Part B

جمعه ۲۴ ,۱ ,۱۴۰۴

بسم الله الرحمن الرحیم

Gravitational field Equation
and Friedmann Equation

Refs:

☆ Chapter 3 Dodelson.

☆ Appendix A : Introduction to the theory of the Early Univers. Gorbunov

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The fundamental equations of cosmology

Cosmology is, essentially, an application of general relativity coupled with statistical mechanics. The only relevant long-range force is gravity, which also provides the background spacetime within which matter moves, as we have seen in the last chapter. Since cosmology deals with the evolution of the entire universe, we are not interested in the fate of individual particles. Instead, we care about the collective, average behavior of matter, which is described by statistical mechanics. This is why essentially all results in cosmology can be derived from the combination of two equations: the Einstein equations on the gravity side, and the Boltzmann equations of statistical mechanics for matter and radiation.

These are formidable equations, and their application can quickly get technical. In this chapter, we will present the general form of the Einstein and Boltzmann equations, and describe their physical content. We will then apply them to the homogeneous universe, which, for the Einstein equations, allows us to derive the Friedmann equation (1.3). These results will also allow us to compute the expansion history and thermal history of the universe in this chapter and the next. Further, with the experience we gain in this chapter, there will be nothing particularly difficult about the subsequent chapters which deal with perturbations in the universe. So, becoming familiar with the framework laid out in this chapter will pay off greatly when going through the rest of the book.

In GR, the energy-momentum tensor **sources gravity** via:

$$G^{\mu\nu} = 8\pi G \, T^{\mu\nu}$$

where:

- $G^{\mu\nu}$ = Einstein tensor (describes spacetime curvature)
- G = gravitational constant

This means:

- **Matter tells spacetime how to curve** ($T^{\mu\nu}$ influences $G^{\mu\nu}$)
- **Spacetime tells matter how to move** (geodesic equation)

Newton	Einstein
Newton 2nd law $\frac{d^2x^i}{dt^2} = -\delta^{ij}\frac{\partial\Phi}{\partial x^j}$	Geodesic equation $\frac{d^2x^\mu}{d^2\sigma} = -\Gamma^\mu{}_{\nu\rho}\frac{\partial x^\rho}{d\sigma}\frac{dx^\nu}{d\sigma}$
Tidal deviation $\frac{d^2\xi^i}{dt^2} = -E^i{}_j\,\xi^j$	Geodesic deviation $\frac{D^2\xi^\mu}{d\sigma^2} = -R^\mu{}_{\nu\rho\sigma}u^\nu u^\sigma \xi^\rho$
1st Bianchi identity $E_{ij} = E_{ji}$	1st Bianchi identitiy $R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0$
2nd Bianchi identity $E^i{}_{[j,l]} = 0$	2nd Bianchi identitiy $\nabla_\kappa R^\mu{}_{\nu\rho\sigma} + \nabla_\sigma R^\mu{}_{\nu\kappa\rho} + \nabla_\rho R^\mu{}_{\nu\sigma\kappa} = 0$
mass density ρ	Energy-momentum tensor $T_{\mu\nu}$
Poisson equation $E^i{}_i = 4\pi G\rho$	Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$
single elliptic equation	10 coupled equations 4 elliptic and 6 hyperbolic
boundary data required	initial and boundary data required

$\nabla\cdot\nabla\phi$

Table 6.1: Newtonian vs Einstenian description of gravity.

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

① $\lim$ GR (Einstine Equation) $\longrightarrow$ Newtonian app.

Weak field (Small scales, low Energy, ...)

② In GR

- Ⓐ $\rightarrow$ Metric & Geodesics
- Ⓑ $\rightarrow$ Energy – Momentum Tensor
  - $\rightarrow$ Continuity Eq.
  - $\rightarrow$ Euler Eq.
- Ⓒ $\rightarrow$ Field Equation
  - $\rightarrow$ Matter $\rightleftarrows$ Geometry

e.g. Ⓐ FLRW Metric (Homogeniety and Isotropy)

Expanding Univ.

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^2(t) & & 0 \\ & 0 & a^2(t) & \\ & & & a^2(t) \end{pmatrix}$$

Ⓑ Energy – Momentum Tensor :

$$T_{\mu\nu} = \rho U_\mu U_\nu + P \gamma_{\mu\nu} + 2 q_\mu U_\nu + \Pi_{\mu\nu}$$

$\rho = T_{\mu\nu} U^\mu U^\nu$ Energy Density

$P = T_{\mu\nu} \dfrac{\gamma^{\mu\nu}}{3}$ Isotropic Pressure

$q^\mu = - T_{\alpha\beta} U^\alpha \gamma^{\beta\mu}$ $\leftarrow$ Energy Flux

$\Pi_{\mu\nu}$ = Anisotropic Pressure

$\gamma_{\mu\nu}$ = 3D Riemannian Metric

Prefect Fluid from Comoving observer

$U : (-1, 0, 0, 0)$

$$T = \begin{pmatrix} -\rho & & & 0 \\ & P & & \\ & & P & \\ 0 & & & P \end{pmatrix}$$

③ In Newtonian approach $\vec{a} = -\vec{\nabla}\phi$

$$\underbrace{\nabla^2 \phi = 4\pi G \rho}_{\text{Scalar Equation}}$$

Scalar

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

Tensor Equation

Einstein Equation in presence of $\underbrace{\text{Cosmological Constant}}_{\Lambda}$

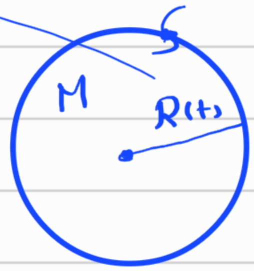
$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$ ← other Form of Einstein Eq.

④ Befor considering GR approach we turn to Newtonian Regim.

Radius ↑

$R \sim \underline{l \ll H^{-1}}$



Birkhoff's theorem (Newton's theorem or Gauss theorem)

$$\boxed{\frac{GM}{Rc^2} \ll 1}$$

$t_{co} \gg t_c = R/c$

$R = 1/2\, a t^2$

$\underline{t_{co}} = (2R/a)^{1/2} \sim \underline{(G\rho)^{-1/2}}$

$R \gg R_{sch.} = \frac{2GM}{c^2}$

$$\boxed{M = \frac{4}{3}\pi R^3 \rho}$$

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} = -\frac{4\pi}{3}\rho G R$$

$$q \equiv \frac{dR}{dt} \qquad q \times \frac{dq}{dt} = -\frac{4\pi}{3}\rho G R \times q$$

$$\underbrace{\int_{q_1}^{q} q' dq' = -GM \int_{R_1}^{R} \frac{dR'}{R^2}}$$

$$\frac{q^2}{2} - \frac{q_1^2}{2} = GM\left[\frac{1}{R} - \frac{1}{R_1}\right]$$

$$\frac{1}{2}\dot{R}^2 = \frac{GM}{R} + B \text{ — A constant.}$$

$$R \equiv a\chi \longrightarrow \underbrace{\dot{R} = \dot{a}\,\chi}$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2B}{\chi^2 a^2}}$$

First Eq. (Friedmann)

Ⓘ If $B > 0 \longrightarrow$ RHS is positive.

$$\dot{a} > 0 \longrightarrow \ddot{a} > 0$$

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Ⓘⓘ If $B = 0$ $\quad t \to \infty \quad \rho(t \to \infty) = 0 \quad \dot{a}(t \to \infty) \longrightarrow 0$

$$\ddot{a} = 0$$

Ⓘⓘⓘ If $B < 0$ $\quad$ For $\dot{a} = 0 \implies \frac{8\pi G}{3}\rho = \frac{2|B|}{\chi^2 a^2}$

$$\boxed{a_{max} = \frac{GM}{\chi|B|}}$$

$$\boxed{K \equiv -\frac{2B}{\chi^2}}$$

Curvature

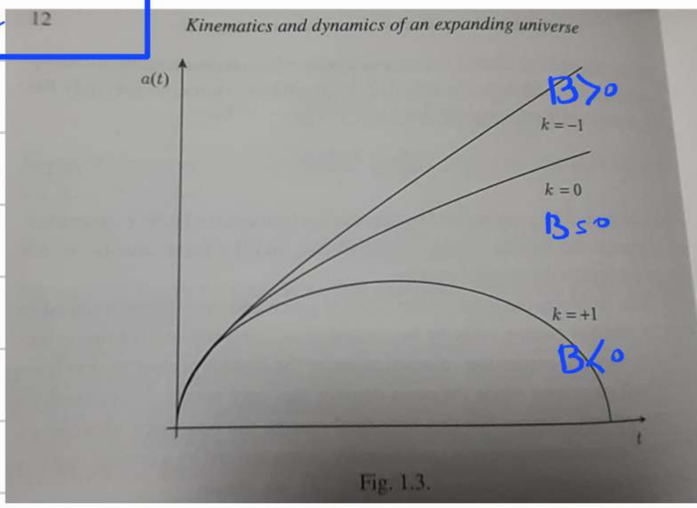

Fig. 1.3.

Deceleration Parameter

$$Q = -\frac{\ddot{a}}{aH^2}$$

$$\begin{cases} B>0 \quad K<0 & \text{open} \quad \ddot{a}>0 \\ B=0 \quad K=0 & \text{Flat} \quad \ddot{a}=0 \\ B<0 \quad K>0 & \text{Closed} \quad \ddot{a}<0 \end{cases}$$

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} \qquad R = ax$$

$$\ddot{a}x = -\frac{4\pi G\rho}{3}\frac{x^3a^3}{x^2a^2} \Rightarrow \boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho}$$

But the correct form of

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}$$

Second friedmann Eqn

$$\rho \longrightarrow \rho_{eff} = \rho + 3P$$

From Thermodynamics

$$dU = T\,dS - P\,dV + \mu\,dN \qquad N = cts$$

$$\left\{ \begin{array}{c} dQ = 0 \\ \downarrow \\ dS = 0 \end{array} \right\}$$

$$dU = -P\,dV$$

$$d(\rho a^3) = -P\,d(a^3) = -d[Pa^3] - a^3\,dP$$

$$\frac{d}{dt}\left\{ a^3(\rho + P) \right\} = \dot{P} a^3 \Rightarrow \boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0}$$

☆ Now an important Point: If the pressure plays as the role of Inertia, So why does it exist in Second Equation but in first Equation the Contribution of P is absent?

To deal with above point, evaluate following exercises

Exercise 1: By Using First Equation and Continuity equation derive Second Equation.

Exercise 2: By using second Equation and Continuity Equation derive the first equation.