بسم الله الرحمن الرحیم

جلسه ۱۴۰۴/۱/۱۷

Continuation : Dynamical aspect of expanding Universe

× # Evolution of Energy (GR View)

Ideal Gas in an Expanding Universe : BBN

① To complete our Mathematical description of our Cosmos, We need to Field Equation in Curved space which is fulfilled by

Poisson Eq.

$\boxed{\nabla^2 \phi = 4\pi G \rho}$ ← Einstein Equation $\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

$G_{\mu\nu}$: Related to Geometry

$T_{\mu\nu}$: Related to matter

② We turn to Energy-Momentum Tensor, $T_{\mu\nu}$

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & 0 & \mathcal{P} \end{pmatrix}$$

T^{00} = energy Density

T^{0i} = Momentum Density

T^{ij} ≡ Stress Part

- ρ = energy density,
- p = pressure,
- u^{μ} = 4-velocity (normalized as $u^{\mu}u_{\mu} = +1$ in this signature),
- $g^{\mu\nu}$ = inverse metric tensor.

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U_\mu U_\nu + P g_{\mu\nu}$$

U^{μ} = Comoving 4-Velocity

- T^{00} = energy density,
- T^{0i} = momentum density (where $i = 1, 2, 3$),
- T^{ij} = stress tensor (pressure and shear forces).

For Comoving observer

$$U^{\mu} = (c, \underset{\substack{\downarrow \\ v}}{0})$$

$$\boxed{\theta \equiv U^{\mu}{}_{;\mu} = 3\,\dot{a}/a} = 3H$$

☆ Conservation means that

$$0 = T^{\mu}{}_{\nu;\mu} = \nabla_\mu T^{\mu}{}_{\nu} \equiv \frac{\partial T^{\mu}{}_{\nu}}{\partial x^{\mu}} + \Gamma^{\mu}{}_{\alpha\mu} T^{\alpha}{}_{\nu} - \Gamma^{\alpha}{}_{\nu\mu} T^{\mu}{}_{\alpha} = 0.$$

$\{\mu, \nu\} = \{0, 1, 2, 3\}$

For each ν we have an equa

□ Temporal Part $\nu = 0$

$$\frac{\partial T^{\mu}{}_{0}}{\partial x^{\mu}} + \Gamma^{\mu}{}_{\alpha\mu} T^{\alpha}{}_{0} - \Gamma^{\alpha}{}_{0\mu} T^{\mu}{}_{\alpha} = 0$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & a^2 & & \\ & & a^2 & \\ 0 & & & a^2 \end{pmatrix}$$

$$\boxed{\frac{\partial \rho}{\partial t} + 3\frac{\dot a}{a}(\rho + P) = 0}$$

For Multi-Component Univers

$$T^{\mu\nu}_{total} = T^{\mu\nu}_{DM} + T^{\mu\nu}_{DE} + T^{\mu\nu}_{B} + T^{\mu\nu}_{nut.} + \cdots$$

$$\boxed{T^{\mu}_{\nu;\mu}\Big|_{total} = 0}$$

Dark Matter Dark Energy

Without Intraction between different compon

we have

$$T^{\mu}_{\nu;\mu}\Big|_{DM} = 0$$

$$T^{\mu}_{\nu;\mu}\Big|_{DE} = 0$$

$$T^{\mu}_{\nu;\mu}\Big|_{B} = 0$$

$$\boxed{\frac{\partial \rho_i}{\partial t} + \frac{3\dot a}{a}(\rho_i + P_i) = 0}$$

$$P_i = \omega_i \rho_i$$

With Intraction

$$\boxed{T^{\mu}_{\nu;\mu}\Big|_{total} = 0}$$

Suppose that we have an Intraction between DM & DE

Therefore we have:

$$\nabla_\mu T^{\mu\nu}_{\rm DM} = +Q^\nu \quad \text{(DM gains/loses energy-momentum)}$$

$$\nabla_\mu T^{\mu\nu}_{\rm DE} = -Q^\nu \quad \text{(DE loses/gains energy-momentum)}$$

$$Q = \begin{cases} > 0 & \text{Dark Energy} \rightarrow \text{Dark Matter (iDEDM regime)} \\ < 0 & \text{Dark Matter} \rightarrow \text{Dark Energy (iDMDE regime)} \\ = 0 & \text{No interaction.} \end{cases}$$

Table 1. Consequences of interacting dark energy models (relative to uncoupled models)

	$Q > 0$	$Q < 0$
Energy flow	DE $\rightarrow$ DM (iDEDM)	DM $\rightarrow$ DE (iDMDE)
Effective equations of state	$\omega^{\rm eff}_{\rm dm} < \omega_{\rm dm}$; $\omega^{\rm eff}_{\rm de} > \omega_{\rm de}$	$\omega^{\rm eff}_{\rm dm} > \omega_{\rm dm}$; $\omega^{\rm eff}_{\rm de} < \omega_{\rm de}$
Coincidence problem	Alleviates ($\zeta_{\rm IDE} < \zeta_{\Lambda\rm CDM}$)	Worsens ($\zeta_{\rm IDE} > \zeta_{\Lambda\rm CDM}$)
Hubble tension	Worsens	Alleviates
S_8 discrepancy	Alleviates	Worsens
Age of universe	Older	Younger
Radiation-matter equality	Later ($z_{\rm IDE} < z_{\Lambda\rm CDM}$)	Earlier ($z_{\rm IDE} > z_{\Lambda\rm CDM}$)
Cosmic jerk ($q = 0$)	Earlier ($z_{\rm IDE} > z_{\Lambda\rm CDM}$)	Later ($z_{\rm IDE} < z_{\Lambda\rm CDM}$)
Matter-dark energy equality	Earlier ($z_{\rm IDE} > z_{\Lambda\rm CDM}$)	Later ($z_{\rm IDE} < z_{\Lambda\rm CDM}$)

Interacting dark energy: clarifying the cosmological implications and viability conditions

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Abstract. In this study, cosmological models are considered, where dark matter and dark energy are coupled and may exchange energy through non-gravitational interactions with one other. These interacting dark energy (IDE) models have previously been introduced to address problems with the standard ΛCDM model of cosmology (which include the coincidence problem, Hubble tension and S_8 discrepancy). However, conditions ensuring positive energy densities have often been overlooked. Assuming two different linear dark energy couplings, $Q = \delta H \rho_{\rm de}$ and $Q = \delta H \rho_{\rm dm}$, we find that negative energy densities are inevitable if energy flows from dark matter to dark energy (iDMDE regime) and that consequently, we should only seriously consider models where energy flows from dark energy to dark matter (iDEDM regime). To additionally ensure that these models are free from early time instabilities, we need to require that dark energy is in the 'phantom' ($\omega < -1$) regime. This has the consequence that model $Q = \delta H \rho_{\rm dm}$ will end with a future big rip singularity, while $Q = \delta H \rho_{\rm de}$ may avoid this fate with the right choice of cosmological parameters.

Keywords: dark energy theory, dark matter theory, cosmology of theories beyond the SM

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As illustration

$Q = \xi H \rho_{DM}$

$Q = \eta H \rho_{DE}$

$Q = \xi H (\rho_{DE} + \rho_{DM})$

$\vdots$

□ Spatial Part $\nu = 1,2,3$ $\longrightarrow$ Euler Eqn
(Momentum Conservation)

Recall GR: According to Spatial Projector $\underset{\sim}{\perp^{\nu}_{\sigma}}$

$$\perp^{\nu}_{\sigma} \equiv \delta^{\nu}_{\sigma} - U^{\nu} U_{\sigma}$$

$$\perp^{\nu}_{\sigma} T^{\mu}_{\nu;\mu} = 0 \longrightarrow \text{Euler Equation}$$

(Momentum Conservation)

$(-,+,+,+) \longrightarrow$ $$\boxed{T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U^{\mu} U^{\nu} + P g^{\mu\nu}}$$

$$U^{\nu} U_{\nu} = -1$$

$c = 1$

$$\boxed{(\rho + P) U^{\mu} U^{\nu}_{,\mu} + P_{,\nu} + U^{\nu} U^{\mu} P_{,\mu} = 0} \quad *$$

Classical Regime $\begin{cases} U^{\mu} \sim (1, \vec{V}) \\ {}_{,\mu} = \partial_{\mu} \\ \rho \gg P \end{cases}$

$$\boxed{\rho \left(\partial_t \vec{V} + (\vec{V} \cdot \vec{\nabla}) \vec{V} \right) = - \vec{\nabla} P}$$

$\vec{F} = ma$

Plays as the role of Force

In the more common $(-,+,+,+)$ signature:

- The energy-momentum tensor is $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}$.
- The orthogonal projector is $\perp^\nu_\sigma = \delta^\nu_\sigma + u^\nu u_\sigma$ (since $u^\nu u_\nu = -1$).

$$(\rho + p)u^\mu \nabla_\mu u^\nu + \nabla^\nu p + u^\nu u^\mu \nabla_\mu p = 0.$$

$$(\rho + p)u^\mu \nabla_\mu u^\nu + \perp^{\nu\mu} \nabla_\mu p = 0.$$

Define the **orthogonal projector** (in $(+,-,-,-)$ signature):

$$\perp^\nu_\sigma = \delta^\nu_\sigma - u^\nu u_\sigma.$$

$u^\nu u_\nu = +1$

Apply it to $\nabla_\mu T^{\mu\nu} = 0$:

$$\perp^\nu_\sigma \nabla_\mu T^{\mu\sigma} = (\rho + p)u^\mu \nabla_\mu u^\nu - \perp^{\nu\mu} \nabla_\mu p = 0.$$

This yields the **relativistic Euler equation**:

$$(\rho + p)u^\mu \nabla_\mu u^\nu = \perp^{\nu\mu} \nabla_\mu p.$$

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu},$$

For $v \ll c$ and weak gravity:

- $u^\mu \approx (1, \mathbf{v})$,
- $\nabla_\mu \approx \partial_\mu$,
- $\rho \gg p$,
- The spatial components reduce to the classical Euler equation:

$$\rho\,(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p.$$

For N-Body System $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$

③ Statistical Mechanics of Ideal Gas in Expanding Universe and at Thermodynamical Equilibrium

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & a^2(t) & & \\ & & a^2(t) & \\ 0 & & & a^2(t) \end{pmatrix} \qquad (-1,+,+,+)$$

$$\underbrace{\rho(\bar{q},\bar{p})}\frac{d^{3N}q\, d^{3N}p}{h^{3N}} \equiv$$ Probability of finding system at $(\bar{q},\bar{p})$ and $(\bar{q}+d\bar{q}, \bar{p}+d\bar{p})$ in its phase space.

- Micro_Canonical Ensemble $\quad \langle \mathcal{H}\rangle = cts = E$

$$\rho \sim \delta_D(\mathcal{H}-E)$$

- Canonical Ensemble: $\quad \rho \sim e^{-\beta\mathcal{H}} \qquad \beta \equiv \frac{1}{K_B T}$

$$N = cts$$

- Grand-Canonical Ensemble: $\quad \rho \sim e^{-\beta(\mathcal{H}-\mu N)}$

$$\langle f\rangle = \int \frac{d^{3N}q\, d^{3N}p}{h^{3N}} f(q,p)\, \bar{\rho}(q,p)$$

$$1 = \int \frac{d^{3N}q\, d^{3N}p}{h^{3N}} \bar{\rho}(q,p)$$

$$\frac{d}{dt}\langle f\rangle = \left\langle \frac{\partial f}{\partial t}\right\rangle + \int \frac{d^{3N}q\, d^{3N}p}{h^{3N}} f \frac{\partial \bar{\rho}}{\partial t}$$

Liouville theorem $\longrightarrow \quad \frac{\partial \rho}{\partial t} = -\{\rho, \mathcal{H}\}$

Von-Neumann $\longrightarrow \quad i\hbar\dot{\rho} = -[\rho, \mathcal{H}]$

$\Longrightarrow \underline{\underline{\rho(\mathcal{H})}}$

In Equilibrium

- State of N-Body System

$$|\psi_N\rangle = |K_1, K_2 \ldots K_N\rangle \triangleq |\overbrace{n_1, n_2, \ldots\ldots}^{\infty \text{ state}}\rangle$$

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$$N = \sum_{i=1}^{\infty} n_i \qquad \underbrace{n_1 > n_2 > n_3 \cdots}$$

In General Case

Total Energy $\quad E = \langle H \rangle = \sum_{i=1}^{N} \langle h_i \rangle = \sum_{i=1}^{N} \epsilon_{K_i}$

Energy of ith Part

(Non-Intractive Regime) $\quad = \sum_{K=1}^{\infty} n_K \epsilon_K$

$$\boxed{\epsilon_{K_i} = \text{energy of ith particle}}$$

n_K: The number of particles at Kth state

$$n_K = \begin{cases} 0, 1 & \text{For Fermi-Dirac} \\ 0, 1, 2, \ldots & \text{For Bose-Einstein}, \ \underline{MB} \end{cases}$$

$$\begin{cases} Z^{MB} = \prod_{K=1}^{\infty} e^{g_K e^{-\beta(\epsilon_K - \mu)}} \\ Z^{FD} = \sum_{\{n_K\}} \left(\prod_{K=1}^{\infty} \frac{g_K!}{n_K!(g_K - n_K)!} \right) e^{-\beta \sum_{K=1}^{\infty} n_K(\epsilon_K - \mu)} \\ Z^{BE} = \sum_{\{n_K\}} \left(\prod_{K=1}^{\infty} \frac{(n_K + g_K - 1)!}{n_K!(g_K - 1)!} \right) e^{-\beta \sum_{K=1}^{\infty} n_K(\epsilon_K - \mu)} \end{cases}$$

$$\boxed{PV = K_B T \ln Z} = \frac{1}{a} \sum_{K=1}^{\infty} \ln\left(1 + a z e^{-\beta \epsilon_K}\right)$$

$$a = \begin{cases} +1 & \text{FD} \\ 0 & \text{MB} \\ -1 & \text{BE} \end{cases}$$

$$\boxed{z \equiv e^{\beta\mu}}$$

$$U = E = \langle H \rangle = \left\langle \sum_{k=1}^{\infty} n_k \epsilon_k \right\rangle = \sum_{k=1}^{\infty} \langle n_k \rangle \epsilon_k$$

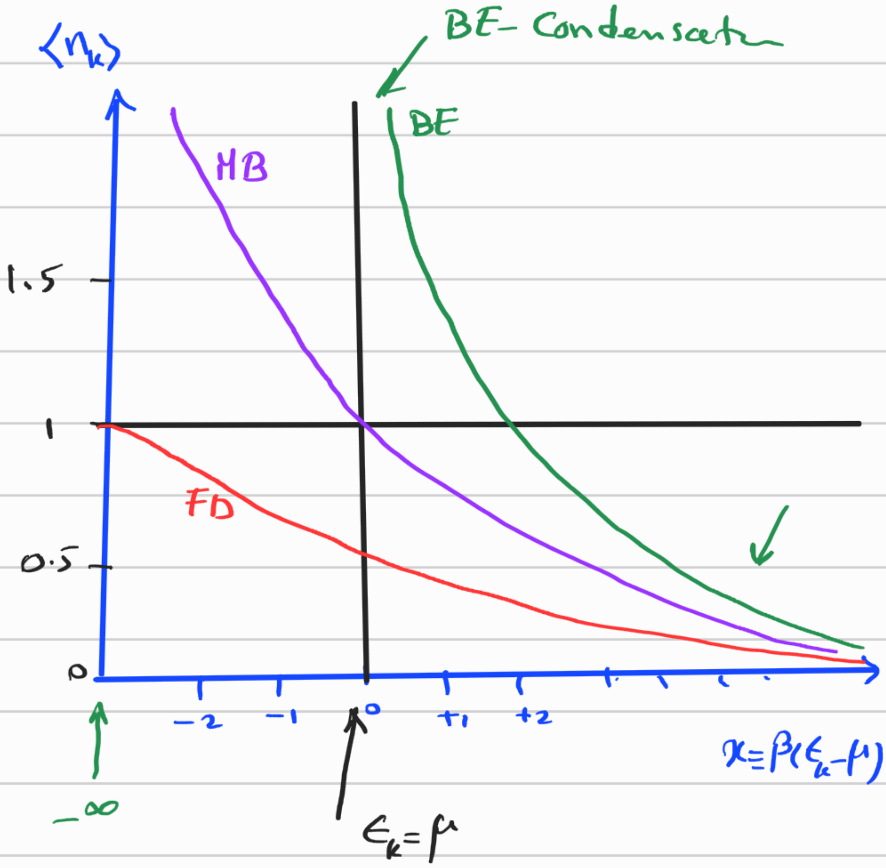
$\downarrow$

Occupation Number

$$\langle n_k \rangle^{MB} = e^{-\beta(\epsilon_k - \mu)}$$

$$\langle n_k \rangle^{BE} = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$$

$$\langle n_k \rangle^{FD} = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$



$$x = -\infty = \frac{(\epsilon_k - \mu)}{k_B T}$$