} = \int_1^{\ell} d\ell \Rightarrow \log\left(\frac{h(\ell)}{h}\right) = \ell - 1 \Rightarrow h(\ell) = he^{\ell - 1}$$

The correlation length is then obtained from ($b = e^l$):

$$\xi(T, h) = e^{\ell} \xi[T(\ell), h(\ell)]$$

We have:

$$\xi(T, h) = e^{\ell} \xi \left[\frac{1}{1/T - (\ell - 1)/2},\ he^{\ell - 1}\right]$$

We can choose $\ell = 2/T - 1$ (for $\frac{1}{1/T - (\ell-1)/2} = 1$), we have:

$$\xi(T, h) = e^{2/T - 1} \xi \left[1,\ he^{2/T - 2}\right]$$

By considering $h = 0$ we can write:

$$\xi(T, 0) \sim e^{2/T}$$

2- (**Kardar's book exercise 6-2**):
Migdal–Kadanoff method: We want to perform this method on Potts spin which is $s_i = (1, 2, \dots, q)$. Hamiltonian is:

$$-\beta\mathcal{H} = K \sum_{\langle ij \rangle} \delta_{s_i, s_j}$$

δ in above is Kronecker delta. In one dimension above becomes:

$$-\beta\mathcal{H} = K \sum_{i} \delta_{s_i, s_{i+1}}$$

Partition function is:

$$\mathcal{Z} = \sum_{S_1=1}^{q} \sum_{S_2=1}^{q} \cdots \sum_{S_N=1}^{q} \exp\left(K \sum_i \delta_{s_i, s_{i+1}} \right) = \sum_{S_1=1}^{q} \sum_{S_2=1}^{q} \cdots \sum_{S_N=1}^{q} \prod_{i=1}^{N} e^{K\delta_{s_i, s_{i+1}}}$$

We can sum over for example S_2 exactly like:

$$\sum_{S_2=1}^{q} e^{K(\delta_{s_1,s_2} + \delta_{s_2,s_3})} = \begin{cases} q - 1 + e^{2K} & \text{if} \quad s_1 = s_3 \\ q - 2 + 2e^{K} & \text{if} \quad s_1 \neq s_3 \end{cases}$$

In above we for case $s_1 = s_2$ we have $q - 1$ term which $\delta_{s_1,s_2} = \delta_{s_2,s_3} = 0$ and one term that $\delta_{s_1,s_2} = \delta_{s_2,s_3} = 1$, so above result obtained. In case $s_1 \neq s_2$ we have $q - 2$ that have property of $\delta_{s_1,s_2} = 0$ or $\delta_{s_2,s_3} = 0$ and two term with property $\delta_{s_1,s_2} = 1$ or $\delta_{s_2,s_3} = 1$. We can find how coupling constant changes before and after of RG transformation by this equation:

$$e^{K'\delta_{s_1,s_3}} e^{K_0} = \begin{cases} q - 1 + e^{2K} & \text{if} \quad s_1 = s_3 \\ q - 2 + 2e^{K} & \text{if} \quad s_1 \neq s_3 \end{cases}$$

from above we can write:

$$e^{K'} e^{K_0} = q - 1 + e^{2K}$$
$$e^{K_0} = q - 2 + 2e^{K}$$

We can find that:

$$e^{K'} = \frac{q - 1 + e^{2K}}{q - 2 + 2e^{K}}, \quad \text{and} \quad e^{K_0} = q - 2 + 2e^{K}$$

For finding fixed points we write above as follows:

$$e^{K'} = \frac{q - 1 + e^{2K}}{q - 2 + 2e^{K}} \Rightarrow x^* = \frac{q - 1 + (x^*)^2}{q - 2 + 2x^*} \Rightarrow (x^*)^2 + x(q - 2) - (q - 1) = 0$$

I solve above with Mathematica an find:

$$\text{Solve}\left[(q - 2)x - (q - 1) + x^2 = 0, x\right] \Rightarrow \{\{x \to 1\}, \{x \to 1 - q\}\}$$

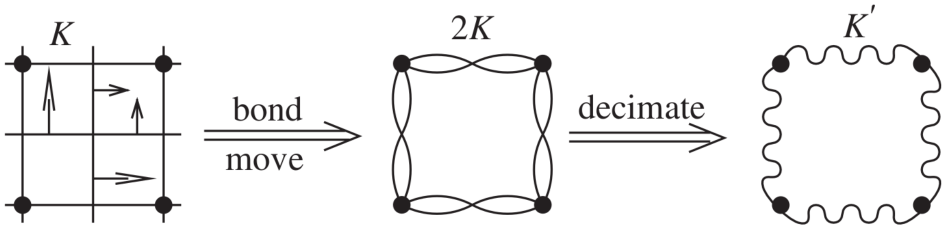
We know that always $q > 1$, so we can write the only fixed point is $K^* = 0$. Now we want to discuss this fixed point is attractive or repulsive. We begin from recursion relation for coupling constant, we can find that if $x \gg 1$ recursion relation become:

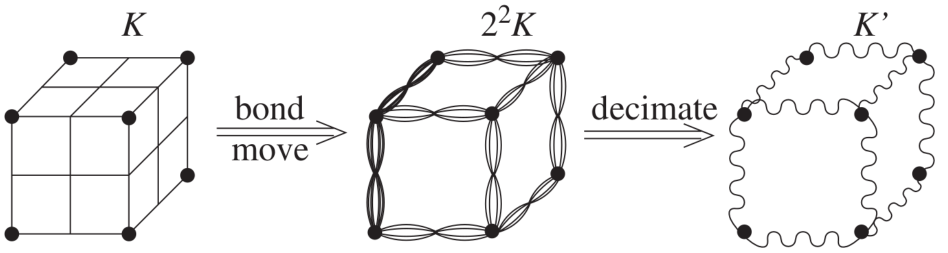
$$x' = \frac{q - 1 + x^2}{q - 2 + 2x} \quad \text{for} \quad x \gg 1 \Rightarrow x' = \frac{x}{2}$$

So new coupling constant is half of old one, therefore $K^* = 0$ is attractive fixed point.
part(b)
We can moving half of bonds and strengthen remains by factor 2^{d-1} it means that $K \longrightarrow 2^{d-1}K$, so we have:

$$e^{K'} = \frac{q - 1 + e^{2^d K}}{q - 2 + 2e^{2^{d-1}K}}$$

Maybe the coefficient 2^{d-1} be confusing, for better understanding I bring two figure of Kardar's[1] book chapter 6.





part(c)
We want to check coupling constant equation in zero and infinity is attractive, this means that there is a unstable(repulsive) fixed point in between zero and infinity. We check for infinity first ($K \gg 1$), we have:

$$e^{K'} = \frac{q - 1 + e^{2^d K}}{q - 2 + 2e^{2^{d-1}K}} \approx \frac{\left(e^K\right)^{2^d}}{\left(e^K\right)^{2^{d-1}}} = \left(e^K\right)^2$$

Above means we start from a big K and put it in coupling constant equation, the result is a bigger K'. This means that infinity is attractive fixed point. Now we perform similar approach on $K \ll 1$ we have:

$$e^{K'} = \frac{q - 1 + e^{2^d K}}{q - 2 + 2e^{2^{d-1}K}} \approx \frac{q - 1 + 1 + 2^d K}{q - 2 + 2 + 2^d K} = 1 \Rightarrow K' = 0$$

Again we start from a very small K and put value in coupling constant equation we obtain new coupling constant becomes zero, this means that $K = 0$ is an attractive fixed point, therefore there is an unstable fixed point in between two trivial fixed points.

[1] *Statistical Physics of Fields by M. Kardar*

part(d)
We have $d = 2$ and $q = 3$ we can find fixed point from below equation:

$$e^{K^*} = \frac{2 + e^{4K^*}}{1 + 2e^{2K^*}}$$

We take $x = e^{K^*}$, we have:

$$x = \frac{2 + x^4}{1 + 2x^2}$$

We can solve this by Mathematica, we have:

$$\text{Solve}\left[x^4 - 2x^3 - x + 2 = 0, x\right] \Rightarrow \{\{x \to 1\}, \{x \to 2\}, \{x \to -(-1)^{1/3}\}, \{x \to (-1)^{2/3}\}\}$$

Real value none trivial fixed point is $K = \log 2$. We can find scaling exponent around fixed point by linearisation of coupling constant equation:

$$K' = \log\left(\frac{2 + e^{4K}}{1 + 2e^{2K}}\right)$$

As always I used Mathematica, we have:

$$\Lambda_\ell^t = 2^{y_t} = \left.\frac{\partial K'}{\partial K}\right|_{K=\log 2} = \left.\frac{\left(2e^{2K} + 1\right)}{e^{4K} + 2}\left(\frac{4e^{4K}}{2e^{2K} + 1} - \frac{4e^{2K}\left(e^{4K} + 2\right)}{\left(2e^{2K} + 1\right)^2}\right)\right|_{K=\log 2} = \frac{16}{9}$$

We should take $b = 2$ because we perform RG transformation on nearest neighbor. finally we have $y_t \simeq 0.830075$.

Now we take $d = 2$ and $q = 1$ coupling constant equation becomes:

$$e^{K'} = \frac{e^{4K}}{2e^{2K} - 1}$$

Fixed points are:

$$\text{Solve}\left[x^4 - 2x^3 + x = 0, x\right] \Rightarrow \left\{\{x \to 0\}, \{x \to 1\}, \left\{x \to \frac{1}{2}\left(1 - \sqrt{5}\right)\right\}, \left\{x \to \frac{1}{2}\left(\sqrt{5} + 1\right)\right\}\right\}$$
$$\{\{x \to 0.\}, \{x \to 1.\}, \{x \to -0.618034\}, \{x \to 1.61803\}\}$$

Fixed points $x = 0$ and $x = 1$ are trivial and none trivial fixed point is $x = 1.61803$ or $K^* = 0.481212$. We have:

$$\Lambda_\ell^t = 2^{y_t} = \left.\frac{\partial K'}{\partial K}\right|_{K=0.481212} = \left.\frac{4\left(e^{2K} - 1\right)}{2e^{2K} - 1}\right|_{K=0.481212} = 1.52786$$

We can find that $y_t = 0.611516$.

Now we take $d = 2$ and $q = 0$ coupling constant equation becomes:

$$e^{K'} = \frac{e^{4K} - 1}{2e^{2K} - 2}$$

Fixed points are:

$$\text{Solve}\left[x^4 - 2x^3 + 2x - 1 = 0, x\right] \Rightarrow \{\{x \to -1\}, \{x \to 1\}, \{x \to 1\}, \{x \to 1\}\}$$

Fixed point is $x = 1$ or $K^* = 0$. Similar to above we have:

$$\Lambda_\ell^t = 2^{y_t} = \left.\frac{\partial K'}{\partial K}\right|_{K=0} = \left.\frac{2e^{2K}}{e^{2K}+1}\right|_{K=0} = 1$$

We can find that $y_t = 0$.

3- (**Kardar's book exercise 6-3**):
Hamiltonian for Potts model is:

$$-\beta\mathcal{H} = K\sum_{i=1}^{N}\delta_{s_i,s_{i+1}} + K\delta_{s_N,s_1}$$

Second term exists for demanding periodic boundary condition $s_{N+1} = s_1$. Partition function can be written as follows:

$$\mathcal{Z}_N(K) = \sum_{s_1}\sum_{s_2}\cdots\sum_{s_N}\exp\left(K\delta_{s_1,s_2}\right)\exp\left(K\delta_{s_2,s_3}\right)\ldots\exp\left(K\delta_{s_N,s_1}\right)$$

We consider periodic boundary condition $s_{N+1} = s_1$, transfer matrix elements are:

$$T_{s_1,s_2} = \exp\left(K\delta_{s_1,s_2}\right)$$

We can write above like:

$$\mathcal{Z}_N(K) = \sum_{s_1}\sum_{s_2}\cdots\sum_{s_N}\langle s_1|\boldsymbol{T}|s_2\rangle\langle s_2|\boldsymbol{T}|s_3\rangle\ldots\langle s_N|\boldsymbol{T}|s_1\rangle = \mathrm{Tr}\,\boldsymbol{T}^N$$

Spin s_i on each site takes q values $s_i = (1,2,\ldots,q)$ so transfer matrix is $q\times q$ and can be written as follows:

$$T = \begin{bmatrix} e^K & 1 & 1 & \ldots & 1 \\ 1 & e^K & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & e^K \end{bmatrix}$$

In above matrix one of A is e^K and others are 1. We can write above in the form:

$$\begin{bmatrix} e^K & 1 & 1 & \ldots & 1 \\ 1 & e^K & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & e^K \end{bmatrix} = \begin{bmatrix} e^K & 0 & 0 & \ldots & 0 \\ 0 & e^K & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & e^K \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{bmatrix}$$

In right hand side of above we have one diagonal matrix (which all of it's eigenvalues are e^K) and a none diagonal (trace less) matrix. We can guess eigenvalues of this trace less matrix by considering that trace in invariant under rotation so we can write this matrix in the form:

$$\begin{bmatrix} q-1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -1 \end{bmatrix}$$

Biggest eigenvalue of above can be any where diagonal elements. So with the aim of above we can find eigenvalues of these two matrix biggest one is $e^K + q - 1$ and other ones are $e^K - 1$. Finally we have:

$$\mathrm{Tr}\,\boldsymbol{T}^N = (e^K + q - 1)^N + (q-1)(e^K - 1)^N$$

part(b)
We can find free energy by:

$$\mathcal{F} = -k_B T \log \mathcal{Z} = -k_B T \left[N \log\left(e^K + q - 1\right) + \log(q-1) + N \log\left(e^K - 1\right) \right]$$

In thermodynamic limit only biggest eigenvalue play crucial role so we can write:

$$\mathcal{F} \approx -k_B T N \log\left(e^K + q - 1\right)$$

Finally free energy per site equal to:

$$\frac{\mathcal{F}}{N} \approx -k_B T \log\left(e^K + q - 1\right)$$

part(c)
We can find that:

$$\xi = \frac{1}{\log\left(\frac{\lambda_1}{\lambda_2}\right)} = \frac{1}{\log\left(\frac{e^K + q - 1}{e^K - 1}\right)}$$

In the limit of $K \to \infty$ we have:

$$\xi = \frac{1}{\log\left(\frac{\lambda_1}{\lambda_2}\right)} = \frac{1}{\log\left(\frac{e^K + q - 1}{e^K - 1}\right)} = \frac{1}{\log\left(1 + \frac{q}{e^K - 1}\right)} \approx \frac{e^K - 1}{q}$$

4- (**Pathria's book exercise 13-1**):
In fact, the energy of a given configuration does not depend on the detailed values of all the variables σ_i ; it depends only on a few numbers such as the total number N_+ of "up" spins, the total number N_{++} of "up–up" nearest-neighbor pairs, and so on. To see this, we define certain other numbers as well: N_- as the total number of "down" spins, N_{--} as the total number of "down–down" nearest-neighbor pairs, and N_{+-} as the total number of nearest-neighbor pairs with opposite spins. The numbers N_+ and N_- must satisfy the relation:

$$N_+ + N_- = N$$

And if q denotes the coordination number of the lattice, that is, the number of nearest neighbors for each lattice site, then we also have the relations:

$$qN_+ = 2N_{++} + N_{+-}$$
$$qN_- = 2N_{--} + N_{+-}$$

With the help of these relations, we can express all our numbers in terms of any two of them, say N_+ and N_{++} . Thus:

$$N_- = N - N_+, \qquad N_{+-} = qN_+ - 2N_{++}, \qquad N_{--} = \frac{1}{2}qN - qN_+ + N_{++};$$

it will be noted that the total number of nearest-neighbor pairs of all types is given, quite expectedly, by the expression:

$$N_{++} + N_{--} + N_{+-} = \frac{1}{2}qN$$

Naturally, the Hamiltonian of the system can also be expressed in terms of N_+ and N_{++}; we have from below Hamiltonian:

$$H\{\sigma_i\} = -J\sum_{n.n.}\sigma_i\sigma_j - \mu B\sum_i \sigma_i \tag{4}$$

With the help of the relations established above:

$$\begin{aligned} H_N(N_+, N_{++}) &= -J(N_{++} + N_{--} - N_{+-}) - \mu B(N_+ - N_-) \\ &= -J\left(\frac{1}{2}qN - 2qN_+ + 4N_{++}\right) - \mu B(2N_+ - N) \end{aligned} \tag{5}$$

Now we can find partition function for Hamiltonian 4 with transfer matrix method and find energy and magnetization of system which is (Eq 13.2.12):

$$\begin{aligned} U(B,T) = -NJ &- \frac{N\mu\sinh(\beta\mu B)}{\left(e^{-4\beta J} + \sinh^2(\beta\mu B)\right)^{1/2}} \\ &+ \frac{2NJe^{-4\beta J}}{\left(\cosh(\beta\mu B) + \left[e^{-4\beta J} + \sinh^2(\beta\mu B)\right]^{1/2}\right)\left(e^{-4\beta J} + \sinh^2(\beta\mu B)\right)^{1/2}} \\ = -NJ &- \frac{N\mu\sinh(\beta\mu B)}{P(\beta,\, B)} + \frac{2NJe^{-4\beta J}}{D(\beta,\, B)} \end{aligned} \tag{6}$$

Where:

$$D(\beta,\, B) = P(\beta,\, B)\,[P(\beta,\, B) + \cosh(\beta\mu B)]\,, \qquad P(\beta,\, B) = \left(e^{-4\beta J} + \sinh^2(\beta\mu B)\right)^{1/2}$$

and magnetization of system is(Eq 13.2.13):

$$M = \frac{N\mu\sinh(\beta\mu B)}{P(\beta,\, B)}$$

We know that $M = (N_+ - N_-)\mu$ and $N_- + N_+ = N$ we can find:

$$\begin{cases} M/\mu = (N_+ - N_-) \\ N_- + N_+ = N \end{cases} \Rightarrow \begin{cases} N_+ = 1/2(N + M/\mu) = N/2\left(1 + \frac{\sinh(\beta\mu B)}{P(\beta, B)}\right) \\ N_- = 1/2(N - M/\mu) = N/2\left(1 - \frac{\sinh(\beta\mu B)}{P(\beta, B)}\right) \end{cases}$$

Now compare equation 5 and 6 with $q = 2$ we can find:

$$-J\left(\frac{1}{2}qN - 2qN_+ + 4N_{++}\right) = -NJ + \frac{2NJe^{-4\beta J}}{D(\beta,\, B)} \Rightarrow N_+ - N_{++} = \frac{Ne^{-4\beta J}}{2D(\beta,\, B)}$$

With above equations we can derived desired result, for N_{++} we can write:

$$\begin{aligned} N_{++} &= N_+ - \frac{Ne^{-4\beta J}}{2D(\beta,\, B)} \\ &= N_+ - N\frac{P^2(\beta,\, B) - \sinh^2(\beta\mu B)}{2D(\beta,\, B)} \\ &= \frac{N}{2}\left(1 + \frac{\sinh(\beta\mu B)}{P(\beta,\, B)}\right) - N\frac{P^2(\beta,\, B) - \sinh^2(\beta\mu B)}{2D(\beta,\, B)} \end{aligned}$$

For other parts we can write:

$$N_{+-} = 2(N_+ - N_{++}) = 2N\frac{P^2(\beta,\, B) - \sinh^2(\beta\mu B)}{2D(\beta,\, B)}$$

and

$$\begin{aligned} N_{--} &= N - 2N_+ + N_{++} \\ &= N - N\left(1 + \frac{\sinh(\beta\mu B)}{P(\beta,\, B)}\right) + \frac{N}{2}\left(1 + \frac{\sinh(\beta\mu B)}{P(\beta,\, B)}\right) - N\frac{P^2(\beta,\, B) - \sinh^2(\beta\mu B)}{2D(\beta,\, B)} \end{aligned}$$

The mathematics to perform desired result is straight forward but it have long way to do, so I ignore them for now.

5- (**Pathria's book exercise 13-2**):

part(a)

We can write Hamiltonian for previous question as follows:

$$H_N(N_+, N_{+-}) = -J\left(\frac{1}{2}qN - 2N_{+-}\right) - \mu B(2N_+ - N)$$

Now we can write partition function with a density of state as follows:

$$\mathcal{Z}_N(B,T) = \sum_{N_+,N_{+-}}^{\prime} g_N(N_+, N_{+-}) \exp\left[H_N(N_+, N_{++})\right]$$

part(b)

In a short paper published in 1925, Ising himself gave an exact solution to the problem in one dimension. He employed a combinatorial approach which was essentially equivalent to the one being presented here. For this, we express the lattice Hamiltonian in terms of the numbers N_+ and N_{+-} rather than in terms of N_+ and N_{++}. We know $2N_{++} = qN_+ - N_{+-}$, we obtain (for $q = 2$):

$$H_N(N_+, N_{+-}) = -J(N - 2N_{+-}) - \mu B(2N_+ - N)$$

The partition function of the system is then given by:

$$\mathcal{Z}_N(B,T) = \sum_{N_+,N_{+-}}^{\prime} \exp\left(-\beta H_N\left(N_+, N_{+-}\right)\right)$$

That is:

$$e^{-\beta A} = e^{\beta N(J-\mu B)} \sum_{N_+=0}^{N} e^{2\beta\mu BN} + \sum_{N_{+-}}^{\prime} g_N(N_+, N_{+-}) e^{-2\beta J N_+}$$

where the primed summation $\sum'$ goes over all values of N_{+-} that are consistent with a fixed value of N_+ and is followed by a summation over all possible values of N_+ viz. from $N_+ = 0$ to $N_+ = N$. The symbol $g_N(N_+, N_{+-})$ here denotes the "number of distinct ways in which the N spins of the (linear) chain can be so arranged as to obtain certain definite values of the numbers N_+ and N_{+-}". To determine $g_N(N_+, N_{+-})$.

we proceed as follows. As soon as we fix the number of "up" spins as N_+ 9 the number of "down" spins is automatically fixed as $(N - N_+)$. The problem then reduces to determining the "number of distinct ways in which N_+ entities of one kind, say A, and $(N - N_+)$ entities of another kind, say B, can be distributed over a row of N sites, such that there occur N_+ links of the type AB or BA in the distribution". For instance, the arrangements shown below indicate two of the many different ways in which eight entities of the type A and seven entities of the type B can be distributed such that there exist nine links of the type AB or BA:

$$AAA|B|A|BB|A|B|A|BB|AA|B \qquad \text{(I)}$$

and

$$B|AA|BB|A|B|A|BB|A|B|AAA \qquad \text{(II)}$$

In arrangements such as (I), we have five links of the type AB and four links of the type BA, while in arrangements such as (II), we have four links of the type AB and five links of the type BA; in this example, we have purposely chosen N_{+-} to be odd. The total number of arrangements in the categories (I) and (II) will indeed be equal, and will be given by the "number of ways in which (a) the $N_+ A$'s can be divided into $\frac{1}{2}(N_{+-}+1)$ groups, each group containing at least one A, and (b) the (NN_+) B's can be divided into $\frac{1}{2}(_{+-}+1)$ groups, each group containing at least one B". We thus have, for odd values of N_{+-},

$$g_N(N_+, N_{+-}) = 2 \cdot \frac{(N_+ - 1)!}{\left[N_+ - \frac{1}{2}(N_{+-}+1)\right]! \left[\frac{1}{2}(N_{+-}-1)\right]!} \times \frac{(N - N_+ - 1)!}{\left[N - N_+ - \frac{1}{2}(N_{+-}+1)\right]! \left[\frac{1}{2}(N_{+-}-1)\right]!}$$

Using Stirling's formula, we obtain

$$\begin{aligned}\log g_N(N_+, N_{+-}) \simeq\ & N_+ \log N_+ + (N - N_+) \log(N - N_+) \\ & - (N_+ - \frac{1}{2}N_{+-}) \log\left(N_+ - \frac{1}{2}N_{+-}\right) \\ & - (N - N_+ - \frac{1}{2}N_{+-}) \log\left(N - N_+ - \frac{1}{2}N_{+-}\right) \\ & - 2\left(\frac{1}{2}N_{+-}\right) \log\left(\frac{1}{2}N_{+-}\right)\end{aligned}$$

A little reflection will show that *asymptotically* the same result holds for even values of N_{+-}.

6- (**Pathria's book exercise 13-7**):
Hamiltonian is:

$$H = -J_1 \sum_i (S_i S_{i+1} + S'_i S'_{i+1}) - J_2 \sum_i S_i S'_i$$

Partition function can be written as follows:

$$\mathcal{Z}_N(h, K_1, K_2) = \sum_{S_1} \sum_{S'_1} \cdots \sum_{S_N} \sum_{S'_N} \exp\left(K_1(S_1 S_2 + S'_1 S'_2) + \frac{K_2}{2}(S_1 S'_1 + S_2 S'_2)\right)$$
$$\ldots \exp\left(K_1(S_{N-1} S_1 + S'_{N-1} S'_1) + \frac{K_2}{2}(S_1 S'_1 + S_{N-1} S'_{N-1})\right)$$

In above $K_1 = \beta J_1$ and $K_2 = \beta J_2$. We can write transfer matrix elements as:

$$T_{S_1 S_2} = \exp\left(K_1(S_1 S_2 + S'_1 S'_2) + \frac{K_2}{2}(S_1 S'_1 + S_2 S'_2)\right)$$

All elements of transfer matrix can be calculated as follows:

	$S'_1 \uparrow S'_2 \uparrow$	$S'_1 \uparrow S'_2 \downarrow$	$S'_1 \downarrow S'_2 \uparrow$	$S'_1 \downarrow S'_2 \downarrow$
$S_1 \uparrow S_2 \uparrow$	$e^{2K_1+K_2}$	1	1	$e^{2K_1-K_2}$
$S_1 \uparrow S_2 \downarrow$	1	$e^{-2K_1+K_2}$	$e^{-2K_1-K_2}$	1
$S_1 \downarrow S_2 \uparrow$	1	$e^{-2K_1-K_2}$	$e^{-2K_1+K_2}$	1
$S_1 \downarrow S_2 \downarrow$	$e^{2K_1-K_2}$	1	1	$e^{2K_1+K_2}$

Finally transfer matrix is:

$$T = \begin{bmatrix} e^{2K_1+K_2} & 1 & 1 & e^{2K_1-K_2} \\ 1 & e^{-2K_1+K_2} & e^{-2K_1-K_2} & 1 \\ 1 & e^{-2K_1-K_2} & e^{-2K_1+K_2} & 1 \\ e^{2K_1-K_2} & 1 & 1 & e^{2K_1+K_2} \end{bmatrix}$$

We name this matrix as:

$$\begin{bmatrix} A & 1 & 1 & B \\ 1 & C & D & 1 \\ 1 & D & C & 1 \\ B & 1 & 1 & A \end{bmatrix}$$

In above:

$$A = e^{2K_1+K_2} \qquad B = e^{2K_1-K_2} \qquad C = e^{-2K_1+K_2} \qquad D = e^{-2K_1-K_2}$$

Now we can use this website [2] to find eigen values. Eigen values are:

$\lambda_1 = C - D$

$\lambda_2 = -\frac{1}{2}\left[\sqrt{D^2 + (2C - 2B - 2A)D + A^2 + (-2B - 2A)C + B^2 + 2AB + A^2 + 16} - D - C - B - A\right]$

$\lambda_3 = \frac{1}{2}\left[\sqrt{D^2 + (2C - 2B - 2A)D + A^2 + (-2B - 2A)C + B^2 + 2AB + A^2 + 16} + D + C + B + A\right]$

$\lambda_4 = A - B$

[2] http://wims.unice.fr/~wims/en_tool~linear~matrix.en.phtml

Since λ_3 is the largest eigenvalue of T, we can approximate partition function as bellow:

$$\mathcal{Z}_N(K) \simeq \lambda_3^N(1 + O(e^{-\alpha N}))$$

Now we can write free energy per spin:

$$\lim_{N\to\infty} \frac{\mathcal{F}(K,T)}{N} = -k_B T \log \lambda_3$$
$$= -k_B T \log \left[\frac{1}{2}\left[\sqrt{D^2 + (2C - 2B - 2A)D + A^2 + (-2B - 2A)C + B^2 + 2AB + A^2 + 16} + D + C + B + A\right]\right]$$

Replacing A, B, C and D, we obtain the question results.

7- (**Pathria's book exercise 13-8**):
We know the Hamiltonian which is:

$$\mathcal{H}_\Omega\{S\} = -J \sum_{\langle ij \rangle} S_i S_j$$

Now we want to calculate partition function in one dimension, we have:

$$\begin{aligned}\mathcal{Z}_N(K) &= \text{Tr } e^{K \sum_i S_i S_{i+1}} \\ &= \sum_{S_1} \cdots \sum_{S_N} \left[e^{K S_1 S_2}\right] \left[e^{K S_2 S_3}\right] \ldots \left[e^{K S_N S_1}\right]\end{aligned}$$

So transfer matrix elements is:

$$T_{S_1 S_2} = e^{K S_1 S_2}$$

We can find elements of transfer matrix by:

	$s_1 = 1$	$s_1 = 0$	$s_1 = -1$
$s_2 = 1$	e^K	1	e^{-K}
$s_2 = 0$	1	1	1
$s_2 = -1$	e^{-K}	1	e^K

So transfer matrix is:

$$T = \begin{bmatrix} e^K & 1 & e^{-K} \\ 1 & 1 & 1 \\ e^{-K} & 1 & e^K \end{bmatrix}$$

Now we have to digonalize this matrix to find eigen values. I used this website [3] and eigen values are:

$$\begin{aligned}\lambda_1 &= \frac{1}{2}\left[(1 + 2\cosh K) + \left(8 + (2\cosh K - 1)^2\right)^{1/2}\right] \\ \lambda_2 &= \frac{1}{2}\left[(1 + 2\cosh K) - \left(8 + (2\cosh K - 1)^2\right)^{1/2}\right] \\ \lambda_3 &= 2\sinh K\end{aligned}$$

Considering that $\lambda_1 > \lambda_2 > \lambda_3$ we can consider λ_1 in thermodynamic limit, so we have:

$$\mathcal{Z}_N(K) \simeq \lambda_1^N (1 + \mathcal{O}(e^{-\alpha N}))$$

Now we can write free energy per spin:

$$\lim_{N\to\infty} \frac{\mathcal{F}(K,T)}{N} = -k_B T \log \lambda_1 = -k_B T \log \left[\frac{1}{2}\left[(1 + 2\cosh K) + \left(8 + (2\cosh K - 1)^2\right)^{1/2}\right]\right]$$

In limit of $T \to 0$ or $K \to \infty$ we can approximate $\cosh K \approx e^K/2$ so above changes as:

$$\frac{1}{N}\mathcal{F}(K,T) = -k_B T \log \lambda_1 = -k_B T \log \left[\frac{1}{2}\left[(1 + e^K) + \left(8 + (e^K - 1)^2\right)^{1/2}\right]\right] \approx -k_B T \log e^K = J$$

[3] http://wims.unice.fr/~wims/en_tool~linear~matrix.en.phtml

Which is energy of perfectly ordered system. In limit of $T \to \infty$ or $K \to 0$ we can approximate $\cosh K \approx 1$ so above changes as:

$$\frac{1}{N}\mathcal{F}(K,T) = -k_B T \log \lambda_1 = -k_B T \log \left[\frac{1}{2}\left[(1+2) + \left(8 + (2-1)^2\right)^{1/2}\right]\right] = -k_B T \log 3$$

Which is the entropy of three state system (consider that in high temperature energy compared to entropy is negligible).

8- (**Pathria's book exercise 14-1**):
We should find $P^2\{K\}$ by:

$$e^{2K_0}\begin{bmatrix} e^{K_1+K_2} & e^{-K_1} \\ e^{-K_1} & e^{K_1-K_2} \end{bmatrix}\begin{bmatrix} e^{K_1+K_2} & e^{-K_1} \\ e^{-K_1} & e^{K_1-K_2} \end{bmatrix} = e^{2K_0}\begin{bmatrix} e^{2K_1+2K_2}+e^{-2K_1} & e^{K_2}+e^{-K_2} \\ e^{K_2}+e^{-K_2} & e^{-2K_1}+e^{2K_1-2K_2} \end{bmatrix}$$

Now we write the desired equation:

$$e^{K_0'}\begin{bmatrix} e^{K_1'+K_2'} & e^{-K_1'} \\ e^{-K_1'} & e^{K_1'-K_2'} \end{bmatrix} = e^{2K_0}\begin{bmatrix} e^{2K_1+K_2}+e^{2K_0-2K_1} & e^{K_2}+e^{-K_2} \\ e^{K_2}+e^{2K_0-K_2} & e^{-2K_1}+e^{2K_1-2K_2} \end{bmatrix}$$

Equation of running coupling constant are:

$$\begin{aligned} e^{K_1'}e^{K_2'}e^{K_0'} &= e^{2K_0}\left[e^{2K_1+2K_2}+e^{-2K_1}\right] = e^{2K_0+K_2}\left[e^{2K_1+K_2}+e^{-2K_1-K_2}\right] = e^{2K_0+K_2}\cosh(2K_1+K_2) \\ e^{K_0'}e^{-K_1'} &= e^{2K_0}\left[e^{K_2}+e^{-K_2}\right] = e^{2K_0}2\cosh(K_2) \\ e^{K_0'}e^{K_1'}e^{-K_2'} &= e^{2K_0}\left[e^{-2K_1}+e^{2K_1-2K_2}\right] = e^{2K_0-K_2}\left[e^{-2K_1+K_2}+e^{2K_1-K_2}\right] = e^{2K_0-K_2}\cosh(2K_1-K_2) \end{aligned}$$

We can find equations 14.2.7a, 14.2.7b and 14.2.7c of Pathria book which is:

$$\begin{aligned} e^{K_0'} &= 2e^{2K_0}\left[\cosh(2K_1+K_2)\cosh(2K_1-K_2)\cosh^2 K_2\right]^{1/4} \\ e^{K_1'} &= \left[\cosh(2K_1+K_2)\cosh(2K_1-K_2)\cosh^2 K_2\right]^{1/4} \\ e^{K_2'} &= e^{K_2}\left[\frac{\cosh(2K_1+K_2)}{\cosh(2K_1-K_2)}\right]^{1/2} \end{aligned}$$

9- (**Pathria's book exercise 14-2**):

part(a)

We put below:

$$f(K_1, 0) = -\log(2\cosh K_1)$$

In:

$$f(K_1, 0) = -\frac{1}{2}\log\left[2\sqrt{\cosh 2K_1}\right] + \frac{1}{2}f\left(\log\left[\sqrt{\cosh 2K_1}\right], 0\right)$$

We have:

$$-\log(2\cosh K_1) = -\frac{1}{2}\log\left[2\sqrt{\cosh 2K_1}\right] - \frac{1}{2}\log\left[2\cosh\left[\log\left(\sqrt{\cosh 2K_1}\right)\right]\right]$$

We use expression $\cosh(\log x) = (x^2+1)/2x$ so we have:

$$-\log(2\cosh K_1) = -\frac{1}{2}\log\left[2\sqrt{\cosh 2K_1}\right] - \frac{1}{2}\log\left[\frac{\cosh 2K_1 + 1}{\sqrt{\cosh 2K_1}}\right]$$

We have:

$$\begin{aligned}
\frac{1}{2}\log\left[2\sqrt{\cosh 2K_1}\right] - \log(2\cosh K_1) &= -\frac{1}{2}\log\left[\frac{\cosh 2K_1 + 1}{\sqrt{\cosh 2K_1}}\right] \\
&= -\frac{1}{2}\log\left[\cosh 2K_1 + 1\right] + \frac{1}{2}\log\left[2\sqrt{\cosh 2K_1}\right] \\
&= -\frac{1}{2}\log\left[(2\cosh K_1)^2\right] + \frac{1}{2}\log\left[2\sqrt{\cosh 2K_1}\right]
\end{aligned}$$

The equation holds for this solution.

part(b)

We should verify that equations:

$$\begin{aligned}
e^{K_1'+K_2'+K_0'} &= e^{2K_0}\left[e^{2K_1+2K_2} + e^{-2K_1}\right] \\
e^{K_0'-K_1'} &= e^{2K_0}\left[e^{K_2} + e^{-K_2}\right] \\
e^{K_0'+K_1'-K_2'} &= e^{2K_0}\left[e^{-2K_1} + e^{2K_1-2K_2}\right]
\end{aligned} \tag{7}$$

with:

$$\begin{aligned}
f(K_1', K_2') &= -\log\left[e^{K_1'}\cosh K_2' + \left(e^{-2K_1'} + e^{2K_1'}\sinh^2 K_2'\right)^{1/2}\right] \\
&= -\log\left[\frac{e^{K_1'+K_2'} + e^{K_1'-K_2'}}{2} + \left(e^{-2K_1'} + \left(\frac{e^{K_1'+K_2'} - e^{K_1'-K_2'}}{2}\right)^2\right)^{1/2}\right]
\end{aligned}$$

Satisfy equation below with $K_0 = 0$, which is:

$$f(K_1, K_2) = \frac{1}{2}f(K_1', K_2')$$

Now we want to substitute equations 7 in above and find right hand side of above equation we have:

$$
\begin{aligned}
&-\frac{1}{2}\log\left[\frac{e^{K_1'+K_2'}+e^{K_1'-K_2'}}{2}+\left(e^{-2K_1'}+\left(\frac{e^{K_1'+K_2'}-e^{K_1'-K_2'}}{2}\right)^2\right)^{1/2}\right]\\
&=-\frac{1}{2}\log\left[\frac{e^{2K_1+2K_2}+e^{-2K_1}+e^{-2K_1}+e^{2K_1-2K_2}}{2}+\left(e^{-2K_1'}+\left(\frac{e^{2K_1+2K_2}+e^{-2K_1}-e^{-2K_1}-e^{2K_1-2K_2}}{2}\right)^2\right)^{1/2}\right]\\
&=-\frac{1}{2}\log\left[e^{-2K_1}+e^{2K_1}\cosh 2K_2+\left(e^{-2K_1'}+\left(e^{2K_1}\sinh 2K_2\right)^2\right)^{1/2}\right]\\
&=-\frac{1}{2}\log\left[e^{-2K_1}+e^{2K_1}\cosh 2K_2+\left(4\cosh^2 K_2+\left(e^{2K_1}\sinh 2K_2\right)^2\right)^{1/2}\right]
\end{aligned}
$$

Now for left hand side we have to organize below equation:

$$
\begin{aligned}
f(K_1,K_2)&=-\log\left[e^{K_1}\cosh K_2+\left(e^{-2K_1}+e^{2K_1}\sinh^2 K_2\right)^{1/2}\right]\\
&=-\frac{1}{2}\log\left[e^{K_1}\cosh K_2+\left(e^{-2K_1}+e^{2K_1}\sinh^2 K_2\right)^{1/2}\right]^2\\
&=-\frac{1}{2}\log\left[e^{2K_1}\cosh^2 K_2+e^{-2K_1}+e^{2K_1}\sinh^2 K_2+2e^{K_1}\cosh K_2\left(e^{-2K_1}+e^{2K_1}\sinh^2 K_2\right)^{1/2}\right]\\
&=-\frac{1}{2}\log\Big[e^{2K_1}(2\cosh^2 K_2-1)+e^{-2K_1}+(4e^{2K_1}\cosh^2 K_2 e^{-2K_1}\\
&\qquad\qquad\qquad+4e^{2K_1}\cosh^2 K_2 e^{2K_1}\sinh^2 K_2)^{1/2}\Big]\\
&=-\frac{1}{2}\log\left[e^{2K_1}\cosh 2K_2+e^{-2K_1}+\left(4\cosh^2 K_2+\left(e^{K_1}\sinh^2 K_2\right)^2\right)^{1/2}\right]
\end{aligned}
$$

Right hand side and left hand side are identical.

10- (**Pathria's book exercise 14-3**):

part a

We should show that equation:

$$f(K_1,\Lambda) = \frac{1}{2}\log\left[\frac{\Lambda+\sqrt{\Lambda^2-K_1^2}}{2\pi}\right] \tag{8}$$

Is a solution for equation:

$$f(K_1,\Lambda) = -\frac{1}{4}\log\left(\frac{\pi}{\Lambda}\right) + \frac{1}{2}f\left(\frac{K_1^2}{2\Lambda},\Lambda-\frac{K_1^2}{2\Lambda}\right)$$

We can find that:

$$f\left(\frac{K_1^2}{2\Lambda},\Lambda-\frac{K_1^2}{2\Lambda}\right) = \frac{1}{2}\log\left[\Lambda-\frac{K_1^2}{2\Lambda}+\sqrt{\left(\Lambda-\frac{K_1^2}{2\Lambda}\right)^2-\left(\frac{K_1^2}{2\Lambda}\right)^2}\right]-\frac{1}{2}\log 2\pi$$

So we substitute 8 in above we have:

$$\begin{aligned}
\frac{1}{2}\log\left[\frac{\Lambda+\sqrt{\Lambda^2-K_1^2}}{2\pi}\right] &= -\frac{1}{4}\log\left(\frac{\pi}{\Lambda}\right)+\frac{1}{4}\log\left[\Lambda-\frac{K_1^2}{2\Lambda}+\sqrt{\left(\Lambda-\frac{K_1^2}{2\Lambda}\right)^2-\left(\frac{K_1^2}{2\Lambda}\right)^2}\right]-\frac{1}{4}\log 2\pi \\
&= -\frac{1}{4}\log\left(\frac{\pi}{\Lambda}\right)+\frac{1}{4}\log\left[\Lambda-\frac{K_1^2}{2\Lambda}+\sqrt{\Lambda^2-K_1^2}\right]-\frac{1}{4}\log 2\pi \\
&= \frac{1}{4}\log\left[\frac{2\Lambda}{2\pi}\cdot\frac{\Lambda-\frac{K_1^2}{2\Lambda}+\sqrt{\Lambda^2-K_1^2}}{2\pi}\right] = \frac{1}{4}\log\left[\left(\frac{\Lambda+\sqrt{\Lambda^2-K_1^2}}{2\pi}\right)^2\right] \\
&= \frac{1}{2}\log\left[\frac{\Lambda+\sqrt{\Lambda^2-K_1^2}}{2\pi}\right]
\end{aligned}$$

Both side of equation is identical.

part(b)

For this part like previous part and previous problem we have to verify that below solution which is:

$$f(K_1,K_2,\Lambda) = \frac{1}{2}\log\left[\frac{\Lambda+\sqrt{\Lambda^2-K_1^2}}{2\pi}\right]-\frac{K_2^2}{4(\Lambda-K_1)} \tag{9}$$

Satisfy:

$$f(K_1,K_2,\Lambda) = -\frac{1}{2}K_0' + \frac{1}{2}f(K_1',K_2',\Lambda') \tag{10}$$

And we know that coupling constant run with below equations:

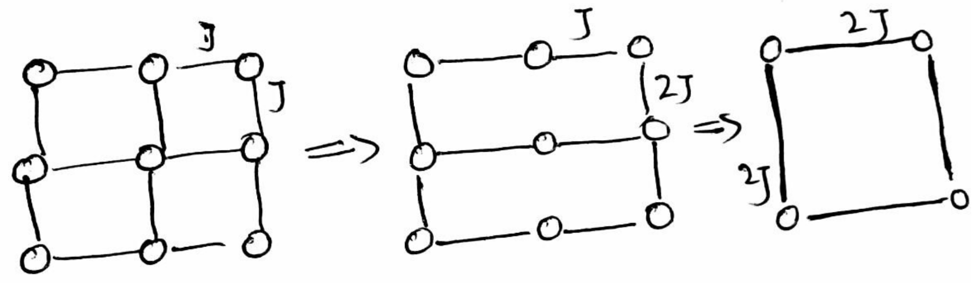
$$K_1' = \frac{K_1^2}{2\Lambda} \qquad K_2' = K_2(1 + \frac{K_1}{\Lambda}) \qquad \Lambda' = \Lambda - \frac{K_1^2}{2\Lambda}$$

Now we put solution 9 in equation 10 with $K_0 = 0$ to verify, we have:

$$\begin{aligned}
f(K_1, K_2, \Lambda) &= -\frac{1}{2}K_0' + \frac{1}{2}f\left(\frac{K_1^2}{2\Lambda}, K_2(1+\frac{K_1}{\Lambda}), \Lambda - \frac{K_1^2}{2\Lambda}\right) \\
&= -\frac{1}{4}\log\left(\frac{\pi}{\Lambda}\right) - \frac{K_2^2}{8\Lambda} + \frac{1}{4}\log\left[\frac{\Lambda - \frac{K_1^2}{2\Lambda} + \sqrt{\left(\Lambda - \frac{K_1^2}{2\Lambda}\right)^2 - \left(\frac{K_1^2}{2\Lambda}\right)^2}}{2\pi}\right] - \frac{\left(K_2(1+\frac{K_1}{\Lambda})\right)^2}{8(\Lambda - \frac{K_1^2}{2\Lambda} - \frac{K_1^2}{2\Lambda})} \\
&= -\frac{1}{4}\log\left(\frac{\pi}{\Lambda}\right) - \frac{K_2^2}{8\Lambda} + \frac{1}{4}\log\left[\frac{\Lambda - \frac{K_1^2}{2\Lambda} + \sqrt{\Lambda^2 - K_1^2}}{2\pi}\right] - \frac{K_2^2}{\Lambda^2}\frac{(\Lambda + K_1)(\Lambda + K_1)}{\frac{8}{\Lambda}(\Lambda - K_1)(\Lambda + K_1)} \\
&= -\frac{1}{4}\log\left(\frac{\pi}{\Lambda}\right) - \frac{K_2^2}{8\Lambda} + \frac{1}{4}\log\left[\frac{\Lambda - \frac{K_1^2}{2\Lambda} + \sqrt{\Lambda^2 - K_1^2}}{2\pi}\right] - \frac{K_2^2}{8\Lambda}\frac{\Lambda + K_1}{\Lambda - K_1} \\
&= \frac{1}{4}\log\left[\frac{2\Lambda}{2\pi} \cdot \frac{\Lambda - \frac{K_1^2}{2\Lambda} + \sqrt{\Lambda^2 - K_1^2}}{2\pi}\right] - \frac{K_2^2}{4(\Lambda - K_1)} \\
&= \frac{1}{2}\log\left[\frac{\Lambda + \sqrt{\Lambda^2 - K_1^2}}{2\pi}\right] - \frac{K_2^2}{4(\Lambda - K_1)}
\end{aligned}$$

Both side of above equation are identical.

11- (**Pathria's book exercise 14-5**):
part(a)
Migdal-Kadanoff method for decimation in two dimension Ising model is shown in below figure:



We use one dimensional decimation in above in x and y direction, so we explain decimation in one dimension. Hamiltonian for 1D Ising model is:

$$H = -J \sum_{i=1} S_i S_{i+1}$$

Partition function is:

$$\mathcal{Z} = \sum_{S_1} \sum_{S_2} \cdots \sum_{S_N} \exp\left(\beta J \sum_{i=1} S_i S_{i+1}\right) = \sum_{S_1} \sum_{S_2} \cdots \sum_{S_N} \prod_{i=1} e^{\beta J S_i S_{i+1}}$$

We define $K = \beta J$ and we can sum over for example S_2 exactly:

$$\sum_{S_2} e^{K S_2 (S_1 + S_3)} = 2 \cosh K(S_1 + S_3)$$

We can find how coupling constant changes before and after of RG transformation by this equation:

$$e^{K' S_1 S_3} e^{K_0} = {}^4 2 \cosh K(S_1 + S_3)$$

Now we have to take all configurations for spins which is:

$$S_1 = S_3 = 1 \qquad S_1 = S_3 = -1 \qquad S_1 = -S_3 = \pm 1$$

We have:

$$(1)\ S_1 = S_3 = 1 \Rightarrow e^{K'} e^{K_0} = 2 \cosh 2K$$

$$(2)\ S_1 = S_3 = -1 \Rightarrow e^{K'} e^{K_0} = 2 \cosh(-2K) = 2 \cosh 2K$$

$$(3)\ S_1 = -S_3 \Rightarrow e^{-K'} e^{K_0} = 2$$

[4] *We take $K_0 = 0$ in original Hamiltonian, but it can NOT be zero after RG transformation because we have two equation and we should have two parameter to determine. We can show that this part is involve to regular(none-singular) part of free energy and hence play no direct role in determining the critical behavior of the system.*

We can find:

$$\left(e^{K'}\right)^2 = \cosh 2K \Rightarrow e^{K'} = \sqrt{\cosh 2K}$$

$$e^{-K'}e^{K_0} = 2 \Rightarrow \frac{e^{K_0}}{\sqrt{\cosh 2K}} = 2 \Rightarrow e^{K_0} = 2\sqrt{\cosh 2K}$$

We can write above equation in the familiar form as follows:

$$e^{K'} = \sqrt{\cosh 2K} = {}^{5}\left(\frac{1+\tanh^2 K}{1-\tanh^2 K}\right)^{1/2} \Rightarrow \tanh^2 K = \frac{e^{2K'}-1}{e^{2K'}+1} = \tanh K'$$

So our coupling constants run with below equations:

$$\tanh K' = \tanh^2 K$$
$$e^{K_0} = 2\sqrt{\cosh 2K}$$

This is called Migdal-Kadanoff transformation if we use above in two dimension Ising model and take $K \to 2K$, so we can write our running equation for coupling constant ($K_x = K_y = K$), we have:

$$e^{K'} = \sqrt{\cosh 4K} \Rightarrow e^{-K'} = \frac{1}{\sqrt{\cosh 4K}} \Rightarrow e^{-2K'} = \frac{2}{e^{4K}+e^{-4K}} \Rightarrow e^{-2K'} = \frac{2e^{-4K}}{1+e^{-8K}}$$

We can find fixed point easily with help of Wolfram Mathematica, we have:

$$\text{Solve}\left[x^4 - 2x + 1 = 0, x\right]$$

$$\{x \to 1\}, \left\{x \to \frac{1}{3}\left(\sqrt[3]{3\sqrt{33}+17} - \frac{2}{\sqrt[3]{3\sqrt{33}+17}} - 1\right)\right\},$$

$$\left\{x \to -\frac{1}{6}\left(1 - i\sqrt{3}\right)\sqrt[3]{3\sqrt{33}+17} - \frac{1}{3} + \frac{1+i\sqrt{3}}{3\sqrt[3]{3\sqrt{33}+17}}\right\}$$

$$\left\{x \to -\frac{1}{6}\left(1 + i\sqrt{3}\right)\sqrt[3]{3\sqrt{33}+17} - \frac{1}{3} + \frac{1-i\sqrt{3}}{3\sqrt[3]{3\sqrt{33}+17}}\right\}$$

With numerical values:

$N[\%1]$

$\{x \to 1.\}, \{x \to 0.543689\}$ None trivial fixed point

$\{x \to -0.771845 + 1.11514i\}$ Imaginary fixed point

$\{x \to -0.771845 - 1.11514i\}$ Imaginary fixed point

part(b)

We know that above is R_ℓ^T with Goldenfeld notation, we can find eigenvalue of this transformation by take a derivative of above equation (by Mathematica):

$$f(x) = \frac{2x^2}{x^4+1} \Rightarrow f'[0.5437] = 1.67857$$

[5] `https://en.wikipedia.org/wiki/Hyperbolic_function`

12- (**Goldenfeld book exercise 9-1**):
part(a)
We know that exponent δ relate external magnetic field and magnetization by this relation:

$$H \sim M^{\delta}$$

From equation (9.33) and (9.35) for correlation function we know that:

$$2(d - y_h) = d - 2 + \eta \Rightarrow y_h = \frac{d + 2 - \eta}{2}$$

If we can relate y_h to δ we can find desired result. We begin from equation (9.26) we have:

$$\bar{m}_\ell = \frac{h_\ell \ell^{-d}}{h}$$

Now we choose ℓ as we please and set $h\ell^{y_h} = 1$ so $\ell = h^{-1/y_h}$, put this in above relation we have:

$$\bar{m}_\ell = \frac{h_\ell h^{d/y_h}}{h} = h^{d/y_h - 1} \Rightarrow h = \bar{m}_\ell^{y_h/(d-y_h)} \Rightarrow \delta = \frac{y_h}{d - y_h}$$

We put relation of y_h in above we have:

$$\delta = \frac{y_h}{d - y_h} = \frac{d + 2 - \eta}{d - 2 + \eta}$$

part(b)
We write free energy after a RG transformation like:

$$f_s(t, h) = \ell^{-d} f_s(t_\ell, h_\ell)$$

We use assumption 2 and use scaling behaviour in the critical region, we have:

$$f_s(t, h) = \ell^{-d} f_s\left(t\ell^{y_t}, h\ell^{y_h}\right)$$

We take $t\ell^{y_t} = 1$ so $\ell = |t|^{-1/y_t}$ we have:

$$f_s(t, h) = t^{d/y_t} f_s\left(1, h|t|^{-y_h/y_t}\right)$$

Finally we can write free energy:

$$f_s(t, h) = t^{d/y_t} F_f\left(\frac{h}{|t|^{y_h/y_t}}\right)$$

We can find from correlation function scaling equation that:

$$\nu = \frac{1}{y_t}$$

Finally:

$$f_s(t, h) = t^{d\nu} F_f\left(\frac{h}{|t|^{\nu y_h}}\right)$$

If we consider that exponent ν can be different around critical point we have to generalize above such as:

$$f_s(t,h) = |t|^{d\bar{\nu}} F_f^{\pm}\left(\frac{h}{|t|^{\bar{\nu}y_h}}\right)$$

with $\bar{\nu} = \nu$ $(t > 0)$ and $\bar{\nu} = \nu'$ $(t < 0)$. For fixed $h \neq 0$, $f_s(t,h)$ should be a smooth function of t, because the only singularity which we expect is at $t = h = 0$ so we can write free energy in none-singular form like:

$$f_s(t,h) = |t|^{d\bar{\nu}} F_f^{\pm}\left(\frac{h}{|t|^{\bar{\nu}y_h}}\right) = h^{-d/y_h}\left(\frac{h}{|t|^{\bar{\nu}y_h}}\right)^{-d/y_h} F_f^{\pm}\left(\frac{h}{|t|^{\bar{\nu}y_h}}\right) = h^{-d/y_h}\phi_f^{\pm}\left(\frac{h}{|t|^{\bar{\nu}y_h}}\right)$$

We can find that exponent $\nu = \nu'$ by demanding that function ϕ should be analytic when $t \longrightarrow 0$, we find two condition that when $t \longrightarrow 0$ function and it's derivative should be continuous.

13- (**Goldenfeld book exercise 9-2**):

part(a)

We know from Ginzburg criterion that for $d > 4$ mean field theory works. Mean field theory violated hyperscaling law ($2 - \alpha = \nu d$) because this law holds just for $d = 4$ NOT for $d > 4$.

part(b)

We have singular part of free energy like:

$$f_s(t, h, \tilde{K}_3) = t^{d/y_t} f_s\left(1, ht^{-y_h/y_t}, \tilde{K}_3 t^{-y_3/y_t}\right)$$

We know that $\tilde{K}_3$ is an irrelevant variable because $y_3 < 0$. We expect that after taking $t \to 0$ irrelevant variables become vanishingly small: $\tilde{K}_3 t^{-y_3/y_t} \to 0$ and we obtain:

$$f_s(t, h, \tilde{K}_3) = t^{d/y_t} f_s\left(1, ht^{-y_h/y_t}, 0\right)$$

But above can NOT happened if limit not be well-defined, it might be the case that:

$$\lim_{z \to 0} f_s(x, y, z) = z^{-\mu} f'(x, y), \qquad (\mu > 0)$$

This odd behaviour results in different hyperscaling law as below:

$$\begin{aligned}\lim_{t \to 0} t^{d/y_t} f_s\left(1, ht^{-y_h/y_t}, \tilde{K}_3 t^{-y_3/y_t}\right) &= \left(\tilde{K}_3 t^{-y_3/y_t}\right)^{-\mu} t^{d/y_t} f_s\left(1, ht^{-y_h/y_t}\right) \\ &\sim t^{(\mu y_3 + d)/y_t} f_s\left(1, ht^{-y_h/y_t}\right)\end{aligned}$$

Above leads to a violation of the Josephson hyperscaling law. Hyperscaling law in above form is $2 - \alpha = (\mu y_3 + d)\nu$.

part(c)

If we take $\mu = 1$ and $d > 4$ we have:

$$2 - \alpha = \frac{\mu y_3 + d}{y_t} = \frac{\mu y_3}{y_t} + \frac{d}{y_t} = -\frac{d}{2} + 2 + \frac{d}{y_t} \Rightarrow \alpha = \frac{d}{2} - \frac{d}{y_t}$$

If we have $\alpha = 0$ and $y_t = 2$ ($\nu = 1/2$) we can say that *"Landau theory does satisfy hyperscaling after all"*.

14- (**Goldenfeld book exercise 9-3**):
part(a)
Hamiltonian is:

$$\mathcal{H} = K\sum_{i=1}^{N} S_i S_{i+1} + h\sum_{i=1}^{N} S_i + NK_0$$

We can write partition function as:

$$\begin{aligned}
\mathcal{Z}_N\{\mathcal{H}\} &= \sum_{S_1}\sum_{S_2}\cdots\sum_{S_N} exp\left(K\sum_{i=1}^{N} S_i S_{i+1} + h\sum_{i=1}^{N} S_i + NK_0\right)\\
&= \sum_{S_1}\sum_{S_2}\cdots\sum_{S_N}\left[e^{\frac{h}{2}(S_1+S_2)+KS_1S_2+K_0}\right]\left[e^{\frac{h}{2}(S_2+S_3)+KS_2S_3+K_0}\right]\ldots\left[e^{\frac{h}{2}(S_N+S_1)+KS_NS_1+K_0}\right]\\
&= \sum_{S_1}\sum_{S_2}\cdots\sum_{S_N}\langle s_1|\boldsymbol{T}|s_2\rangle\langle s_2|\boldsymbol{T}|s_3\rangle\ldots\langle s_N|\boldsymbol{T}|s_1\rangle
\end{aligned}$$

Where:

$$\langle s_1|\boldsymbol{T}|s_2\rangle = e^{\frac{h}{2}(S_1+S_2)+KS_1S_2+K_0}$$

We want to sum over all even spins so we can perform this for for example S_2 we have:

$$\langle s_1|\boldsymbol{T}|s_2\rangle\langle s_2|\boldsymbol{T}|s_3\rangle = e^{\frac{h}{2}(S_1+S_2)+KS_1S_2+K_0}e^{\frac{h}{2}(S_2+S_3)+KS_2S_3+K_0} = e^{\frac{h}{2}(S_1+S_3)+hS_2+KS_2(S_1+S_3)+2K_0}$$

We know that $S_2 = \pm 1$ so we can write:

$$\begin{aligned}
\langle s_1|\boldsymbol{T}|s_3\rangle = \sum_{S_2}\langle s_1|\boldsymbol{T}|s_2\rangle\langle s_2|\boldsymbol{T}|s_3\rangle &= e^{\frac{h}{2}(S_1+S_3)+h+K(S_1+S_3)+2K_0} + e^{\frac{h}{2}(S_1+S_3)-h-K(S_1+S_3)+2K_0}\\
&= e^{2K_0}e^{\frac{h}{2}(S_1+S_3)}2\cosh\left(K(S_1+S_3)+h\right)
\end{aligned}$$

Above is new effective transfer matrix with twice the lattice spacing as the original system. Now we want to obtain how coupling constant would run with this RG transformation? We use Kadanoff assumption which is shape of Hamiltonian doesn't change with RG transformation we consider spins with new lattice constant as S' and write:

$$e^{\frac{h'}{2}(S_1'+S_2')+K'S_1'S_2'+K_0'} = e^{2K_0}e^{\frac{h}{2}(S_1'+S_2')}2\cosh\left(K(S_1'+S_2')+h\right)$$

The various choice being $S_1' = S_2' = +1$, $S_1' = S_2' = -1$ and $S_1' = -S_2' = \pm 1$ we obtain from above:

$$\begin{aligned}
e^{h'+K'+K_0'} &= e^{2K_0+h}\cdot 2\cosh(2K+h)\\
e^{-h'+K'+K_0'} &= e^{2K_0-h}\cdot 2\cosh(2K-h)\\
e^{-K'+K_0'} &= e^{2K_0}\cdot 2\cosh(h)
\end{aligned}$$

We have three equation and three unknown parameter so we can solve above equations and we get:

$$\begin{aligned}
e^{K_0'} &= 2e^{2K_0}\left(\cosh(2K+h)\cosh(2K-h)\cosh^2 h\right)^{1/4}\\
e^{K'} &= \left(\frac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2 h}\right)^{1/4}\\
e^{h'} &= e^{h}\left(\frac{\cosh(2K+h)}{\cosh(2K-h)}\right)^{1/2}
\end{aligned}$$

We demand that our new Hamiltonian have original symmetries and I don't understand what is the meaning of this part of question. Mathematically we need K_0 because we have three equation and we need three parameter to obtain. If we take $K_0 = 0$ a $K_0' \neq 0$ is essential for a proper representation of the transformed system.

part(b)

From second equation with $h = 0$ we have:

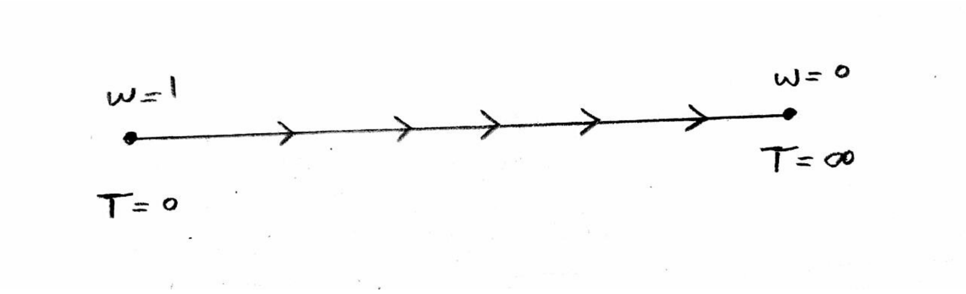
$$e^{K'} = \left(\cosh^2(2K)\right)^{1/4} \Rightarrow e^{2K'} = \sqrt{\cosh^2(2K)} \Rightarrow e^{4K'} = \cosh^2(2K)$$

$$e^{4K'} = \cosh^2(2K) \Rightarrow e^{-2K'} = \frac{1}{\cosh 2K} \Rightarrow e^{-2K'} = \frac{2}{e^{2K} + e^{-2K}} \Rightarrow e^{-2K'} = \frac{2e^{-2K}}{1 + e^{-4K}}$$

We can define $w = e^{-2K}$ and for fining fixed point we have $w' = w$, so:

$$w = \frac{2w}{1 + w^2} \Rightarrow w + w^3 = 2w \Rightarrow w(w^2 - 1) = 0 \Rightarrow w = 0, w = \pm 1$$

We have two physical fixed point for above equation which is $w = 0$ correspond to system with high temperature (paramagnetic phase) and $w = 1$ correspond to zero temperature which is ferromagnetic phase or ordered phase. We can consider a physical reasoning for what is RG flow direction is, if we have a RG transformation on system surely we make fluctuation bigger that is correspond to high temperature so if we have a RG transformation definitely temperature take bigger value with respect to original one. RG flow is shown in below figure:



part(c)

We know that recursion relation for coupling constant is:

$$e^{4K'} = \cosh^2(2K)$$

Now we can find transformation that take previous temperature and give next temperature which is:

$$K' = \frac{1}{4}\log\left(\cosh^2(2K)\right) = \frac{1}{2}\log\left(\cosh(2K)\right)$$

For linearisation we take a derivative with respect to K we have:

$$\Lambda_\ell^t = \frac{\partial K'}{\partial K}\bigg|_{K=K^*=\infty} = \frac{\partial}{\partial K}\left(\frac{1}{2}\log\left(\cosh(2K)\right)\right)\bigg|_{K=K^*=\infty} = \tanh 2K\bigg|_{K=K^*=\infty} = \lim_{K\to\infty} \tanh 2K = 1$$

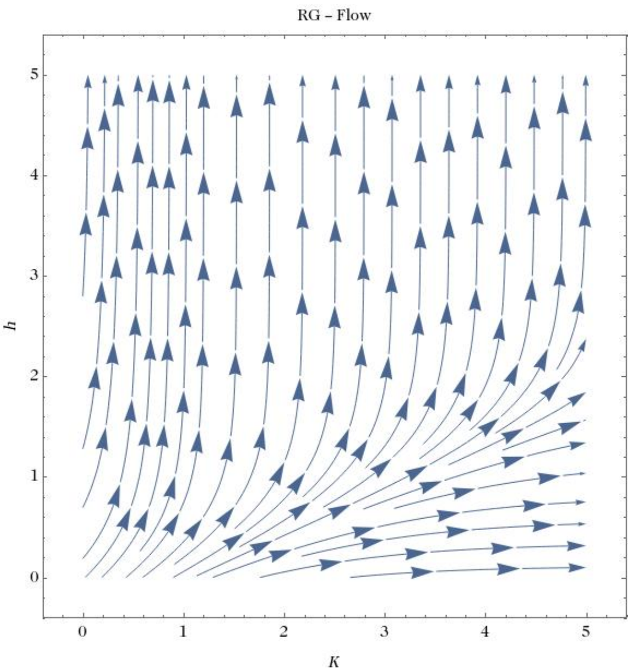
We know that $\Lambda_\ell^t = \ell^{y_t}$ and we can find $y_t = 0$.

part(c)

Equations for coupling constants are:

$$e^{K'} = \left(\frac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2 h}\right)^{1/4}$$
$$e^{h'} = e^h \left(\frac{\cosh(2K+h)}{\cosh(2K-h)}\right)^{1/2}$$

We can find all fixed points from above equations, we know that 1D Ising model doesn't have phase transition so we can say that we have just trivial fixed points. We can use Mathematica for drawing RG-flow.



For linearisation we consider $K = 0$ in:

$$e^{h'} = e^h \left(\frac{\cosh h}{\cosh h}\right)^{1/2} \Rightarrow h' = h \Rightarrow \Lambda_\ell^h = \frac{\partial h'}{\partial h}\bigg|_{h=h^*=\infty} = 1$$

We know that $\Lambda_\ell^h = \ell^{y_h}$ and we can find $y_h = 0$.

part(e)

In above we calculate new coupling constants after RG transformation, equations are:

$$e^{K_0'} = 2e^{2K_0} \left(\cosh(2K+h)\cosh(2K-h)\cosh^2 h\right)^{1/4}$$
$$e^{K'} = \left(\frac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2 h}\right)^{1/4}$$
$$e^{h'} = e^h \left(\frac{\cosh(2K+h)}{\cosh(2K-h)}\right)^{1/2}$$

By linearisation in we find that both $y_h = 0$ and $y_t = 0$ are zero, this means that in one dimensional Ising model system doesn't have any phase transition and we don't expect to have scaling behaviour.