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Einstein Field Equation

① GR Field Eqn : Some Keywords:

- ☆ Riemannian geometry (Curved space)
- ☆ General Covariance (No Preferred Refrence frame)

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  "Components in F.E. are in the form of Tensor"

- ☆ Gravity as Geometry
- ☆ Lagrangian Formalism → Field Theory (Effective Theory)

② $g_{\mu\nu}$ → Coordinate Information

$g_{\mu\nu}$ → Depends on gravitational potential: Perturbation

$g(\phi, \ldots)$:

③ $T_{\mu\nu}$: → Mass, Energy..

④ ②, ③ ⇒

$$g_{\mu\nu} = K\, T_{\mu\nu} \longrightarrow \underbrace{g_{\mu\nu;\mu} = K\, T_{\mu\nu;\mu}}_{\text{It is Not Correct}}$$

Energy Conserved $T^{\mu}_{\nu;\mu} = 0$

⑤ $g_{\mu\nu} \longrightarrow G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

$\uparrow$

Einstein Tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

⑥ Derivation of Poisson Equation $\nabla^2 \phi = 4\pi G \rho$
through Einstein Eq in $\underbrace{\text{Weak-Field approximation}}$

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

perturbation

$$\bar{g}_{\mu\nu} \longrightarrow g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \quad , \quad \boxed{g_{00} = 1 + h_{00}}$$

Static Gravitational field $\quad \partial_0 h_{\mu\nu} = 0$

$$P \ll \rho$$

$$\boxed{T_{\mu\nu} \sim \rho U_\mu U_\nu}$$

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \longrightarrow ?$$

$R_{\mu\nu} = \checkmark$

$\Rightarrow G = \checkmark$

$R = \checkmark$

$$R_{00} = -\partial_i \Gamma^i_{00} \quad , \quad \Gamma^i_{00} = \frac{1}{2} \delta^{ij} \partial_j h_{00}$$

$$R_{00} = -\frac{1}{2} \nabla^2 h_{00}$$

$$R \sim \partial_i \partial_j h^{ij} - \nabla^2 h$$

$$\rho c^2$$

$$G_{00} = -\tfrac{1}{2} \nabla^2 h_{00} = \frac{8\pi G}{c^4} T_{00}$$

$$-\tfrac{1}{2} \nabla^2 h_{00} = \frac{8\pi G}{c^4} \rho c^2$$

$$\boxed{\nabla^2 h_{00} = -\frac{16\pi G}{c^2} \rho} \quad , \quad \boxed{\nabla^2 \phi = 4\pi G \rho}$$

$$\boxed{g_{00} = 1 + h_{00}}$$

$$\boxed{h_{00} = -\frac{2\phi}{c^2}}$$

We achieve Poisson Equn

(7) Another approach to derive Einstein Eq. is utilizing Lagrangian Formalism in curved space and It is based on

Principle of least action:

☆ $$e^{-\beta F} = Z = \sum_{\{q,p\}} e^{-\beta \mathcal{H}(\{q,p\})}$$

$$= Z = \int \widetilde{D\phi}\, e^{-\beta L[\phi]}$$

$$L[\phi] = \int d^d x \, \underbrace{\mathcal{L}[\phi, \partial\phi, \ldots]}_{\text{Symmetry, Stability, Locality} \ldots}$$

e.g. $\mathcal{L}[\phi ..] = \frac{1}{2}(\nabla\phi)^2 + m^2\phi^2 + \lambda\phi^4 + [(\nabla\cdot\nabla)\phi]^2 - \vec{B}\cdot\vec{P}$

$$\frac{\partial\phi}{\partial x} = \lim_{\Delta x\to 0} \frac{\phi(x+\Delta x) - \phi(x)}{\Delta x}$$

☆ Classical Mechanics : $S = \int dt\, L(q, \dot{q})$

$L = T - V$

$$\delta S = 0 \Longrightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\ddot{q} = -\frac{dV}{dq}$$

☆ Field Theory. $q \longrightarrow \phi^i(x^a),\ \phi^i(x,y,z)$

$$L = \int d^3x\, \mathcal{L}(\phi^i, \partial_x\phi \ldots)$$

$$S = \int dt\, L = \int d^4x\, \mathcal{L}$$

Lagrangian Density

$\delta S = 0 \Longrightarrow$ Euler - Lagrange Eq

$$\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\right) = 0$$

Exercise : $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu \longrightarrow$ Maxwell Eq

☆ GR, Einstein Eq

$$S_{GR} = \int \underbrace{d^4x\sqrt{-g}}\ \mathcal{L}_{GR}$$

Here : $g \equiv \mathrm{Det}(g_{\mu\nu})$ and show the invariant volume Element

What is proper action for GR ? Einstein – Hilbert Action

$$\mathcal{L}_{GR} = R \longrightarrow \text{For empty Universe.}$$

$$\mathcal{L}_{EH} = R \longrightarrow \text{The Geometric Part}$$

$$S_{GR} = S_{EH} + S_m$$

$$= \int d^4x\sqrt{-g}\,(\mathcal{L}_{EH} + \mathcal{L}_m \dots)$$

$$= \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\,R + \int d^4x\sqrt{-g}\,\mathcal{L}_m$$

$$\delta S_{GR} = 0 = \delta S_{EH} + \delta S_m = 0 \Longrightarrow \boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}}$$

Einstein Eq.

$$\boxed{T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\,\mathcal{L}_m)}{\delta g_{\mu\nu}}} \leftarrow \text{Energy-Momentum Tensor}$$

☆ Action with Cosmological Constant

$$S_{GR+\Lambda} = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\,(R - 2\Lambda) + S_m$$

$\downarrow$

$$\delta S_{GR+\Lambda} = 0 \Rightarrow \boxed{R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

$$T^{\Lambda}_{\mu\nu} = -\frac{\Lambda c^4}{8\pi G} g_{\mu\nu} \Rightarrow P_\Lambda = -\rho c^2$$

$$\downarrow$$

$$\boxed{w_\Lambda = -1}$$

$$T_{(\Lambda)}{}^{\mu}{}_{\nu} = -\frac{\Lambda}{8\pi G}\delta^{\mu}_{\nu} = \begin{pmatrix} -\rho_\Lambda & 0 & 0 & 0 \\ 0 & -\rho_\Lambda & 0 & 0 \\ 0 & 0 & -\rho_\Lambda & 0 \\ 0 & 0 & 0 & -\rho_\Lambda \end{pmatrix}, \quad \text{where} \quad \rho_\Lambda \equiv \frac{\Lambda}{8\pi G} \qquad (3.10)$$

More about Cosmological Constant.

$$\nabla^2\phi = 4\pi G\rho \quad , \quad \nabla\phi = -\vec{g}$$

For Static Universe $\phi = cts \rightarrow \bar{g} = 0$

$\downarrow$

$\rho = 0$

$$\nabla^2\phi + \Lambda = 4\pi G\rho$$

For $\underline{\Lambda = 4\pi G\rho} \Rightarrow \phi = cts \longrightarrow \bar{g} = 0 \ , \ \rho \neq 0$

$$T_{00} = \rho \longrightarrow \bar{T}_{00} = \rho + \frac{\Lambda}{8\pi G}$$

$$P \longrightarrow \bar{P} = P - \frac{\Lambda}{8\pi G}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

$$\rightarrow \ddot{a} > 0 \Rightarrow \rho + 3P < 0$$

$$P < -\tfrac{1}{3}\rho$$

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⑧ Einstein Eqn and Friedmann Eqs:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

For Homogeneous and Isotropic Expanding Univers.

without Perturbation (Prefect Fluid).

$$R_{00} - \tfrac{1}{2} R g_{00} = \frac{8\pi G}{c^4} T_{00}$$

time-Time component

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho}$$

$$K = 0 \qquad \Omega_{total} = \frac{\rho_{tot}}{\rho_c} = 1$$

$$\rho_c \equiv \frac{3H_0^2}{8\pi G}$$

$$R_{ij} - \tfrac{1}{2} R g_{ij} = \frac{8\pi G}{c^4} T_{ij}$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}$$

Space-Space Component

(See Exercise 3.4)

For General Case:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{K}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}$$

$$\dot{\rho}_i + 3\frac{\dot{a}}{a}(\rho_i + P_i) = 0$$

$$\rho = \sum \rho_i$$

$$P = \sum P_i$$

For $K=0 \Rightarrow H^2 = \frac{8\pi G}{3}\rho \quad , \rho = \rho_m + \rho_r + \rho_\Lambda$

$$\rho_c = \frac{3H_0^2}{8\pi G} \simeq 1.12 \times 10^{-5} h^2 \text{ Proton/cm}^3$$

For Radiat. $\quad P_r = w_r \rho_r \quad , w_r = 1/3$

$$\rho_r(a) = \rho_r(a=1)\, a^{-4}$$

$P_m = w_m \rho_m = 0 \longrightarrow \rho_m(a) = \rho_m(a=1)\, a^{-3}$

$w_m = 0$

$P_\Lambda = w_\Lambda \rho_\Lambda \ , \ w_\Lambda = -1 \longrightarrow \rho_\Lambda(a) = \rho_\Lambda(a=1) = cts$

Recall that $\quad P = w(a)\rho \longrightarrow \rho(a) = \rho(a=1)\, a^{-3(1+\bar{w}(a))}$

$$\star \quad \bar{W}(a) = \frac{\int d\ln a\, w(a)}{\int d\ln a} = \frac{\int da \frac{w(a)}{a}}{\int \frac{da}{a}}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \qquad P = w(a)\rho$$

$$\boxed{\frac{d\rho}{\rho} + 3\frac{da}{a}(1 + w(a)) = 0}$$

$$\Omega_i(a) \equiv \frac{\rho_i(a)}{\rho_c(a=1)} = \frac{\rho_i(a=1)\, a^{-(1+\bar{w}(a))3}}{\rho_c(a=1)}$$

$$\boxed{\Omega_i(a) = \Omega_i(a=1)\, a^{-(1+\bar{w}(a))3}}$$

$K=0$

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left(\rho_r(a) + \rho_m(a) + \rho_\Lambda\right)$$

$$= \frac{8\pi G}{3}\frac{\rho_c(a=1)}{\rho_c(a=1)}\left[\rho_r a^{-4} + \rho_m a^{-3} + \rho_\Lambda\right]$$

$$H^2 = H_0^2\left[\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda\right]$$

$\vdots$