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Answer to Exercise Set #4 of Critical Phenomena

1- (**Goldenfeld book exercise 3-1**):

part(a)

We want to nationalize this matrix:

$$\boldsymbol{T} = \begin{bmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{bmatrix}$$

We should find $\boldsymbol{S}$ (rotation matrix) to which diagonals above with $\boldsymbol{S}^{-1}\boldsymbol{T}\boldsymbol{S}$, we know rotation matrix is of a form:

$$\boldsymbol{S} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

We can find easily inverse above and finally:

$$\boldsymbol{T'} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$\boldsymbol{T'}$ should be diagonal and off diagonal terms should be zero, finally we find:

$$e^{-K}\cos 2\phi = \frac{1}{2}\,\sin 2\phi\; e^{K}\; 2\sinh \boldsymbol{h}$$

$$\cot 2\phi = e^{2K}\sinh \boldsymbol{h}$$

part(b)

We know that:

$$\langle S_i \rangle = \frac{1}{\mathcal{Z}} \sum_{S_1} \cdots \sum_{S_N} e^{-\beta\mathcal{H}_\Omega} S_i$$

$$= \frac{1}{\mathcal{Z}} \sum_{S_1} \cdots \sum_{S_N} \left[\boldsymbol{T}_{S_1S_2}\boldsymbol{T}_{S_2S_3} \ldots \boldsymbol{T}_{S_{i-1}S_i}\; S_i\; \boldsymbol{T}_{S_{i+1}S_i} \ldots \right]$$

Sum in $\boldsymbol{i}$ is a matrix A like:

$$A_{ab} = \sum_{S_i} \boldsymbol{T}_{aS_i}\; \boldsymbol{T}_{S_ib}\; S_i$$

We can write above expression as following:

$$\boldsymbol{A} = \boldsymbol{T} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{T}$$

Finally we have:

$$\langle S_i \rangle = \frac{1}{\mathcal{Z}}\, \mathrm{Tr}\,(\,\sigma_z \boldsymbol{T}^N\,)$$

We can write above in diagonal space of matrix $\boldsymbol{T}$, we have:

$$\langle S_i \rangle = \frac{\mathrm{Tr}\left[\boldsymbol{S}^{-1}\ \sigma_z\ \boldsymbol{S}\ (\boldsymbol{T'}^N)\right]}{\mathrm{Tr}\ (\boldsymbol{T'})^N} = \frac{\mathrm{Tr}\left[\boldsymbol{S}^{-1}\ \sigma_z\ \boldsymbol{S}\ (\boldsymbol{T'}^N)\right]}{\mathrm{Tr}\ (\boldsymbol{T})^N}$$

In last equation we know that $\mathrm{Tr}\ (\boldsymbol{T})^N$ is equal to $\mathrm{Tr}\ (\boldsymbol{T'})^N$.

Now we should calculate $\langle S_i \rangle$ with part a assumptions we have:

$$\boldsymbol{S}^{-1}\ \sigma_z\ \boldsymbol{S} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} c^2 - s^2 & -2cs \\ -2cs & -c^2 + s^2 \end{bmatrix} = \begin{bmatrix} \cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{bmatrix}$$

We have:

$$\begin{aligned}
\langle S_i \rangle &= \frac{1}{\mathrm{Tr}\ (\boldsymbol{T'})^N}\ \mathrm{Tr}\ \left(\begin{bmatrix} \cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{bmatrix} \right) \\
&= \frac{1}{\mathrm{Tr}\ (\boldsymbol{T'})^N}\ \mathrm{Tr}\ \left(\begin{bmatrix} \lambda_1^N \cos 2\phi & -\lambda_2^N \sin 2\phi \\ -\lambda_1^N \sin 2\phi & -\lambda_2^N \cos 2\phi \end{bmatrix} \right) \\
&= \cos 2\phi\ \frac{\lambda_1^N - \lambda_2^N}{\lambda_1^N + \lambda_2^N} = \cos 2\phi\ \left(\frac{\lambda_1^N}{\lambda_1^N + \lambda_2^N} - \frac{\lambda_2^N}{\lambda_1^N + \lambda_2^N} \right)
\end{aligned}$$

When we take the limit $N \to \infty$ and we know $\lambda_1 > \lambda_2$ we have $\langle S_i \rangle \to \cos 2\phi$.

We now that correlation function is:

$$G(i,j) = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle$$

So we should calculate $\langle S_i S_j \rangle$ we have:

$$\begin{aligned}
\langle S_i S_j \rangle &= \frac{\mathrm{Tr}\left[(\boldsymbol{T'})^{i-1} \boldsymbol{S}^{-1}\ \sigma_z\ \boldsymbol{S}\ (\boldsymbol{T'})^{i+j-i}\ \boldsymbol{S}^{-1}\ \sigma_z\ \boldsymbol{S}\ (\boldsymbol{T'})^{N-j-i+1}\right]}{\mathrm{Tr}\ (\boldsymbol{T'})^N} \\
&= \frac{\mathrm{Tr}\left[\boldsymbol{S}^{-1}\ \sigma_z\ \boldsymbol{S}\ (\boldsymbol{T'})^{j}\ \boldsymbol{S}^{-1}\ \sigma_z\ \boldsymbol{S}\ (\boldsymbol{T'})^{N-j}\right]}{\mathrm{Tr}\ (\boldsymbol{T'})^N}
\end{aligned}$$

We have:

$$\begin{aligned}
\langle S_i S_j \rangle &= \frac{1}{\mathrm{Tr}\ (\boldsymbol{T'})^N}\ \mathrm{Tr}\ \left(\begin{bmatrix} \cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \lambda_1^j & 0 \\ 0 & \lambda_2^j \end{bmatrix} \begin{bmatrix} \cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \lambda_1^{N-j} & 0 \\ 0 & \lambda_2^{N-j} \end{bmatrix} \right) \\
&= \frac{1}{\mathrm{Tr}\ (\boldsymbol{T'})^N}\ \mathrm{Tr}\ \left(\begin{bmatrix} \lambda_1^j \cos 2\phi & -\lambda_2^j \sin 2\phi \\ -\lambda_1^j \sin 2\phi & -\lambda_2^j \cos 2\phi \end{bmatrix} \begin{bmatrix} \lambda_1^{N-j} \cos 2\phi & -\lambda_2^{N-j} \sin 2\phi \\ -\lambda_1^{N-j} \sin 2\phi & -\lambda_2^{N-j} \cos 2\phi \end{bmatrix} \right) \\
&= \frac{1}{\lambda_1^N + \lambda_2^N} \left(\lambda_1^N \cos^2 2\phi + (\frac{\lambda_2}{\lambda_1})^j \lambda_1^N \sin^2 2\phi + (\frac{\lambda_1}{\lambda_2})^j \lambda_2^N \sin^2 2\phi + \lambda_2^N \cos^2 2\phi \right) \\
&= \frac{\lambda_1^N}{\lambda_1^N + \lambda_2^N} \left(\left[\cos^2 2\phi + (\frac{\lambda_2}{\lambda_1})^j \sin^2 2\phi \right] + (\frac{\lambda_2}{\lambda_1})^N \left[(\frac{\lambda_1}{\lambda_2})^j \sin^2 2\phi + \cos^2 2\phi \right] \right)
\end{aligned}$$

When we take the limit $N \to \infty$ and we know $\lambda_1 > \lambda_2$ we have:

$$\begin{aligned}
\langle S_i S_j \rangle &\longrightarrow \cos^2 2\phi + (\frac{\lambda_2}{\lambda_1})^j \sin^2 2\phi \\
\langle S_i \rangle \langle S_j \rangle &\longrightarrow \cos^2 2\phi
\end{aligned}$$

Finally:

$$G(i,j) = (\frac{\lambda_2}{\lambda_1})^j \sin^2 2\phi$$

part(c)

We have magnetization and we want to derive an expression for susceptibility, we have:

$$\begin{aligned}
\chi_T &= \frac{\partial M}{\partial H}\Big|_T = \beta\frac{\partial M}{\partial h}\Big|_T = \beta\frac{\partial}{\partial h}\left(\frac{\sinh h}{\sqrt{\sinh^2 h + w^2}}\right) \\
&= \beta\,\frac{\cosh h(\sinh^2 h + w^2)^{1/2} - \sinh h\cosh h(\sinh^2 h + w^2)^{-1/2}}{\sinh^2 h + w^2} \\
&= \beta\left[\frac{\cosh h}{(sinh^2 h + w^2)^{1/2}} - \frac{\sinh^2 h\cosh h}{(sinh^2 h + w^2)^{3/2}}\right] \\
&= \beta\left[\frac{\cosh h\sinh^2 h + w^2\cosh h - \sinh^2 h\cosh h}{(\sinh^2 h + w^2)^{3/2}}\right] \\
&= \beta\left[\frac{w^2\cosh h}{(\sinh^2 h + w^2)^{3/2}}\right]
\end{aligned}$$

Now we want to calculate susceptibility with summation (equation 3.159) of correlation function we have:

$$\begin{aligned}
\chi_T &= \frac{1}{k_B T}\sum_{j=-\infty}^{\infty} G(i,i+j) = \frac{1}{k_B T}\sum_{j=-\infty}^{\infty}(\frac{\lambda_2}{\lambda_1})^j \sin^2 2\phi \\
&= \frac{\sin^2 2\phi}{k_B T}\sum_{j=-\infty}^{\infty}(\frac{\lambda_2}{\lambda_1})^j = \frac{\sin^2 2\phi}{k_B T}\left[1 + 2\sum_{j=1}^{\infty}(\frac{\lambda_2}{\lambda_1})^j\right] \\
&= \frac{\sin^2 2\phi}{k_B T}\left[1 + \frac{2\lambda_2}{\lambda_1 - \lambda_2}\right] = \frac{\sin^2 2\phi}{k_B T}\left[\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\right] \\
&= \frac{\sin^2 2\phi}{k_B T}\left[\frac{\cosh h}{\sqrt{\sinh^2 h + e^{-4K}}}\right]
\end{aligned} \tag{1}$$

We know that:

$$\sin^2 2\phi = \frac{1}{1 + \cot^2 2\phi} = \frac{1}{1 + e^{4K}\sinh^2 h} = \frac{1}{e^{4K}\left(\sinh^2 h + e^{-4K}\right)}$$

Finally we have:

$$\begin{aligned}
\chi_T &= \frac{1}{k_B T}\frac{1}{e^{4K}\left(\sinh^2 h + e^{-4K}\right)}\left[\frac{\cosh h}{\sqrt{\sinh^2 h + e^{-4K}}}\right] \\
&= \frac{1}{k_B T}\frac{e^{-4K}\cosh h}{\left(\sinh^2 h + e^{-4K}\right)^{3/2}}
\end{aligned}$$

We take $w^2 = e^{-4K}$ so we have:

$$\chi_T = \frac{1}{k_B T}\frac{w^2\cosh h}{\left(\sinh^2 h + w^2\right)^{3/2}}$$

Above is equal to equation 1 results.

part(d)

We know that partition function with free boundary condition is:

$$\mathcal{Z} = \sum_{S_1} \cdots \sum_{S_N} e^{h(s_1+\cdots+S_N)} e^{K(S_1S_2+\cdots+S_{N-1}S_N)}$$

Above summation is NOT simply something in power of N we should use a technique to do this. We can write partition function as below:

$$\begin{aligned}
\mathcal{Z}_N(h,K) &= \sum_{S_1} \cdots \sum_{S_N} e^{h(s_1+\cdots+S_N)} e^{K(S_1S_2+\cdots+S_{N-1}S_N)} \\
&= \sum_{S_1} \cdots \sum_{S_N} e^{\frac{h}{2}(S_1+S_2)+KS_1S_2} \ldots e^{\frac{h}{2}(S_{N-1}+S_N)+KS_{N-1}S_N} e^{\frac{h}{2}(S_1+S_N)}
\end{aligned}$$

We put the last exponential to correct our summation. We can write above in this way:

$$\mathcal{Z}_N(h,K) = \text{Tr}\left(\begin{bmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{bmatrix}^{N-1} \begin{bmatrix} e^h & 1 \\ 1 & e^{-h} \end{bmatrix}\right)$$

We use part a transformation, we have:

$$\begin{aligned}
\mathcal{Z}_N(h,K) &= \text{Tr}\left(\begin{bmatrix} \lambda_1^{N-1} & 0 \\ 0 & \lambda_2^{N-1} \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} e^h & 1 \\ 1 & e^{-h} \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\right) \\
&= \text{Tr}\left(\begin{bmatrix} \lambda_1^{N-1} & 0 \\ 0 & \lambda_2^{N-1} \end{bmatrix} \begin{bmatrix} c^2e^h + 2sc + s^2e^{-h} & -sce^h + c^2 - s^2 + sce^{-h} \\ -sce^h + c^2 - s^2 + sce^{-h} & -s^2e^h - 2sc + c^2e^{-h} \end{bmatrix}\right) \\
&= \text{Tr}\left(\begin{bmatrix} \lambda_1^{N-1} & 0 \\ 0 & \lambda_2^{N-1} \end{bmatrix} \begin{bmatrix} \sin 2\phi + e^h\cos^2\phi + e^{-h}\sin^2\phi & \cos 2\phi - \sin 2\phi \sinh h \\ \cos 2\phi - \sin 2\phi \sinh h & -\sin 2\phi + e^h\cos^2\phi - e^{-h}\sin^2\phi \end{bmatrix}\right) \\
&= \text{Tr}\left(\begin{bmatrix} \lambda_1^{N-1}\left[\sin 2\phi + e^h\cos^2\phi + e^{-h}\sin^2\phi\right] & \lambda_1^{N-1}\left[\cos 2\phi - \sin 2\phi \sinh h\right] \\ \lambda_2^{N-1}\left[\cos 2\phi - \sin 2\phi \sinh h\right] & \lambda_2^{N-1}\left[-\sin 2\phi + e^h\cos^2\phi - e^{-h}\sin^2\phi\right] \end{bmatrix}\right) \\
&= \lambda_1^{N-1}\left[\sin 2\phi + e^h\cos^2\phi + e^{-h}\sin^2\phi\right] + \lambda_2^{N-1}\left[-\sin 2\phi + e^h\cos^2\phi - e^{-h}\sin^2\phi\right] \\
&= \lambda_1^{N-1}\left[\sin 2\phi + e^h\cos^2\phi + e^{-h}\sin^2\phi + \left(\frac{\lambda_2}{\lambda_1}\right)^{N-1}\left(-\sin 2\phi + e^h\cos^2\phi - e^{-h}\sin^2\phi\right)\right]
\end{aligned}$$

We can find free energy as follows:

$$\begin{aligned}
\mathcal{F}_N &= -k_BT\log\mathcal{Z}_N(h,K)\\
&= -k_BT(N-1)\log\lambda_1 - k_BT\log\Big[\sin 2\phi + e^h\cos^2\phi + e^{-h}\sin^2\phi\\
&\qquad + \left(\frac{\lambda_2}{\lambda_1}\right)^{N-1}\Big(-\sin 2\phi + e^h\cos^2\phi - e^{-h}\sin^2\phi\Big)\Big]\\
&= -k_BT(N-1)\log\lambda_1 - k_BT\log\Big[\sin 2\phi + e^h\cos^2\phi + e^{-h}\sin^2\phi\Big]\\
&\qquad - k_BT\log\left[1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{N-1}\left(\frac{-\sin 2\phi + e^h\cos^2\phi - e^{-h}\sin^2\phi}{\sin 2\phi + e^h\cos^2\phi + e^{-h}\sin^2\phi}\right)\right]
\end{aligned}$$

We know that free energy is given by:

$$\mathcal{F}_N = Nf_b(h,K) + f_s(h,K) + F_{fs}(N,h,K)$$

where f_b is the bulk free energy, f_s is the surface free energy due to the boundaries, and F_{fs} is an intrinsically finite size contribution. With above definition we have:

$$\begin{aligned}
f_b(h,K) &= -k_BT\log\lambda_1\\
f_s(h,K) &= -k_BT\log\Big[\sin 2\phi + e^h\cos^2\phi + e^{-h}\sin^2\phi\Big] + k_BT\log\lambda_1\\
F_{fs}(N,h,K) &= -k_BT\log\left[1 + \left(\frac{\lambda_1}{\lambda_2}\right)^{N-1}\left(\frac{-\sin 2\phi + e^h\cos^2\phi - e^{-h}\sin^2\phi}{\sin 2\phi + e^h\cos^2\phi + e^{-h}\sin^2\phi}\right)\right]
\end{aligned}$$

From part (a) we know that $\phi = \cot^{-1}(e^{2K}\sinh h)/2$, put this in finite size free energy we can find function $C(h,K)$.

part(e)

We know that surface free energy is:

$$f_s(h,K) = -k_BT\log\left[\frac{\sin 2\phi + e^h\cos^2\phi + e^{-h}\sin^2\phi}{\lambda_1}\right]$$

If we take $h = 0$ we have $\phi = \pi/4$, for above we have:

$$f_s(h,K) = -k_BT\log\left[\frac{1}{\cosh K}\right] \tag{2}$$

For next part we should find:

$$\begin{aligned}
f_s(h,K) &= \lim_{N\to\infty}[\mathcal{F}^{\text{free}} - \mathcal{F}^{\text{periodic}}]\\
&= -k_BT\lim_{N\to\infty}\left[\log\left(2(2\cosh K)^{N-1}\right) - \log\left((2\cosh K)^N + (2\sinh K)^N\right)\right]\\
&= -k_BT\lim_{N\to\infty}\left[\log 2 + (N-1)\log(2\cosh K) - N\log(2\cosh K) - \log\left((1 + (2\tanh K)^N\right)\right]
\end{aligned}$$

We know that $\tanh x < 1$ so in limit this term approach to zero, so we have:

$$f_s(h,K) = -k_BT\left[\log 2 - \log(2\cosh K)\right] = -k_BT\log\left[\frac{1}{\cosh K}\right]$$

Which is the same with equation 2 result.

2- (**Goldenfeld book exercise 3-3**):

part(a)

We should calculate this integral:

$$\frac{1}{(2\pi)^{N/2}}\int_{-\infty}^{\infty} d^N X \exp\Big(-\frac{1}{2}\, X^\dagger\cdot A\cdot X + X^\dagger\cdot B\Big)$$

If D is diagonalized matrix of A with transformation $A = Q^{-1}DQ$, with change of variable in the form $Y = QX$ we can obtain:

$$\begin{aligned}\int_{-\infty}^{\infty} d^N X \exp\Big(-\frac{1}{2}\, X^\dagger\cdot A\cdot X + X^\dagger\cdot B\Big) &= \int_{-\infty}^{\infty} d^N Y\Big|\frac{d^N X}{d^N Y}\Big| \exp\Big(-\frac{1}{2}\, Y^\dagger\cdot D\cdot Y + Y^\dagger Q\cdot B\Big)\\ &= \int_{-\infty}^{\infty}\prod_{i=1}^{N} dY_i\Big|\frac{dX_i}{dY_i}\Big| \exp\Big(-\frac{1}{2}Y_i^2 D_{ii} + (Y_i Q_{ij})B_j\Big)\\ &= \prod_{i=1}^{N}\int_{-\infty}^{\infty} dY_i{}^1\Big|\frac{dX_i}{dY_i}\Big| \exp\Big(-\frac{1}{2}Y_i^2 D_{ii} + (Y_i Q_{ij})B_j\Big)\end{aligned}$$

Finally:

$$\begin{aligned}\frac{1}{(2\pi)^{N/2}}\prod_{i=1}^{N}\int_{-\infty}^{\infty} dY_i \exp\Big(-\frac{1}{2}Y_i^2 D_{ii} + (Y_i Q_{ij})B_j\Big)^2 &= \frac{1}{(2\pi)^{N/2}}\prod_{i=1}^{N}\Big[\frac{2\pi}{D_{ii}}\Big]^{1/2}\exp(\frac{1}{2}\, B_j\, Q_{ij}\, D_{ii}^{-1}\, Q_{ij}\, B_j)\\ &= \frac{1}{(2\pi)^{N/2}}\Big[(2\pi)^N\prod_{i=1}^{N}\frac{1}{D_{ii}}\Big]^{1/2}\exp(\frac{1}{2}\, B\cdot A^{-1}\cdot B)\\ &= \frac{1}{\sqrt{\text{Det}\, A}} e^{\frac{1}{2}\, B\cdot A^{-1}\cdot B}\end{aligned}$$

part(b)

We know that Hamiltonian is:

$$\begin{aligned}\mathcal{H}_\Omega = -\beta H_\Omega &= \frac{1}{2}\sum_{i\neq j} J_{ij}S_iS_j + \sum_i H_iS_i\\ &= \frac{1}{2}\sum_{ij} J_{ij}S_iS_j - \frac{1}{2}\sum_i J_{ii}S_i^2 + \sum_i H_iS_i\end{aligned}$$

In above identity if we choose $X_i = \beta S_i$ and $A = J/\beta$ we have:

$$\begin{aligned}\mathcal{H}_\Omega &= \frac{1}{2}(\beta S_i)(\frac{J_{ij}}{\beta})(\beta S_j) - \frac{1}{2}(\beta S_i)(\frac{J_{ii}}{\beta})(\beta S_i) + H_i(\beta S_i)\\ &= \frac{1}{2}(\beta S_i)\,(\frac{J_{ij}}{\beta})\,(\beta S_j) - \frac{\beta}{2}\,\text{Tr}(J) + H_i(\beta S_i)\end{aligned}$$

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[1] *We have* $Det(1/Q^T) = Det(1/Q) = Det(Q^{-1}) = Det(Q^T) = +1$.

[2] *This integral is helpful:*

$$\int_{-\infty}^{\infty} e^{-uy^2} e^{vy} dy = \sqrt{\frac{\pi}{\alpha}}\, e^{v^2/4u}$$

We know that $S_i^2 = 1$ so we can choose $\frac{\beta}{2}$ Tr(J) as zero energy, and apply part (a) identity.

part(c)

We use part (b) Hamiltonian and use part (a) identity to calculate partition function, we know that identity is:

$$\int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(\frac{d\psi_i}{\sqrt{2\pi}}\right)\exp\left(-\frac{1}{2}\psi_i A_{ij}\psi_j + \psi_i B_i\right) = \frac{1}{\sqrt{\text{Det } A}}e^{\frac{1}{2}B_i(A^{-1})_{ij}B_j}$$

We take $A_{ij}^{-1} = J/\beta$ and $B_i = \beta S_i$. We have to change above in order to use it in relation of partition function, we have [3] :

$$\sum_{S_1}\cdots\sum_{S_N} e^{\frac{1}{2}(\beta S_i)\frac{J_{ij}}{\beta}(\beta S_j)}e^{H_i(\beta S_i)} = \sqrt{\text{Det } \beta J^{-1}}\sum_{S_1}\cdots\sum_{S_N} e^{H_i(\beta S_i)}\int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(\frac{d\psi_i}{\sqrt{2\pi}}\right)\exp\left(-\frac{\beta}{2}\psi_i(J^{-1})_{ij}\psi_j + \psi_i(\beta S_i)\right)$$

We can write For partition function with part (b) Hamiltonian we have:

$$\begin{aligned}
\mathcal{Z}_\Omega &= \sum_{S_1}\cdots\sum_{S_N} e^{\frac{1}{2}(\beta S_i)\,(\frac{J_{ij}}{\beta})\,(\beta S_j)}e^{-\frac{\beta}{2}\text{ Tr}(J)}e^{H_i(\beta S_i)} \\
&= e^{-\frac{\beta}{2}\text{ Tr}(J)}\sum_{S_1}\cdots\sum_{S_N} e^{\frac{1}{2}(\beta S_i)\,(\frac{J_{ij}}{\beta})\,(\beta S_j)}e^{H_i(\beta S_i)} \\
&= e^{-\frac{\beta}{2}\text{ Tr}(J)}\sqrt{\text{Det } \beta J^{-1}}\sum_{S_1}\cdots\sum_{S_N} e^{H_i(\beta S_i)}\int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(\frac{d\psi_i}{\sqrt{2\pi}}\right)\exp\left(-\frac{\beta}{2}\psi_i(J^{-1})_{ij}\psi_j + \psi_i(\beta S_i)\right) \\
&= e^{-\frac{\beta}{2}\text{ Tr}(J)}\sqrt{\text{Det } \beta J^{-1}}\int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(\frac{d\psi_i}{\sqrt{2\pi}}\right)e^{-\frac{\beta}{2}\psi_i(J^{-1})_{ij}\psi_j}\left[\sum_{S_1}\cdots\sum_{S_N} e^{S_i\beta(H_i+\psi_i)}\right] \\
&= e^{-\frac{\beta}{2}\text{ Tr}(J)}\sqrt{\text{Det } \beta J^{-1}}\int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(\frac{d\psi_i}{\sqrt{2\pi}}\right)e^{-\frac{\beta}{2}\psi_i(J^{-1})_{ij}\psi_j}\left[\prod_{i=1}^{N}\left(e^{\beta(H_i+\psi_i)} + e^{-\beta(H_i+\psi_i)}\right)\right] \\
&= e^{-\frac{\beta}{2}\text{ Tr}(J)}\sqrt{\text{Det } \beta J^{-1}}\int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(\frac{d\psi_i}{\sqrt{2\pi}}\right)e^{-\frac{\beta}{2}\psi_i(J^{-1})_{ij}\psi_j}\left[\prod_{i=1}^{N}\left(2\cosh\beta[H_i+\psi_i]\right)\right] \\
&= e^{-\frac{\beta}{2}\text{ Tr}(J)}\sqrt{\text{Det } \beta J^{-1}}\int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(\frac{d\psi_i}{\sqrt{2\pi}}\right)e^{-\frac{\beta}{2}\psi_i(J^{-1})_{ij}\psi_j}e^{\sum_i\log(2\cosh\beta[H_i+\psi_i])}
\end{aligned}$$

We can write above in this fashion:

$$S = \frac{1}{2}\psi_i J_{ij}^{-1}\psi_j - \frac{1}{\beta}\sum_i \log\left(2\cosh\beta[H_i+\psi_i]\right)$$

We can shift integration variable like $\psi_i \to \psi_i - H_i$ to have:

$$S = \frac{1}{2}(\psi_i - H_i)J_{ij}^{-1}(\psi_j - H_j) - \frac{1}{\beta}\sum_i \log\left(2\cosh\beta\psi_i\right)$$

[3] *I multiply the identity by $e^{H_i(\beta S_i)}$ and summed in all degree of freedom*

part(d)

For this part we use ***saddle point approximation*** or ***steepest descents*** method, for approximating partition function. If function S in previous part is minimum with respect to ψ_i then we can say the exponential is maximum (consider minus sign), variation of function S is:

$$\begin{aligned}\delta S &= \frac{1}{2}\delta\psi_i\, J_{ij}^{-1}(\psi_j - H_j) + \frac{1}{2}(\psi_i - H_i)J_{ij}^{-1}\delta\psi_j - \frac{1}{\beta}\sum_i \beta\delta\psi_i \tanh(\beta\psi_i)\\ &= \sum_i \delta\psi_i \left[J_{ij}^{-1}(\psi_i - H_i) - \tanh(\beta\psi_i)\right]\end{aligned}$$

So we have:

$$J_{ij}^{-1}(\bar{\psi}_i - H_i) - \tanh(\beta\bar{\psi}_i) = 0 \tag{3}$$

So we approximate partition function as follows:

$$\mathcal{Z}_\Omega \approx e^{-\beta S\{\bar{\psi}_i\}}$$

We can Helmholtz free energy like:

$$\mathcal{F} = -k_B T \log \mathcal{Z}_\Omega \approx S\{\bar{\psi}_i\}$$

Magnetization is:

$$\begin{aligned}m_i &= -\frac{\partial \mathcal{F}}{\partial H_i} \approx -\frac{\partial S}{\partial H_i} = -\frac{\partial}{\partial H_i}\left[\frac{1}{2}(\psi_i - H_i)J_{ij}^{-1}(\psi_j - H_j) - \frac{1}{\beta}\sum_i \log\left(2\cosh\beta\psi_i\right)\right]\\ &= (\bar{\psi}_i - H_i)J_{ij}^{-1}\end{aligned}$$

By using equation 3 we have:

$$m_i = \tanh(\beta\bar{\psi}_i)$$

From above we can find $\bar{\psi}_i$ as a function of m_i we have:

$$m_i = \frac{e^{\beta\bar{\psi}_i} - e^{-\beta\bar{\psi}_i}}{e^{\beta\bar{\psi}_i} + e^{-\beta\bar{\psi}_i}} \Rightarrow e^{\beta\bar{\psi}_i}\left[m_i - 1\right] = e^{-\beta\bar{\psi}_i}\left[m_i + 1\right]$$

We find:

$$\bar{\psi}_i = \frac{1}{2\beta}\log\left(\frac{1+m_i}{1-m_i}\right)$$

We put above in equation 3 to find $H_i\{m_i\}$, we have:

$$\begin{aligned}J_{ij}^{-1}(\bar{\psi}_i - H_i) = \tanh(\beta\bar{\psi}_i) &\Rightarrow \left[\frac{1}{2\beta}\log\left(\frac{1+m_i}{1-m_i}\right) - H_i\right] = m_i J_{ij}\\ H_i &= \frac{1}{2\beta}\log\left(\frac{1+m_i}{1-m_i}\right) - m_i J_{ij}\end{aligned} \tag{4}$$

Remember that we use Einstein summation convention in all calculation.

part(e)
We use previous part result and put it in Helmholtz free energy function, we have:

$$\mathcal{F} = -k_BT \log \mathcal{Z}_\Omega \approx S\{\bar{\psi_i}\} = \frac{1}{2}(\bar{\psi_i} - H_i)J_{ij}^{-1}(\bar{\psi_j} - H_j) - \frac{1}{\beta}\sum_i \log\left(2\cosh\beta\bar{\psi_i}\right)$$
$$= \frac{1}{2}\left[\frac{1}{2\beta}\log\left(\frac{1+m_i}{1-m_i}\right) - \frac{1}{2\beta}\log\left(\frac{1+m_i}{1-m_i}\right) - m_iJ_{ij}\right]J_{ij}^{-1}\left[\frac{1}{2\beta}\log\left(\frac{1+m_j}{1-m_j}\right) - \frac{1}{2\beta}\log\left(\frac{1+m_j}{1-m_j}\right) - m_jJ_{ij}\right]$$
$$- \frac{1}{\beta}\sum_i \log\left(2\cosh\left[\beta\frac{1}{2\beta}\log\left(\frac{1+m_i}{1-m_i}\right)\right]\right)$$
$$= \frac{1}{2}J_{ij}m_im_j - \frac{1}{\beta}\sum_i \log\left(2\cosh\left[\frac{1}{2}\log\left(\frac{1+m_i}{1-m_i}\right)\right]\right)$$
$$= \frac{1}{2}J_{ij}m_im_j - \frac{1}{\beta}\sum_i \log\left(\sqrt{\frac{1+m_i}{1-m_i}} + \sqrt{\frac{1-m_i}{1+m_i}}\right)$$
$$= \frac{1}{2}J_{ij}m_im_j - \frac{1}{\beta}\sum_i \log\left(\frac{2}{\sqrt{1-m_i^2}}\right)$$

Finally:

$$\bar{S}(\{m_i\}) = \frac{1}{2}J_{ij}m_im_j - \frac{1}{\beta}\sum_i \log\left(\frac{2}{\sqrt{1-m_i^2}}\right)$$

We know that Gibbs free energy is:

$$\Gamma\{m_i\} = \bar{S}(\{m_i\}) + \sum_i H_i(\{m_j\})m_i$$

Put function of $\bar{S}(\{m_i\})$ and $H_i(\{m_j\})$ in above, we have:

$$\Gamma\{m_i\} = \frac{1}{2}J_{ij}m_im_j - \frac{1}{\beta}\sum_i \log\left(\frac{2}{\sqrt{1-m_i^2}}\right) + \sum_i\left(\frac{1}{2\beta}\log\left(\frac{1+m_i}{1-m_i}\right) - m_iJ_{ij}\right)m_i$$
$$= -\frac{1}{2}J_{ij}m_im_j + \frac{1}{\beta}\sum_i\left[\frac{m_i}{2}\log\left(\frac{1+m_i}{1-m_i}\right) - \log\left(\frac{2}{\sqrt{1-m_i^2}}\right)\right]$$

We can verify equation of state by $H_i = \partial\Gamma\{m_i\}/\partial m_i$, we have:

$$H_i = -J_{ij}m_j + \frac{1}{\beta}\sum_i\left[\frac{1}{2}\log\left(\frac{1+m_i}{1-m_i}\right) + \frac{m_i}{1-m_i^2} - \frac{m_i}{1-m_i^2}\right]$$
$$= -J_{ij}m_j + \frac{1}{\beta}\sum_i\left[\frac{1}{2}\log\left(\frac{1+m_i}{1-m_i}\right)\right]$$
$$= H_i$$

In last part we use equation 4.

3- (**Goldenfeld book exercise 5-2**):

part(a)

We should derivative form the Landau free energy with $h = 0$, so we have:

$$\frac{\partial \mathcal{L}}{\partial \eta} = \eta(a + b\eta^2 + c\eta^4) = 0 \Rightarrow \eta = 0 \quad and \quad \eta^2 = \eta_s^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2c}$$

I remove one of answer that have imaginary answer. For stability we should take second derivative so we have:

$$\frac{\partial^2 \mathcal{L}}{\partial^2 \eta} = a + 3b\eta^2 + 5c\eta^4$$

$$for \quad \eta = 0 \quad \Rightarrow \quad \frac{\partial^2 \mathcal{L}}{\partial^2 \eta} = a \quad \Rightarrow a < 0 \quad so\ it\ is\ unstable$$

$$for \quad \eta = +\left(\frac{-b + \sqrt{b^2 - 4ac}}{2c}\right)^{1/2} \Rightarrow \quad \frac{\partial^2 \mathcal{L}}{\partial^2 \eta} = \sqrt{b^2 - 4ac}\left[\frac{\sqrt{b^2 - 4ac}}{c} - \frac{b}{c}\right]$$

$$for \quad \eta = -\left(\frac{-b + \sqrt{b^2 - 4ac}}{2c}\right)^{1/2} \Rightarrow \quad \frac{\partial^2 \mathcal{L}}{\partial^2 \eta} = \sqrt{b^2 - 4ac}\left[\frac{\sqrt{b^2 - 4ac}}{c} - \frac{b}{c}\right]$$

For last two parts we have this:

$$\begin{cases} \text{if } b > 0 & \frac{\partial^2 \mathcal{L}}{\partial^2 \eta} < 0 \quad \text{Unstable} \\ \text{if } b < 0 & \frac{\partial^2 \mathcal{L}}{\partial^2 \eta} > 0 \quad \text{Stable} \end{cases}$$

part (b)

We should have $\eta_s^2 = positive\ real\ number$, that is impossible if we have $a > 0$ and $b > 0$. If consider the case that $b^2 - 4ac > 0$ then expression $-b + \sqrt{b^2 - 4ac}$ is always negative, so in this region we have only one answer that is $\eta_s = 0$.

part (c)

We know that $b < 0$, $a > 0$ and $c > 0$ so we can write:

$$\frac{\partial^2 \mathcal{L}}{\partial^2 \eta} = a + 3b\eta^2 + 5c\eta^4$$

$$for \quad \eta = 0 \Rightarrow \quad \frac{\partial^2 \mathcal{L}}{\partial^2 \eta} = a \quad \Rightarrow a > 0 \quad so\ it\ is\ Stable$$

$$for \quad \eta = +\left(\frac{-b + \sqrt{b^2 - 4ac}}{2c}\right)^{1/2} \Rightarrow \quad \frac{\partial^2 \mathcal{L}}{\partial^2 \eta} = \sqrt{b^2 - 4ac}\left[\frac{\sqrt{b^2 - 4ac}}{c} - \frac{b}{c}\right]$$

$$for \quad \eta = -\left(\frac{-b + \sqrt{b^2 - 4ac}}{2c}\right)^{1/2} \Rightarrow \quad \frac{\partial^2 \mathcal{L}}{\partial^2 \eta} = \sqrt{b^2 - 4ac}\left[\frac{\sqrt{b^2 - 4ac}}{c} - \frac{b}{c}\right]$$

So we have:

$$\begin{cases} \text{if } b > 0 & \frac{\partial^2 \mathcal{L}}{\partial^2 \eta} < 0 \quad \text{Unstable} \\ \text{if } b < 0 & \frac{\partial^2 \mathcal{L}}{\partial^2 \eta} > 0 \quad \text{Stable} \end{cases}$$

part (d)

We can solve Landau free energy density for different value of . We can sketch diagram in a-b plane like below:

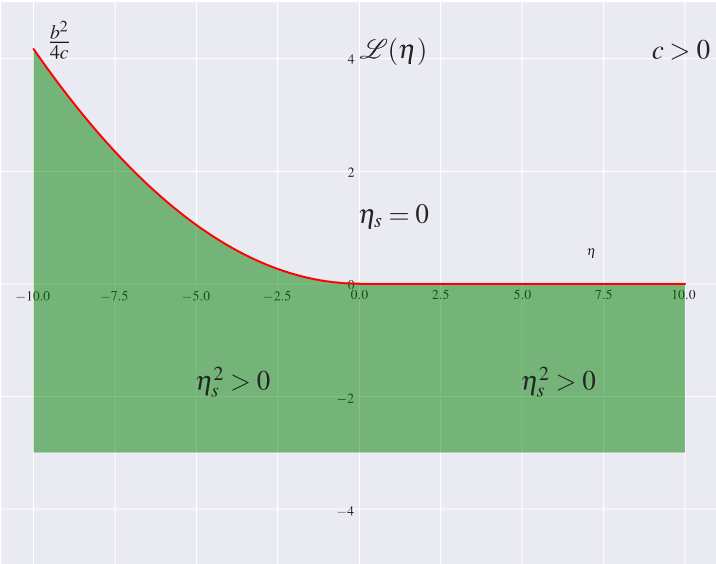

Figure 1

In the curvature ($a = b^2/4c$) we have first order transition because above the parabolic line we have on stable answer and below this line we have three stable answer. If we sketch for all possible a and b sign and values we have four category that is shown in below diagrams:

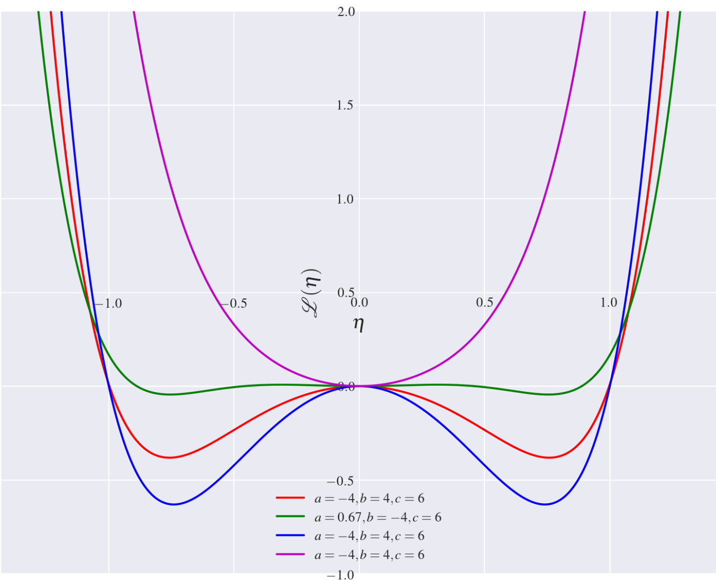

Figure 2

Point $\eta = 0$ may called triciritical point because it can reached by changing three parameter like T, P and h.

part (e)

We can calculate critical exponent by using Landau free energy equation, for β we have:

$$\mathcal{L} = \frac{1}{2}a\eta^2 + \frac{1}{6}c\eta^6 - h\eta \quad \text{for} \quad b = 0,\ h = 0$$

$$\frac{\partial \mathcal{L}}{\partial \eta} = \eta(a + c\eta^4) = 0 \Rightarrow \eta = \Big(\frac{-a}{c}\Big)^{1/4} = \Big(\frac{-a_1 t + a_2 p}{c}\Big)^{1/4} \Rightarrow \beta = \frac{1}{4}$$

for α we know that $C_v = -T\partial^2\mathcal{L}/\partial T^2$ and so we have:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial T} &= \frac{1}{T_c}\frac{\partial \mathcal{L}}{\partial t} \\
&= \frac{a_1}{2}\Big(\frac{-a_1 t + a_2 p}{c}\Big)^{1/2} + \frac{1}{4}(-a_1 t + a_2 p)\Big(\frac{-a_1 t + a_2 p}{c}\Big)^{-1/2}(\frac{-a_1}{c}) + \frac{c}{4}\Big(\frac{-a_1 t + a_2 p}{c}\Big)^{1/2}(\frac{-a_1}{c}) \\
&\propto (-a_1 t + a_2 p)^{1/2} \\
\frac{\partial^2 \mathcal{L}}{\partial T^2} &= \frac{1}{T_c^2}\frac{\partial^2 \mathcal{L}}{\partial t^2} \propto (-a_1 t + a_2 p)^{-1/2} \Rightarrow \quad \alpha = 1/2
\end{aligned}$$

for δ we have(consider that in tricritical point a=0):

$$\mathcal{L} = \frac{1}{2}a\eta^2 + \frac{1}{6}c\eta^6 - h\eta \quad for \quad b = 0$$

$$\frac{\partial \mathcal{L}}{\partial \eta} = 2a\eta + c\eta^5 - h = 0 \Rightarrow 2a\eta + c\eta^5 = h \quad \Rightarrow h \sim \eta^5 \quad \delta = 5$$

for γ we have:

$$\begin{aligned}
\frac{\partial}{\partial h}\Big(2a\eta + c\eta^5\Big) &= 1 \\
\chi_T(h) = \frac{\partial \eta(h)}{\partial h} &= \frac{1}{2a + 5c\eta^4} \\
&= \frac{1}{2a_1 t + a_2 p + 5c\eta^4} \Rightarrow \gamma = \gamma' = 1
\end{aligned}$$

We can find ν by using scaling law derived in chapter 9 ($2 - \alpha = \nu d$). I don't know any way to derive ν explicitly.

part (f)

We know that the only difference between triciritical and ordinary behavior is additional term in Landau free energy $1/6c\eta^6$. So if this term can merge to other terms the cross over can happen, so if we have:

$$1/6c\eta^6 \sim 1/4b\eta^4 \Rightarrow b \sim c\eta^2$$

In ordinary critical behavior we have stable order parameter in zero and $\eta = (-at/b)^{1/2}$, if we use this in above equation we have:

$$b \sim c\eta^2 \Rightarrow b^2 \sim c(-at/b) \Rightarrow b^2 \approx -ac$$

4- (**Goldenfeld book exercise 5-3**):
part(a)
We have relation like this in the chapter five text, we have:

$$M(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty} e^{iq_n x} M_n, \quad M_n = \int_L M(x) e^{-iq_n x} dx$$

We put left to right equation so we get:

$$\begin{aligned} M(x) &= \frac{1}{L}\sum_{n=-\infty}^{\infty} e^{iq_n x}\Big[\int_L M(x') e^{-iq_n x'} dx'\Big] \\ &= \int_L M(x')\Big[\frac{1}{L}\sum_{n=-\infty}^{\infty} e^{iq_n(x-x')}\Big] dx' \end{aligned}$$

So we have:

$$\delta(x-x') = \frac{1}{L}\sum_{n=-\infty}^{\infty} e^{iq_n(x-x')}$$

If we put right to left we can get:

$$\begin{aligned} M_n &= \int_L \Big[\frac{1}{L}\sum_{n=-\infty}^{\infty} e^{iq_{n'} x} M_{n'}\Big] e^{-iq_n x} dx \\ &= \sum_{n=-\infty}^{\infty} M_{n'}\Big[\frac{1}{L}\int_L e^{i(q_{n'}-q_n)x} dx\Big] \\ &= \sum_{n=-\infty}^{\infty} M_{n'}\Big[\frac{1}{L}\int_L e^{i(n'-n)2\pi x/L} dx\Big] \end{aligned}$$

So we can define Kronecker delta function:

$$\delta_{nn'} = \frac{1}{L}\int_L e^{i(q_{n'}-q_n)x} dx = \frac{1}{L}\int_L e^{i(n'-n)2\pi x/L} dx \tag{5}$$

If L goes to infinity we should consider density of states ($L/(2\pi)$).

part (b)
I want to transform all terms in Landau free energy to Fourier space. For first term we have:

$$\int_L atM^2 dx = \frac{at}{L^2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} M_n M_m \int_L e^{i(q_n+q_m)x} dx = \frac{at}{L}\sum_{n=-\infty}^{\infty} M_n M_{-n} = \frac{at}{L}\sum_{n=-\infty}^{\infty} |M_n|^2$$

for second term we have:

$$\begin{aligned} \frac{b}{2}\int_L M^4 dx &= \frac{b}{2L^4}\int_L \Big(\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} M_n M_{n'} e^{i(q_n+q_{n'})x}\Big)\Big(\sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty} M_m M_{m'} e^{i(q_m+q_{m'})x}\Big) dx \\ &= \frac{b}{2L^3}\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty} M_n M_{n'} M_m M_{m'} \delta_{n+m+n'+m',0} \end{aligned}$$

for third term we have:

$$\int_L \frac{\gamma}{2}(\frac{\partial M}{\partial x})^2 = \frac{\gamma}{2L^2}\int_L \Big(\sum_{n=-\infty}^{\infty} M_n i q_n e^{iq_n x}\Big)\Big(\sum_{m=-\infty}^{\infty} M_m (i) q_m e^{iq_m x}\Big)dx$$
$$= -\frac{\gamma}{2L^2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} q_n q_m M_n M_m \Big(\int_L e^{-i(q_n+q_m)x}dx\Big)$$
$$= \frac{\gamma}{2L}(\frac{2\pi}{L})^2 \sum_{n=-\infty}^{\infty} n^2|M_n|^2$$

for the last term we have:

$$\int_L \frac{\sigma}{2}(\frac{\partial^2 M}{\partial x^2})^2 = \frac{\sigma}{2L^2}\int_L \Big(\sum_{n=-\infty}^{\infty} M_n(-q_n^2)e^{iq_n x}\Big)\Big(\sum_{m=-\infty}^{\infty} M_m(-q_m^2)e^{iq_m x}\Big)dx$$
$$= \frac{\sigma}{2L^2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} q_n^2 q_m^2 M_n M_m \Big(\int_L e^{-i(q_n+q_m)x}dx\Big)$$
$$= \frac{\sigma}{2L}(\frac{2\pi}{L})^4 \sum_{n=-\infty}^{\infty} n^4|M_n|^2$$

Finally Landau free energy in Fourier space is:

$$\mathscr{L}_{Landau} = \sum_{n=-\infty}^{\infty}\Big[\frac{at}{L}|M_n|^2 + \frac{b}{2L^3}\sum_{n'=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty} M_n M_{n'} M_m M_{m'}\delta_{n+m+n'+m',0}$$
$$+ \frac{\gamma}{2L}(\frac{2\pi}{L})^2 n^2|M_n|^2 + \frac{\sigma}{2L}(\frac{2\pi}{L})^4 n^4|M_n|^2\Big]$$

Second term make calculation very difficult but we can make it easier by assume that $m = -n$ and $m' = n'$, we have:

$$\mathscr{L}_{Landau} = \sum_{n=-\infty}^{\infty}\Big[\frac{at}{L}|M_n|^2 + \frac{b}{2L^3}\sum_{m=-\infty}^{\infty}|M_n|^2|M_m|^2 + \frac{\gamma}{2L}(\frac{2\pi}{L})^2 n^2|M_n|^2 + \frac{\sigma}{2L}(\frac{2\pi}{L})^4 n^4|M_n|^2\Big]$$

part (c)

Now we want to take derivative for M_n. Derivative of all terms in $\mathscr{L}_{Landau}$ have ordinary behavior except second term. Now I want to derivative second term:

$$\frac{\partial}{\partial|M_n|}\Big[\frac{b}{2L^3}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}|M_n|^2|M_m|^2\Big] = \frac{b}{L^3}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\Big[|M_n||M_m|^2 + |M_n|^2|M_m|\frac{\partial|M_m|}{\partial|M_n|}\Big]$$
$$= \frac{b}{L^3}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\Big[|M_n||M_m|^2 + |M_n|^2|M_m|\delta_{nm}\Big]$$
$$= \frac{b}{L^3}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}|M_n||M_m|^2 + \frac{b}{L^3}\sum_{n=-\infty}^{\infty}|M_n|^3$$

Finally we can write derivative for M_n, we have:

$$\frac{\partial \mathscr{L}_{Landau}}{\partial |M_n|} = \sum_{n=-\infty}^{\infty} \left[\frac{2at}{L}|M_n| + \frac{b}{L^3}|M_n| \sum_{m=-\infty}^{\infty} |M_m|^2 + \frac{b}{L^3}|M_n|^3 + \frac{\gamma}{L}(\frac{2\pi}{L})^2|M_n|n^2 + \frac{\sigma}{L}(\frac{2\pi}{L})^4|M_n|n^4 \right]$$
$$= \sum_{n=-\infty}^{\infty} |M_n| \Big[\frac{2at}{L} + \frac{b}{L^3} \sum_{m=-\infty}^{\infty} |M_m|^2 + \frac{b}{L^3}|M_n|^2 + \frac{\gamma}{L}(\frac{2\pi}{L})^2 n^2 + \frac{\sigma}{L}(\frac{2\pi}{L})^4 n^4 \Big] = 0 \tag{6}$$

Now we want to derivate with respect to n we have:

$$\frac{\partial \mathscr{L}_{Landau}}{\partial n} = \sum_{n=-\infty}^{\infty} \left[\frac{\gamma}{2L}(\frac{2\pi}{L})^2|M_n|(2n) + \frac{\sigma}{2L}(\frac{2\pi}{L})^4|M_n|(4n^3) \right]$$
$$= \frac{4\pi^2}{L^3} \sum_{n=-\infty}^{\infty} |M_n| \left[\gamma + 2\sigma(\frac{2\pi}{L})^2 n^2 \right] n = 0 \tag{7}$$

From equation 7 we can find two condition that can make equation equal to zero. First, if we have one point sum and $n = 0$, second, we have $|M_n|$ behave specialty like $|M_n| = |M_{-n}|$ (which we assume it have because I can't calculate the summation). For first condition ($n = 0$) by put the condition in 6 we have:

$$\frac{\partial \mathscr{L}_{Landau}}{\partial |M_n|} = |M_0| \left[\frac{2at}{L} + \frac{2b}{L^3}|M_0|^2 \right] = 0 \Rightarrow |M_0| = 0 \quad \text{and} \quad |M_0| = L\sqrt{\frac{-at}{b}}$$

We find stationary solution in Fourier space, by using inverse Fourier transformation we find that $M = 0$ and $M = \sqrt{-at/b}$ is solution in real space. If we put condition ($M_n = M_{-n}$) in 6 third and fourth term in summation will diverge and I have no idea what is wrong.

part (d)

I can't solve this part.

5- (**Goldenfeld book exercise 6-4**):

part(a)

I use generating function for proof above. For nominator we can write generating function form:

$$\frac{\partial}{\partial J_q}\frac{\partial}{\partial J_r}\int DXe^{(-\frac{1}{2}X^TAX+J^TX)} = \frac{\partial}{\partial J_q}\frac{\partial}{\partial J_r}\left[\sqrt{\frac{(2\pi)^N}{\det A}}e^{-\frac{1}{2}J^TA^{-1}J}\right]\Bigg|_{J=0}$$
$$= A^{-1}_{qr}\sqrt{\frac{(2\pi)^N}{\det A}}$$

for denominator we have:

$$\int DXe^{-\frac{1}{2}X^TAX} = \sqrt{\frac{(2\pi)^N}{\det A}}$$

by dividing above we find:

$$\langle x_q x_r\rangle = \frac{A^{-1}_{qr}\sqrt{\frac{(2\pi)^N}{\det A}}}{\sqrt{\frac{(2\pi)^N}{\det A}}} = A^{-1}_{qr}$$

part (b)

I want to prove that right hand side of equation equal to left hand side so:

$$\langle x_a x_b\rangle = \frac{1}{\sqrt{\frac{(2\pi)^N}{\det A}}}\frac{\partial^2}{\partial J_a\partial J_b}\int DXe^{(-\frac{1}{2}X^TAX+J^TX)}$$
$$\langle x_a x_d\rangle = \frac{1}{\sqrt{\frac{(2\pi)^N}{\det A}}}\frac{\partial^2}{\partial J_a\partial J_d}\int DXe^{(-\frac{1}{2}X^TAX+J^TX)}$$
$$\langle x_a x_c\rangle = \frac{1}{\sqrt{\frac{(2\pi)^N}{\det A}}}\frac{\partial^2}{\partial J_a\partial J_c}\int DXe^{(-\frac{1}{2}X^TAX+J^TX)}$$

We know that the Gaussian integral is:

$$\int DXe^{(-\frac{1}{2}X^TAX+J^TX)} = \sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^TA^{-1}J}$$

by using part (a) we know the result of above integrals and finally all two point functions are:

$$\langle x_a x_b\rangle = A^{-1}_{ab} \quad \langle x_c x_d\rangle = A^{-1}_{cd}$$
$$\langle x_a x_d\rangle = A^{-1}_{ad} \quad \langle x_b x_c\rangle = A^{-1}_{bc}$$
$$\langle x_a x_c\rangle = A^{-1}_{ac} \quad \langle x_b x_d\rangle = A^{-1}_{bd}$$

So RHS is:

$$\langle x_a x_b\rangle\langle x_c x_d\rangle + \langle x_a x_d\rangle\langle x_b x_c\rangle + \langle x_a x_c\rangle\langle x_b x_d\rangle = A^{-1}_{ab}A^{-1}_{cd} + A^{-1}_{ad}A^{-1}_{bc} + A^{-1}_{ac}A^{-1}_{bd}$$

Now we want to calculate left hand side of part (b), so we have:

$$\langle x_a x_b x_c x_d\rangle = \frac{1}{\sqrt{\frac{(2\pi)^N}{\det A}}}\frac{\partial^4}{\partial J_a\partial J_b\partial J_c\partial J_d}\int DXe^{(-\frac{1}{2}X^TAX+J^TX)}$$
$$= \frac{\partial^4}{\partial J_a\partial J_b\partial J_c\partial J_d}e^{\frac{1}{2}J^TA^{-1}J}$$

We can derivative exponential function by two pair of variables, so we can write:

$$\begin{aligned}\langle x_a x_b x_c x_d \rangle &= \frac{\partial^4}{\partial J_a \partial J_b \partial J_c \partial J_d} e^{\frac{1}{2} J^T A^{-1} J} \\ &= A^{-1}_{ab} A^{-1}_{cd} + A^{-1}_{ad} A^{-1}_{bc} + A^{-1}_{ac} A^{-1}_{bd}\end{aligned}$$

6- (**Goldenfeld book exercise 7-1**):
part(a)
Landau free energy is:

$$L = \int d^d x \left\{ \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2 + \frac{u_n}{n!}\phi^n \right\}$$

If we write effective Hamiltonian with above, we have:

$$H_{\text{eff}}\{\phi\} \equiv \beta L = \int d^d x \left\{ \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2 + \frac{u_n}{n!}\phi^n \right\}$$

We know that dimension of effective Hamiltonian is one ($[H_{\text{eff}}] = 1$) so we should have:

$$\left[\int d^d x \, (\nabla\phi)^2\right] = 1 \;\longrightarrow L^d L^{-2}[\phi]^2 = 1 \longrightarrow \; [\phi] = L^{1-d/2}$$

$$\left[\int d^d x \; r_0\phi^2\right] = 1 \;\longrightarrow L^d L^{2-d}[r_0] = 1 \longrightarrow [r_0] = L^{-2}$$

$$\left[\int d^d x \; u_n\phi^n\right] = 1 \;\longrightarrow L^d L^{n-nd/2}[u_n] = 1 \longrightarrow [u_n] = L^{(nd-2d-2n)/2}$$

We have already seen that Gaussian functional integrals are easy to do. So we will write the partition function as a Gaussian functional integral with a modification, which we treat by perturbation theory. We define the following dimensionless variables:

$$\varphi = \frac{\phi}{L^{1-d/2}}; \qquad \bar{u_n} = \frac{u_n}{L^{(nd-2d-2n)/2}}; \qquad L = r_0^{-1/2}$$

We have to calculate partition function for all purposes, so we have:

$$\mathcal{Z}(\bar{u_n}) = \int D\varphi \; \exp\left[-H_0\{\varphi\} - H_{int}\{\varphi\}\right] \tag{8}$$

Where:

$$H_0 = \int d^d x \left\{ \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2} r_0 \varphi^2 \right\}$$

$$H_{int} = \int d^d x \left\{ \frac{\bar{u_n}}{n!}\varphi^n \right\}$$

If $H_{\text{int}} = 0$, the integral 8 is just the Gaussian approximation, which is exactly soluble. The partition function has, however, a contribution from the interactions, H_{int}. We might imagine that if $\bar{u_n} \ll 1$, then we could use perturbation theory:

$$\begin{aligned}
\mathcal{Z} &= \int D\varphi \; e^{-H_0} e^{-H_{\text{int}}} \\
&= \int D\varphi \; e^{-H_0}\left(1 - H_{int} + \frac{1}{2!}(H_{int})^2 - \ldots\right)
\end{aligned}$$

The important point is that the partition function depends on one dimensionless parameter u_n; this is our perturbation parameter. Written out explicitly,

$$\bar{u_n} = u_n L^{(-nd+2d+2n)/2} = u_n r_0^{(nd-2d-2n)/4}$$

r_0 is a characteristic length for the system which is correlation length and it varies between finite value to infinite value so for $d > 2n/(n-2)$, $\bar{u_n} \longrightarrow \infty$ and perturbation theory becomes meaningless! On the other hand, for $d < 2n/(n-2)$, $\bar{u_n} \longrightarrow 0$, and mean field theory becomes increasingly accurate as $T \longrightarrow T_c^+$.

part(b)

In problem 5-2 we have a term with $n = 6$ and we can find that $d > (2 \times 6)/(6-2) = 3$, perturbation theory for this problem will be accurate for dimension $d > 3$.

7- **Exercise # 2 of set # 4**

First, we define the two-point function:

$$G(\boldsymbol{r}_i - \boldsymbol{r}_j) = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle$$

Partition function is:

$$\mathcal{Z}_\Omega = \text{Tr } e^{-\beta \mathcal{H}_\Omega} = \text{Tr } \exp \left[\beta J \sum_{\langle ij \rangle} S_i S_j + \beta H \sum_i S_i \right]$$

We can obtain averages by differentiating from partition function:

$$\sum_i \langle S_i \rangle = \frac{1}{\mathcal{Z}_\Omega} \text{Tr} \left[\sum_i S_i \right] e^{-\beta \mathcal{H}_\Omega} = \frac{1}{\beta \mathcal{Z}_\Omega} \frac{\partial \mathcal{Z}_\Omega}{\partial H}$$

$$\sum_{ij} \langle S_i S_j \rangle = \frac{1}{\mathcal{Z}_\Omega} \text{Tr} \left[\sum_{ij} S_i S_j \right] e^{-\beta \mathcal{H}_\Omega} = \frac{1}{\beta^2 \mathcal{Z}_\Omega} \frac{\partial^2 \mathcal{Z}_\Omega}{\partial H^2}$$

Now we want to calculate susceptibility:

$$\begin{aligned}
\chi_T = \frac{\partial M}{\partial H} &= -\frac{\partial^2 \mathcal{F}}{\partial H^2} = \frac{1}{N\beta} \frac{\partial^2 \log \mathcal{Z}_\Omega}{\partial H^2} \\
&= \frac{1}{N\beta} \frac{\partial}{\partial H} \left[\frac{1}{\mathcal{Z}_\Omega} \frac{\partial \mathcal{Z}_\Omega}{\partial H} \right] \\
&= \frac{1}{N\beta} \left[\frac{1}{\mathcal{Z}_\Omega} \frac{\partial^2 \mathcal{Z}_\Omega}{\partial H^2} - \left(\frac{1}{\mathcal{Z}_\Omega} \frac{\partial \mathcal{Z}_\Omega}{\partial H} \right)^2 \right] \\
&= \frac{\beta}{N} \left[\sum_{ij} \langle S_i S_j \rangle - \left(\sum_i \langle S_i \rangle \right)^2 \right]
\end{aligned}$$

In mean field theory we have:

$$\sum_{ij} \langle S_i S_j \rangle = \left(\sum_i \langle S_i \rangle \right)^2 = M^2$$

So we have $G(\boldsymbol{r}_i - \boldsymbol{r}_j) = 0$. According to Goldenfeld text result of none zero correlation function comes from Landau theory which is solving the equation of correlation function.

8- **Exercise # 3 of set # 4**

The Helmholtz free energy given by:

$$e^{-\beta\mathcal{F}} = \int \mathcal{D}\eta \ e^{-\beta\mathcal{H}\{\eta(r)\}}$$

where the integral $\int \mathcal{D}\eta$ is a functional integral over all degrees of freedom associated with η, instead of an integral over all microstate. Landau's assumption is that we can replace the entire partition function by the following:

$$e^{-\beta\mathcal{F}} \approx \int \mathcal{D}\eta \ e^{-\beta\mathcal{L}\{\eta(r)\}} \tag{9}$$

For example, if η is the mean magnetization, a given value for the magnetization can be determined by many different microstates. It is assumed that all of this information is contained in $\mathcal{L}\{\eta(r)\}$. This is a non-trivial assumption which can nonetheless be proven for certain systems. The conversion of the degree of freedom from $S$to η is known as *coarse-graining*, and is at the heart of the relationship between statistical mechanics and thermodynamics. The next step is to minimize $\mathcal{L}\{\eta(r)\}$(to maximize integrated), performing a saddle point approximation (or steepest descent) to the functional integral in 9, giving:

$$e^{-\beta\mathcal{F}} \approx e^{-\beta\mathcal{L}_{\min}\{\eta(r)\}}$$

this is relation between Helmholtz free energy and Landau free energy.