بسم الله الرحمن الرحیم

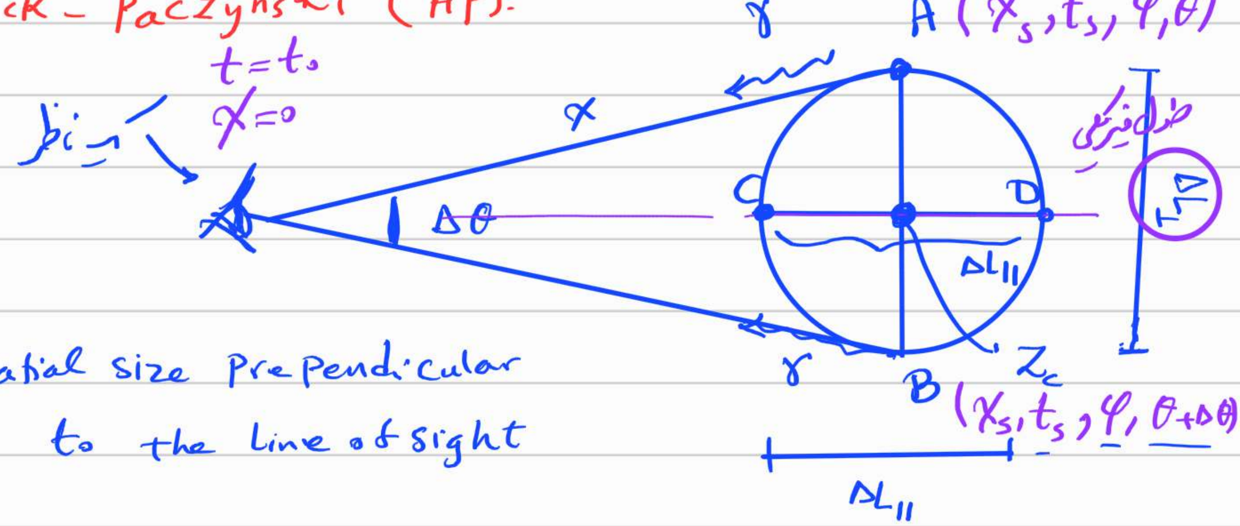
جلسه ۲۸ ، ۱۲ ، ۱۴۰۳

Continuation : Geometrical measures.

Part A:

source: physical Foundation of cosmology: By Mukhanov

⑥ Alcock - Paczynski (AP).



$\Delta L_\perp$: Spatial size prependicular to the line of sight

$\Delta L_{\parallel}$: Size along the line of sight

Z_c = Redshift at extended source

$ds^2 = 0\,,\; d\chi^2 = 0\,,\; d\varphi^2 = 0$

Physical size

$$\Delta L_\perp = \frac{S(\chi)\,\Delta\theta}{(1+z_c)} \longrightarrow \frac{\Delta L_\perp}{\Delta\theta} = \frac{S(\chi)}{1+z_c} = d_A$$

$ds^2 = 0,\; d\Omega^2 = 0$

$$d\chi = \frac{dt}{a(t)} = \frac{dz}{H(z)}$$

$$\boxed{\Delta\chi = \frac{\Delta z}{H(z)}\Big|_{z = z_c}}$$

$$\Delta L_{\parallel} = \frac{\Delta\chi(z)}{(1+z_c)} = \frac{\Delta z}{(1+z_c)\,H(z_c)}$$

$$1 = \frac{\Delta L_\perp}{\Delta L_{\parallel}} \Rightarrow \boxed{\frac{\Delta z}{\Delta\theta} = H(z)\,S(\chi)} \;*$$

we suppose that

observation ⟷ Theory

★ But in observation we measure following quantities

$$\underline{v_{\parallel}} = v_H + \underline{v_{pec}}$$

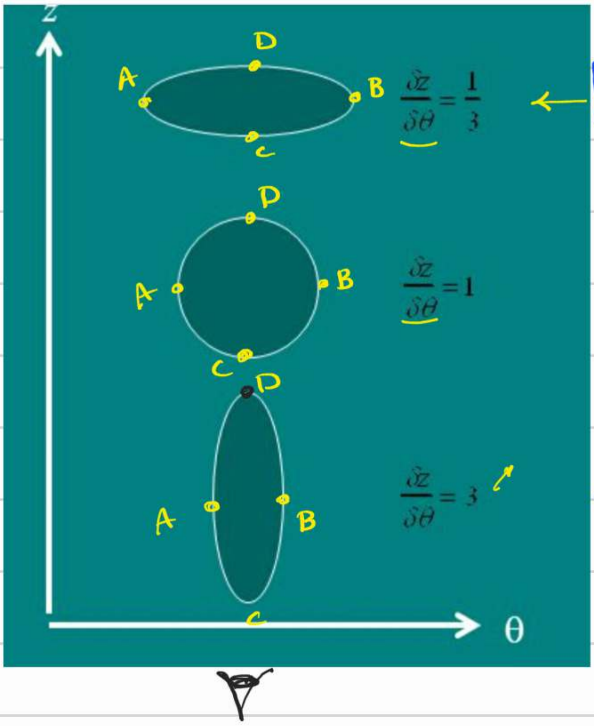
$\Delta v_{pec} \sim 0 \leftarrow$ we assume that

$$\Delta v_{\parallel} = \Delta v_H + \Delta v_{pec} \sim \Delta v_H = \frac{H(z)\,\Delta\chi}{(1+z)} = \frac{\cancel{H(z)}}{1+z_c}\,\frac{\Delta z}{\cancel{H(z)}}$$

$$\Delta v_{\perp} = H(z)\,\Delta L_{\perp}$$

$$\frac{H(z)\,\Delta\chi}{1+z} = \frac{H(z)\,S(\chi)\,\Delta\theta}{1+z}$$

we assume that $\Delta v_{\parallel} = \Delta v_{\perp}$



$\leftarrow$ FoG Effect

$\leftarrow$ Kaiser Effect

$$\boxed{\frac{\Delta z}{\Delta\theta} = H(z)\,S(\chi)}$$

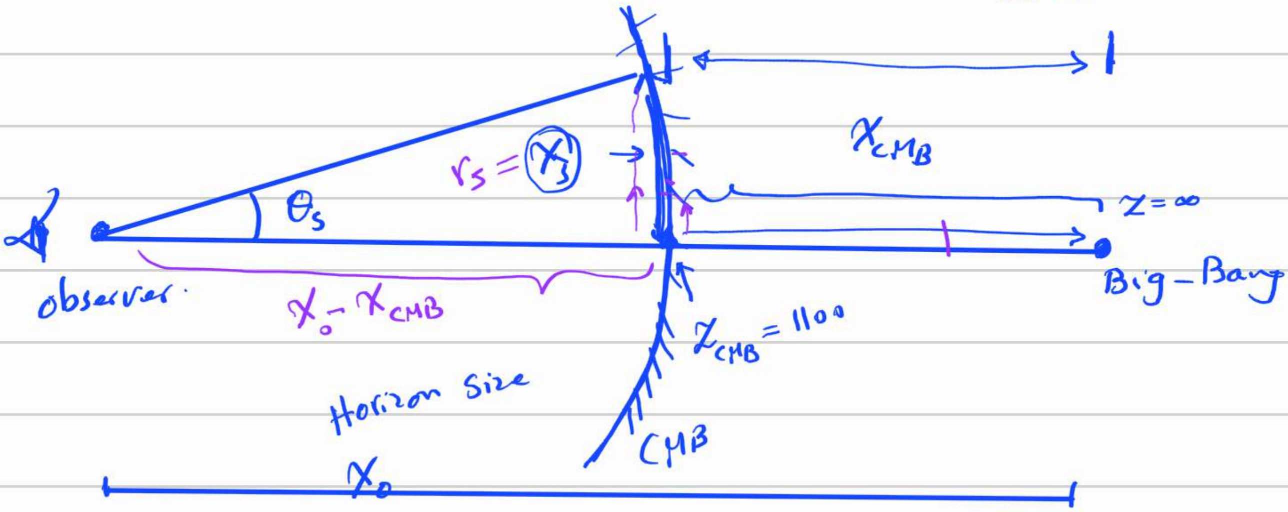
This is another observable quantities for Theory-Based Data modeling.

⑦ CMB Shift Parameter

$\leftarrow$ this is related to the acoustic peak of CMB.

قله صوتی $\leftarrow$

z_{CMB}



$v_s \neq c$ $\quad \boxed{v_s^2 = \frac{1}{3} c^2}$ $\quad v_s^2 = \frac{\partial P}{\partial \rho}$

$$\chi_s = \int_{z_{CMB}}^{\infty} \frac{v_s \, dz}{H(z)}$$

افق موج صوتی

Acoustic Horizon

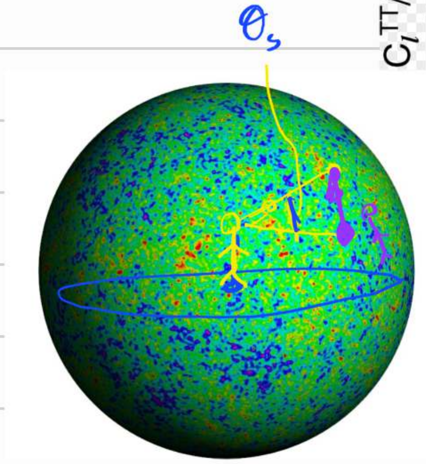
$$\boxed{S(\chi_0 - \chi_{CMB})\, \theta_s = \chi_s}$$

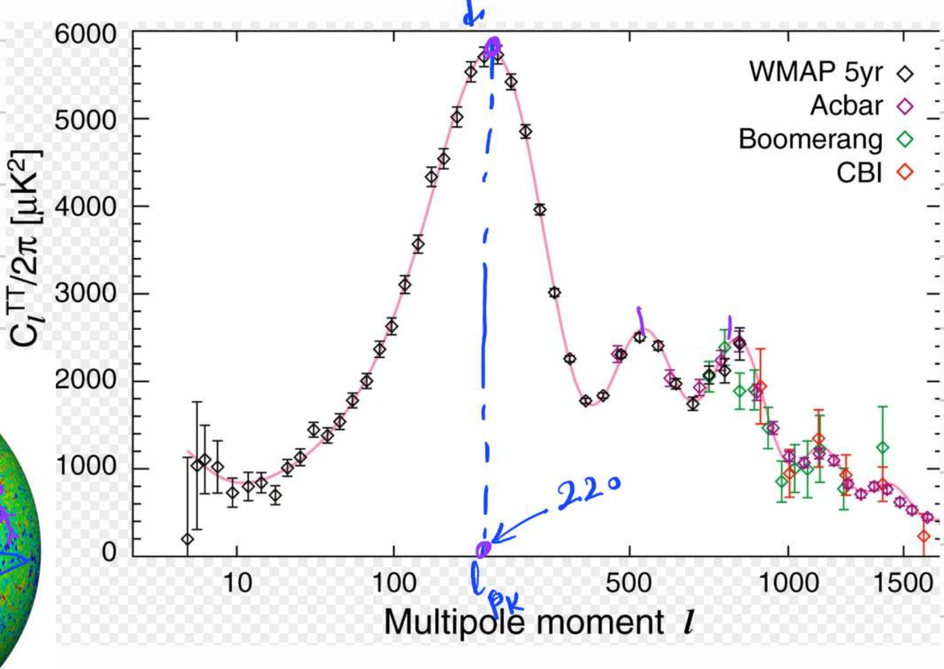
$$\theta_s = \frac{\chi_s}{S(\chi_0 - \chi_{CMB})}$$

r_s

$$\theta_s = \frac{\int_{z_{CMB}}^{\infty} \frac{v_s \, dz}{H(z)}}{\frac{c}{H_0} \frac{1}{\sqrt{|k|}} \sin\left[\sqrt{|k|} \int_0^{z_{CMB}} \frac{dz}{H(z)}\right]}$$

$l_s = \frac{\pi}{\theta_s}$ $\quad$ Acoustic Peak $\quad \theta_s \sim \frac{\pi}{220} \times \frac{180}{\pi} = 0.8^\circ$





To do Cosmological Inference Robustly:

$K=0 \qquad \Omega_r, \Omega_m \qquad \Omega_m = 1 - \Omega_r$

$$* \ \tilde{\theta}_s^{Flat} = \frac{v_s}{c}\underbrace{\left(\sqrt{\Omega_r + \frac{1}{1+z_{CMB}}} - \sqrt{\Omega_r}\right)}_{} \ * \quad \boxed{\text{CMB - Shift Parameter}}$$

$$\Gamma' \equiv \frac{\theta_s^{obs}}{\theta_s^{Flat}} , \qquad R \equiv \frac{1}{\Gamma' A} \ *$$

$\Omega_r \simeq$ ولی $K=0$ و θ_s^{fl}

$$* \ A \equiv \frac{\sqrt{\Omega_r + \frac{1}{1+z_{CMB}}} - \sqrt{\Omega_r}}{2\sqrt{\frac{\Omega_r}{\Omega_m} + \frac{1}{1+z_{CMB}}} - \sqrt{\frac{\Omega_r}{\Omega_m}}} \ *$$

$$R = \frac{\sqrt{\Omega_m}}{1+z_{CMB}} \, d_L(z_{CMB}) = 1.1716 \pm 0.0062$$

WMAP.

$$R_{Markov} \equiv 1.483^{+0.132}_{-0.103} \ *$$

$$\theta_{Pen} = \frac{v_s \int_{z_{CMB}}^{\infty} \frac{dz}{H(z)}}{\frac{c}{H_0}\frac{1}{\sqrt{|k|}} \sin\left[\sqrt{|k|}\int_0^{z_{CMB}} \frac{dz}{H(z)}\right]} = \quad , \quad H^2 = H_0^2\left[\Omega_r a^{-4} + \Omega_m a^{-3} - (\Omega_{tot} - 1)a^{-2}\right]$$

$$\theta_{Pen}^{Flat} = \frac{v_s}{c}\left(\sqrt{\Omega_r + \frac{1}{1+z_{cm}}} - \sqrt{\Omega_r}\right) \longrightarrow \boxed{H^2 = H_0^2\left[\Omega_r a^{-4} + \overbrace{(1-\Omega_r)}^{\Omega_m} a^{-3}\right]}$$

$\Omega_{total} = 1$

8 Baryon Acoustic oscillation (BAO)

$$\underline{\beta} \equiv \frac{D_V(\widetilde{z = 0.35})}{D_V(\underset{\sim}{z = 0.2})} , \quad D_V \equiv \left[\frac{(1+z)^2 d_A^2 \, cz}{H(z)}\right]^{1/3}$$

$\beta_{phs} = 1.812 \pm 0.06$ *

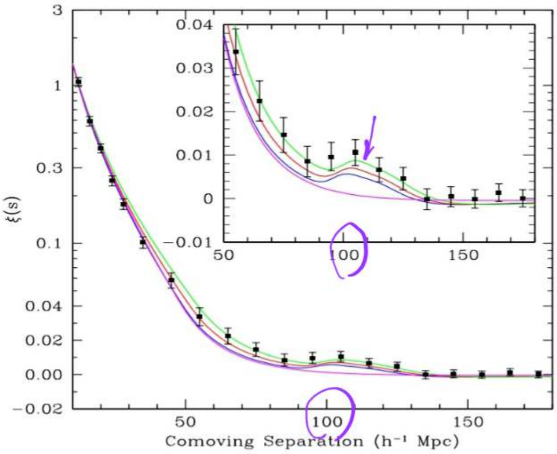


Fig. 1.1. The Baryon Acoustic Peak (BAP) in the correlation function – the BAP is visible in the clustering of the SDSS LRG galaxy sample, and is sensitive to the matter density (shown are models with $\Omega_m h^2 = 0.12$ **(top)**, 0.13 **(second)** and 0.14 **(third)**, all with $\Omega_b h^2 = 0.024$). The bottom line without a BAP is the correlation function in the pure CDM model, with $\Omega_b = 0$. From Eisenstein *et al.*, 2005 (52).

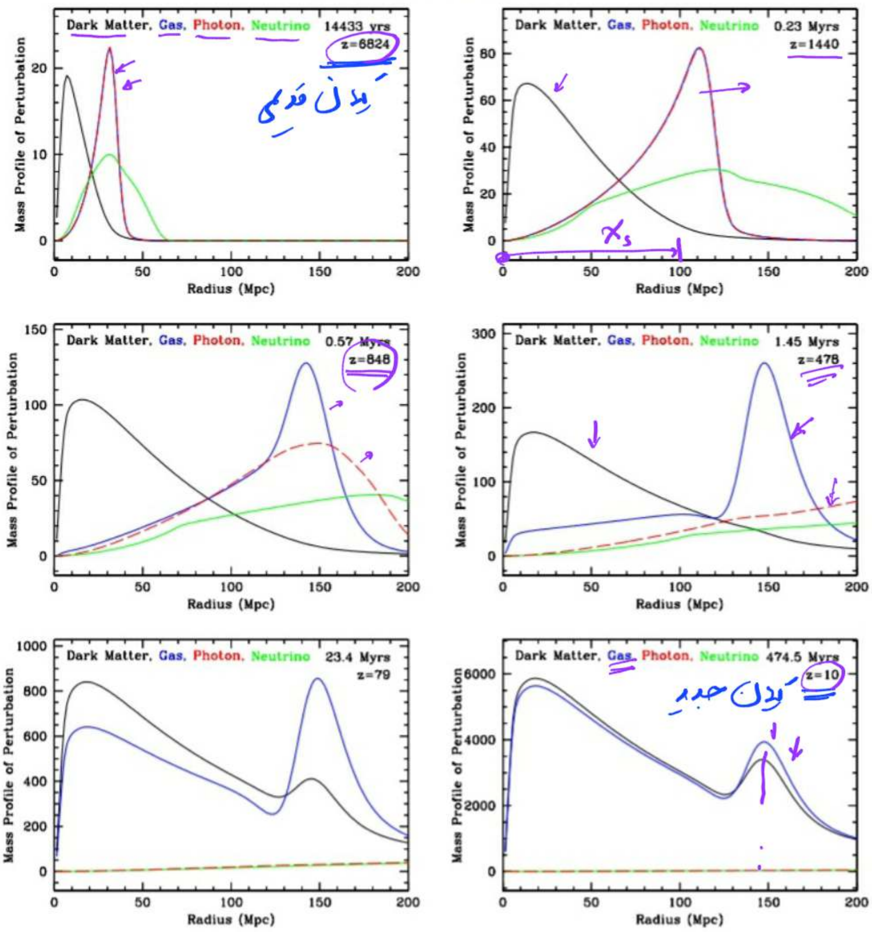


Fig. 1.10. **Snapshots of an evolving spherical density perturbation** – the radial mass profile as a function of comoving radius for an initially point-like overdensity located at the origin. The perturbations in the dark matter (black curve), baryons (blue curve), photons (red) and neutrinos (green) evolve from early times ($z = 6824$, top left) to long after decoupling ($z = 10$, bottom right). Initially the density perturbation propagates through the photons and baryons as a single pulse (top left-hand panel). The drag of the photons and baryons on the dark matter is visible in the top right panel; the dark matter only interacts gravitationally and therefore its perturbation lags behind that of the tightly coupled plasma. During recombination, however, the photons start to "leak" away from the baryons (middle left panel); and once recombination is complete ($z = 470$, middle right) the photons freely steam away leaving only a density perturbation in the baryons around 150Mpc, and a dark matter perturbation near the origin. In the bottom two panels we see the how the gravitational interaction between the dark matter and the baryons affects the peak: dark matter pulls the baryons to the peak in the density near zero radius, while the baryons continue to drag the dark matter overdensity towards the 150Mpc peak (bottom left), finally yielding a peak in the radial mass profile of the dark matter at the scale set by the distance the baryon-photon acoustic wave could have travelled in the time before recoupling. Figure taken from Eisenstein *et al.* (50).

astro-ph/0910.5224

Baryon Acoustic Oscillations from galaxy surveys

Baryon Acoustic Oscillations from galaxy surveys

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Abstract: We conducted a review of the fundamental aspects of describing and detecting the Baryon Acoustic Oscillation (BAO) feature in galaxy surveys, emphasizing the optimal tools for constraining this probe based on the type of observation. Additionally, we included new results with two spectroscopic datasets to determine the best-fit model for the power spectrum, $P(k)$. Using the framework described in a previous analysis, we applied this to a different sub-sample of the BOSS survey, specifically galaxies with redshifts $0.3 < z < 0.65$. We also examined the eBOSS dataset with redshifts $0.6 < z < 1.0$, adjusting the number of parameters in the traditional polynomial fit to account for the higher redshift range. Our results showed that the dilation scale parameter α derived from the BOSS dataset had smaller error bars compared to the eBOSS dataset, attributable to the larger number of luminous red galaxies (LRGs) in the BOSS sample. We also compared our findings with other surveys such as WiggleZ, DES Y6, and DESI III, noting that photometric surveys typically yield larger error bars due to their lower precision. The DESI III results were in good agreement with ours within 1σ, with most bins close to unity. The variation of α with respect to the redshift is an unresolved issue in the field, appearing in both three-dimensional and angular tomographic analyses.

Keywords: *Observational Cosmology, BAO, LSS, galaxy surveys*

1 Introduction

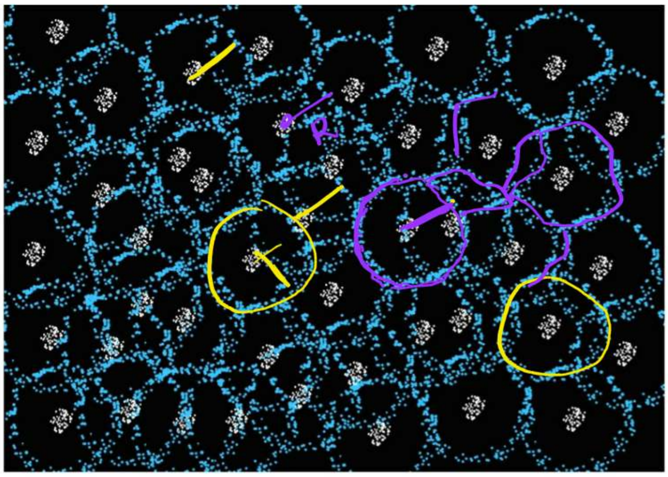


Figure 1: BAO representation from point sources. Each point can be thought of as a galaxy. The blue ones are found in the BAO feature, while the white ones are clustered due to Dark Matter after decoupling

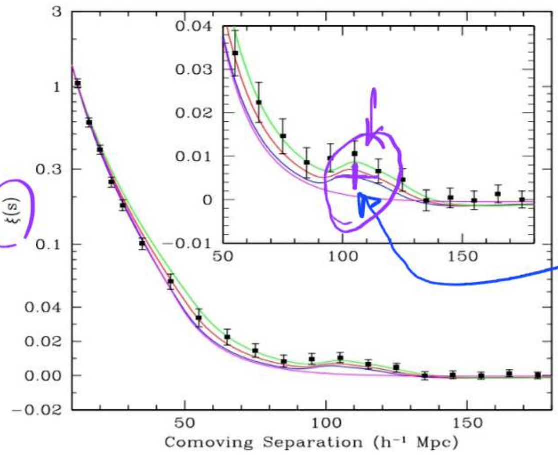
$$\xi_{gg} = \langle \delta_g(r)\, \delta_g(r') \rangle$$

$$R = |r - r'|$$

احتمال خوشه شدن در

$R \sim 150\,\mathrm{Mpc}$

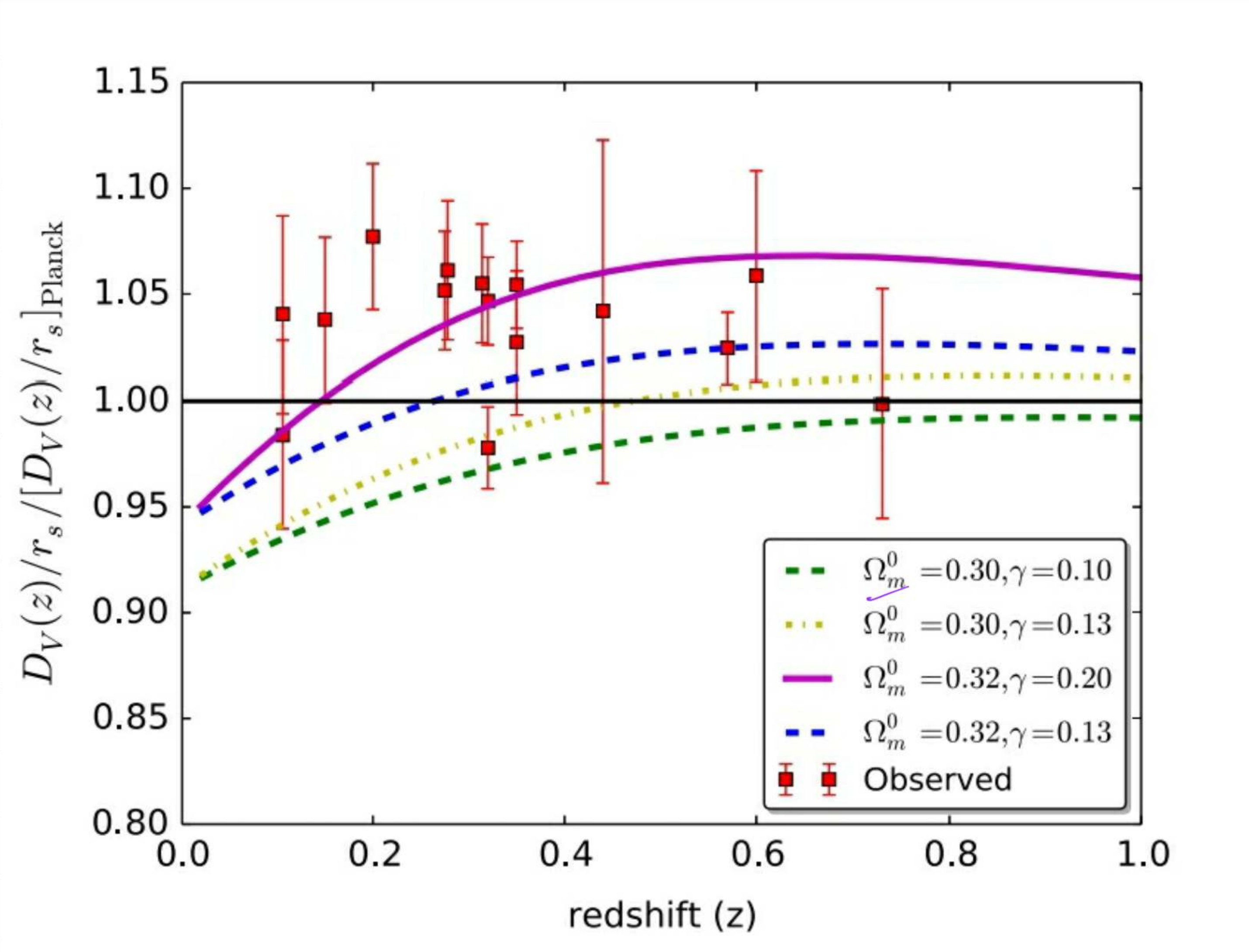
بیشتر است.



$$\delta_g = b\, \delta_m$$

Bias Factor

Fig. 1.1. The Baryon Acoustic Peak (BAP) in the correlation function – the BAP is visible in the clustering of the SDSS LRG galaxy sample, and is sensitive to the matter density (shown are models with $\Omega_m h^2 = 0.12$ **(top)**, 0.13 **(second)** and 0.14 **(third)**, all with $\Omega_b h^2 = 0.024$). The bottom line without a BAP is the correlation function in the pure CDM model, with $\Omega_b = 0$. From Eisenstein *et al.*, 2005 (52).



Where r_s is Sound Horizon at CMB

$$D_V(z;\{\Theta_p\}) = \left[(1+z)^2 D_A^2(z;\{\Theta_p\})\frac{cz}{H(z;\{\Theta_p\})}\right]^{1/3}, \tag{49}$$

Model's Free Parameter

$$d_{\rm BAO}(z;\{\Theta_p\}) \equiv \frac{r_s(z;\{\Theta_p\})}{D_V(z;\{\Theta_p\})}, \tag{50}$$

$$r(z;\{\Theta_p\}) = \int_0^z \frac{dz'}{H(z';\{\Theta_p\})}, \tag{31}$$

9 Cosmographics quantities:

$$j \equiv \frac{1}{a H^3} \frac{d^3 a}{dt^3}, \quad s \equiv \frac{1}{a H^4} \frac{d^4 a}{dt^4},$$
$$l \equiv \frac{1}{a H^5} \frac{d^5 a}{dt^5}, \quad m \equiv \frac{1}{a H^6} \frac{d^6 a}{dt^6}. \tag{36}$$

these parameters are called *jerk*, *snap*, *lerk* and *maxout*,

$$\begin{aligned}
\dot{H} &= -H^2(1+q),\\
\ddot{H} &= H^3(j+3q+2),\\
\dddot{H} &= H^4[s-4j-3q(q+4)-6],\\
\ddddot{H} &= H^5[l-5s+10(q+2)j+30(q+2)q+24],\\
H^{(5)} &= H^6\{m-10j^2-120j(q+1)\\
&\quad -3[2l+5(24q+18q^2+2q^3-2s-qs+8)]\}.
\end{aligned} \tag{37}$$

For ΛCDM model. and Flat Univer

$$H^2 = H_0^2\left[\Omega_r (1+z)^4 + \Omega_m (1+z)^3 + \Omega_\Lambda\right]$$

also we have $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_{total}$

First F. Eq

$q \equiv -\frac{\ddot{a} a}{\dot{a}^2}$ is Deceleration parameter

$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \leftarrow$ Second F. Eq

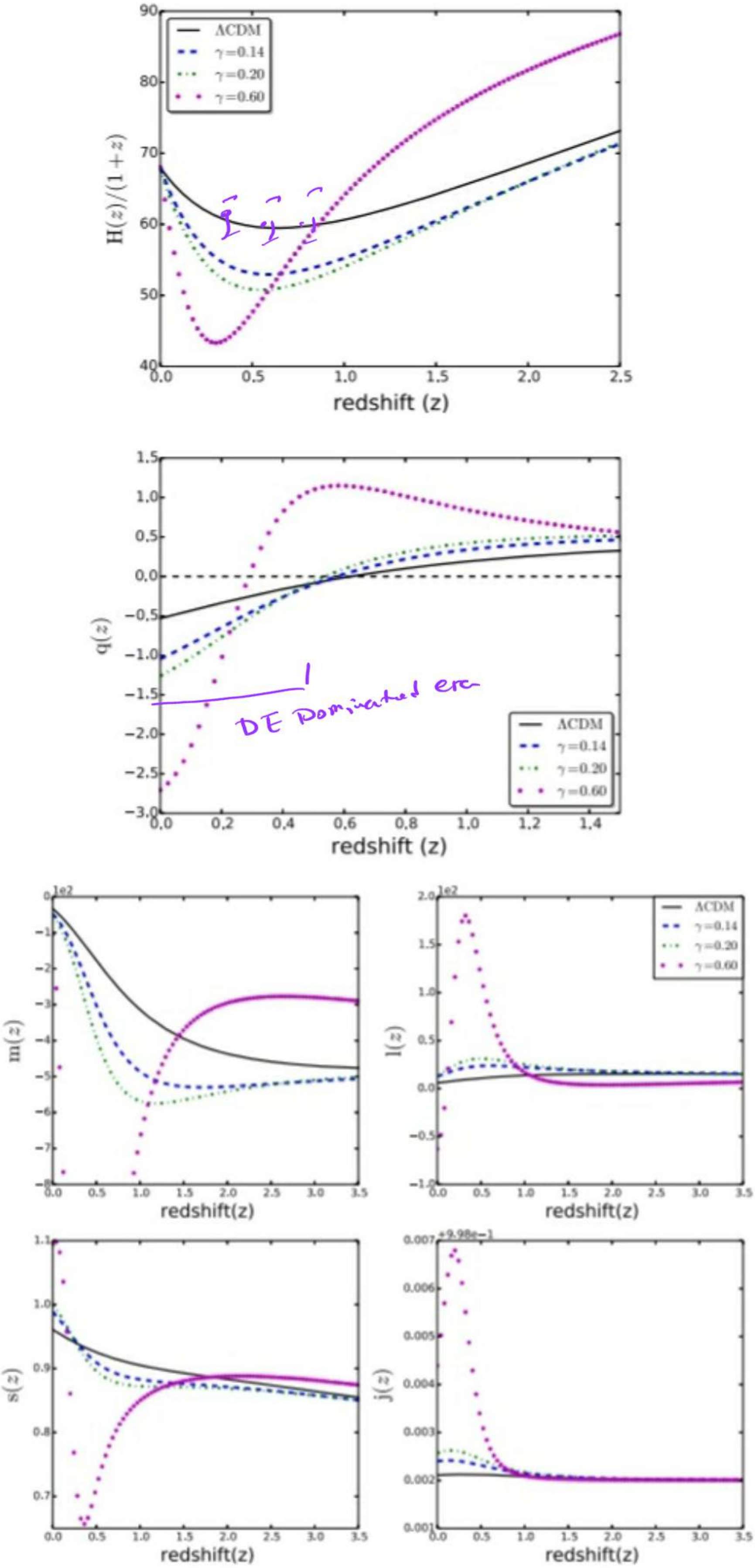
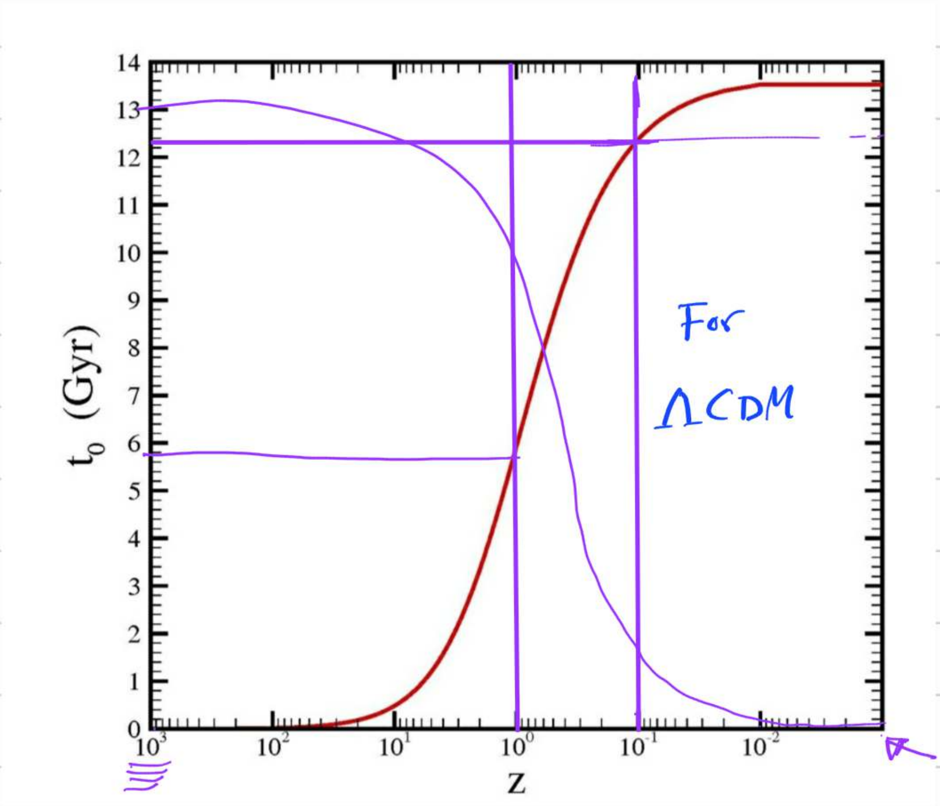


Fig. 9 *Upper panel* the $\frac{H(z)}{(1+z)}$ as a function of the redshift. *Middle panel* deceleration probe diagnostic. *Lower panel jerk*, *snap*, *lerk* and *maxout* parameters for the bulk viscous model with respect to ΛCDM model. *Solid lines* represents corresponding quantity for ΛCDM. *Other lines* are associated with different values for viscosity. The rest of free parameters have been fixed according to JLA observation at 1σ confidence level

10 Age of Univers

$$t = \int_0^t dt' = \int_0^a \frac{dt}{da'}\, da' = \int \frac{da}{a\,\dot{a}/a} = \int_0^z \frac{dz'}{(1+z')H(z')}$$



observer

Age Crisis

$$\tau(z_i; \{\Theta_p\}) = \frac{t(z_i; \{\Theta_p\})}{t_{obs}(z_i)}, \quad i = 1 \cdots 42,$$

We expect That $\tau \geq 1$ To have consistent Results.

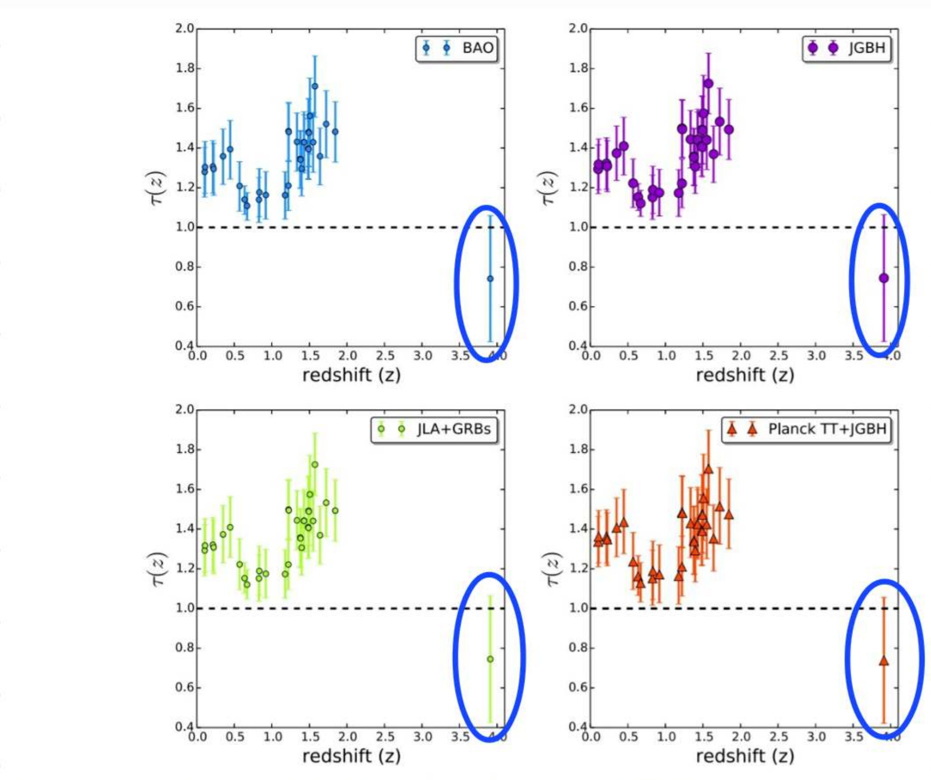
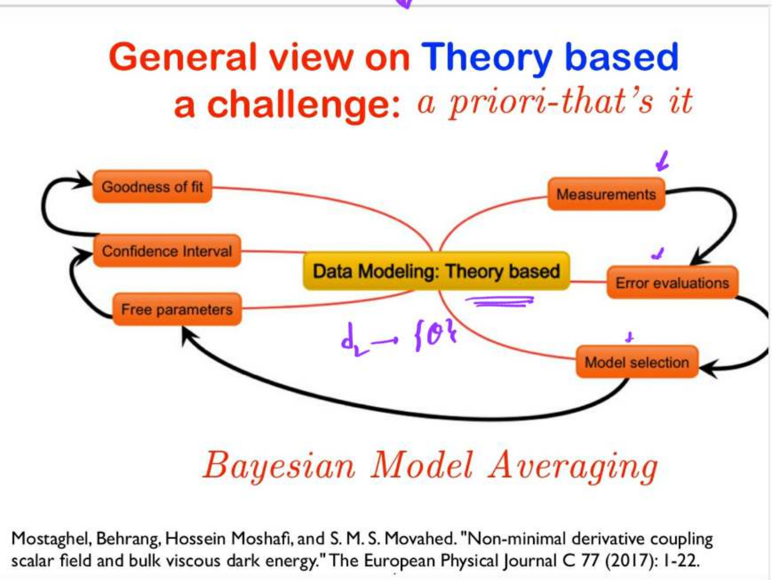
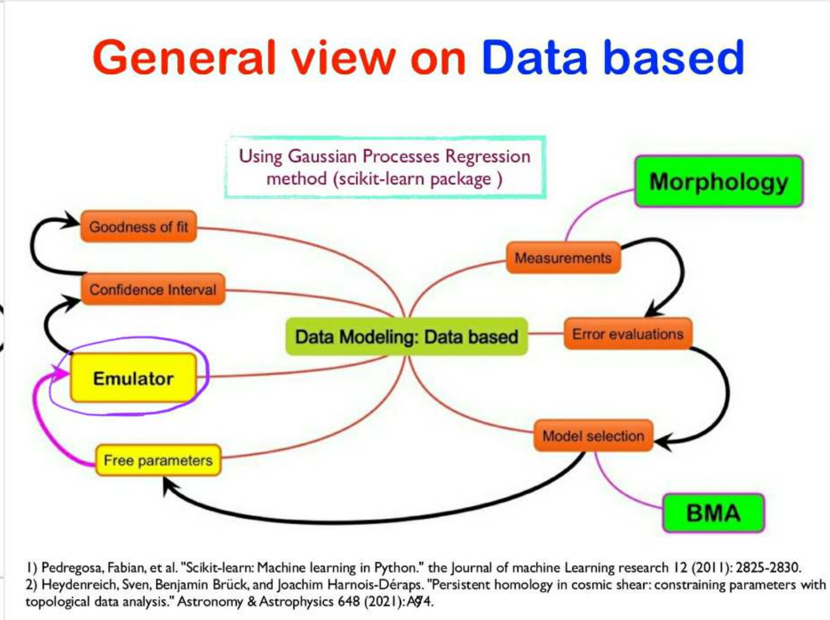


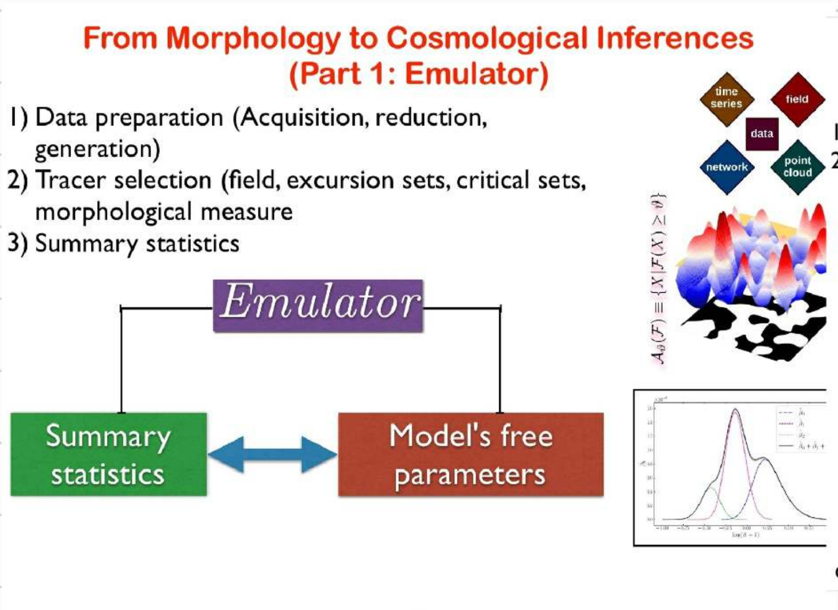
Fig. 19 τ as a function of the redshift for 32 old objects. Data have been given from [96,97,99]

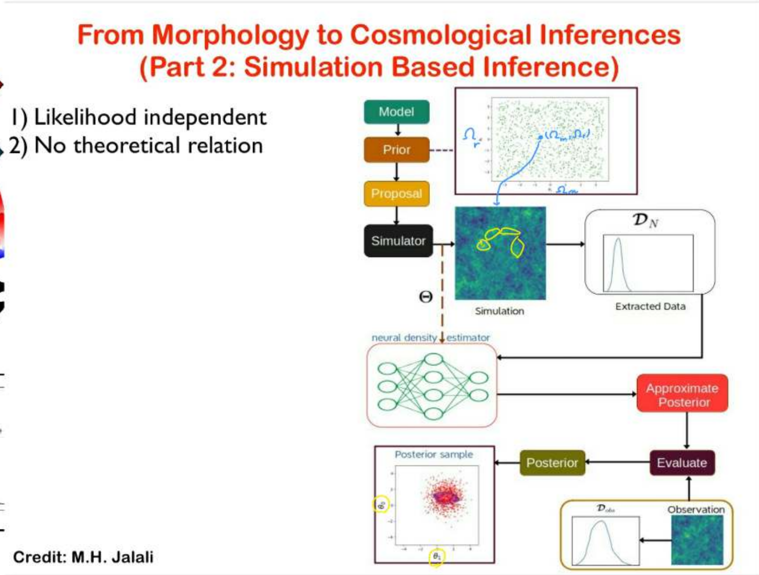
But for some astronomical objects we obtain that $t_{obs} > t_{Univers}$. This means that The associated object was created before The existence of Univers at given Z.

Reminder : Theory_Based VS. Data_Based approaches









Part B: Dynamical aspect of Background of Universe

2.3: Evolution of Energy (Dodelson's Book)

& Chapter 2: Cosmology Daniel Baumann

Previously we examine the Kinematics of Expanding Universe, Now we turn to Dynamical aspect of expanding Universe.

Ⓐ Matter Distribution $\longrightarrow$ Gravitational field

Ⓑ Gravitational field $\longrightarrow$ Behavior of Matter

$$\vec{a} = \frac{\vec{F}}{m}$$

$$\downarrow$$

$$\vec{a}_g = -\vec{\nabla}\phi \quad \leftarrow \text{Newtonian approach}$$

$$\downarrow$$

What about Covariant Form ?

$$\nabla^2\phi = +4\pi G\rho \quad \leftarrow \text{Euclidean space.}$$

classical Mechanic

Recall that: chapter 3. A short course in GR By: James Foster

$t \equiv$ Coordinate time

$\tau \equiv$ Proper time

$$\gamma \equiv \frac{dt}{d\tau} = (1 - v^2/c^2)^{-1/2}$$

$v^\mu = u^\mu/\gamma$

$$v^\mu = \frac{dx^\mu}{dt} \quad : \text{ Coordinate Velocity.}$$

$$u^\mu = \frac{dx^\mu}{d\tau} \quad : \text{ World Velocity.}$$

$$p^\mu = \text{4-Momentum} = m_0 u^\mu = \gamma m_0 v^\mu$$

$$= (E/c, \vec{P})$$

We Introduce Energy-Momentum tensor

For prefect fluid $\rightarrow$ $T^{\mu\nu} = (\rho - P/c^2) u^\mu u^\nu + P \eta^{\mu\nu}$

$$\eta \equiv \begin{pmatrix} -1 & & & \\ & +1 & & 0 \\ & 0 & +1 & \\ & & & +1 \end{pmatrix}$$

Some properties of $T^{\mu\nu}$:

Symmetric $T^{\mu\nu} = T^{\nu\mu}$

$$T^{\mu\nu} u_\nu = c^2 \rho u^\mu$$

$$T^{\mu\nu}{}_{;\mu} = 0 \quad \longrightarrow$$

$\uparrow$ Covariant Derivative

2.3.1 Perfect Fluids

We have seen how the spacetime geometry of the universe is constrained by homogeneity and isotropy. Now, we will determine which types of matter are consistent with these symmetries. We will find that the coarse-grained energy-momentum tensor is required to be that of a **perfect fluid**,

$$\boxed{T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U_\mu U_\nu + P\, g_{\mu\nu}}\,, \tag{2.85}$$

where ρc^2 and P are the energy density and the pressure in the rest frame of the fluid, and U^μ is its four-velocity relative to a comoving observer.

$\eta : (-1, +1, +1, +1)$

$$(\rho u^\mu)_{,\mu} u^\nu + \rho u^\mu u^\nu_{,\mu} + (P/c^2) u^\mu_{,\mu} u^\nu + (P/c^2) u^\mu u^\nu_{,\mu} + \frac{P_{,\mu}}{c^2} u^\mu u^\nu - P_{,\mu}\eta^{\mu\nu} = 0$$

General case

$$① \quad T_{\mu\nu} = \rho u_\mu u_\nu + P\gamma_{\mu\nu} + 2q_\mu u_\nu + \Pi_{\mu\nu}$$

Anisotropic Pressure Tensor

$$T^{\mu\nu}{}_{;\mu} = 0$$

3D Reminnian Metric

continuity Eq.

$$② \quad u_\nu T^{\mu\nu}{}_{;\mu} = 0$$

$$③ \quad \gamma_\nu^\lambda T^{\mu\nu}{}_{,\mu} = 0$$

Euler Eq.

$$P = \frac{T_{\mu\nu}\gamma^{\mu\nu}}{3} \leftarrow \text{Isotropic Pressure}$$

$$\rho = T_{\mu\nu} u^\mu u^\nu \leftarrow \text{Energy Density.}$$

$$q^\mu \equiv -T_{\alpha\beta} u^\alpha \gamma^{\beta\mu} \leftarrow \text{Energy Flux}$$

$$q_\mu u^\mu = 0 \qquad \Pi_{\mu\nu} u^\mu = 0$$

For Prefect Fluid $\quad \Pi_{\mu\nu} = 0 \quad q^\mu = 0$

$$T^{\mu\nu} = (\rho - P/c^2) u^\mu u^\nu + P\eta^{\mu\nu}$$

Continuity Eq: ? $\qquad$ In classical approach $\boxed{\frac{\partial\rho}{\partial t} + \bar{\nabla}\cdot(\rho\vec{v}) = 0}$

Relativistic Continuity Eq:

$$\boxed{(\rho u^\mu)_{;\mu} + (P/c^2) u^\mu{}_{;\mu} = 0}$$

?

For $\frac{P}{c^2} \to 0 \quad (\rho u^\mu)_{;\mu} = 0 \to \boxed{(\rho c)_{;0} + (\rho v^i)_{;i} = 0}$

$$u^\mu : (c, \vec{v})$$

Euler Eq: In classical Regime. $v/c \to 0$

$$\rho\left(\frac{\partial}{\partial t} + \bar{v}\cdot\bar{\nabla}\right)\bar{v} = -\bar{\nabla}\underset{\smile}{P}$$

For Dust $P=0$ $\quad \rho \frac{\partial}{\partial t}\bar{v} + (\bar{v}\cdot\bar{\nabla})\bar{v} = 0 \leftarrow$ Navier-Stokes Eq

GR: $\boxed{(\rho + P/c^2)\, u^{\nu}{}_{,\mu}\, u^{\mu} = \left(\eta^{\mu\nu} - \frac{u^{\mu}u^{\nu}}{c^2}\right) P_{,\mu} = 0}$

More about Covariant Derivative:

$\phi(x)$: scalar

$\downarrow$

$\rightarrow A_\mu \equiv \dfrac{\partial\phi}{\partial x^\mu}$: A covariant Vector

$\downarrow$

$$\underline{A'_\mu} \equiv \frac{\partial\phi'}{\partial x'^\mu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial}{\partial x^\alpha}\phi = \underline{\frac{\partial x^\alpha}{\partial x'^\mu}A_\alpha}$$

as a covariant Vector

$$M_{\mu\nu} \overset{?}{\equiv} \frac{\partial A_\nu}{\partial x^\mu},$$

is this a Covariant Tensor? *

$$\frac{\partial}{\partial x^\mu}A_\nu = \frac{\partial}{\partial x^\mu}\left[\frac{\partial\phi}{\partial x^\nu}\right] = \underbrace{\frac{\partial}{\partial x^\mu}\overbrace{\frac{\partial}{\partial x^\nu}\phi}^{A_\nu}}_{M_{\mu\nu}}$$

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\phi \longrightarrow \frac{\partial}{\partial x'^\mu}\underbrace{\frac{\partial}{\partial x'^\nu}\phi'(x')}_{A'_\nu} = \;?$$

$$\frac{\partial}{\partial x'^\mu} A'_\nu$$

$$\frac{\partial A'_\nu}{\partial x'^\mu} = \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x^\alpha}{\partial x'^\nu} A_\alpha\right)$$

$$\boxed{\frac{\partial A'_\nu}{\partial x'^\mu} = \underbrace{\frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial A_\alpha}{\partial x'^\mu}} + A_\alpha \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu}}$$

$$M_{\nu\mu} \longrightarrow M'_{\nu\mu} = \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu} M_{\alpha\beta}$$

$$\boxed{M'_{\nu\mu} = \underbrace{\frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\left(\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\phi\right)}}$$

Compare together

$$\boxed{M_{\mu\nu} = A_{\mu;\nu} = \partial_\nu A_\mu - \Gamma^\alpha_{\mu\nu} A_\alpha}$$

$$\underline{T^\mu_{\nu;\mu}} = \frac{\partial T^\mu_\nu}{\partial x^\mu} + \underbrace{\Gamma^\mu_{\alpha\mu} T^\alpha_\nu} - \underbrace{\Gamma^\alpha_{\mu\nu} T^\mu_\alpha}$$

Ex: Homogeneous and Isotropic Expanding Universe and Prefect Fluid.

$$g_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -a^2(t) & & 0 \\ & 0 & -a^2(t) & \\ & & & -a^2(t) \end{pmatrix} \qquad \left\{ \begin{array}{l} \Gamma^0_{00} = 0 \\ \Gamma^0_{0i} = 0 \\ \Gamma^0_{ij} = \delta_{ij}\, \dot{a} a \\ \Gamma^i_{0j} = \delta_{ij}\, \dot{a}/a \end{array} \right.$$

$$T^i_0 = 0$$

$$\frac{\partial T^\mu_0}{\partial x^\mu} + \Gamma^\mu_{\alpha\mu} T^\alpha_0 - \Gamma^\alpha_{0\mu} T^\mu_\alpha = 0$$

$$T^\mu_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

For $\mu = 0 \Rightarrow$ $-\frac{\partial \rho}{\partial t} - \Gamma^\mu_{0\mu}\rho - \Gamma^\alpha_{0\mu} T^\mu_\alpha = 0$

$$\boxed{\frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a}(\rho + P) = 0}$$

Continuity Eq.

فرض

$P = \omega \rho$ $\qquad \omega = cts$

☆ DM $\quad \omega = 0 \Rightarrow$ $\frac{\partial \rho_m}{\partial t} + 3\frac{\dot{a}}{a}(\rho_m + 0) = 0$

$$\frac{\partial \rho_m}{\rho_m} = -3\frac{da}{a} \longrightarrow \boxed{\rho_m \sim a^{-3}}$$

☆ Radiat $\omega = 1/3 \Rightarrow \frac{\partial \rho_r}{\partial t} + 3\frac{\dot a}{a}\left(\rho_r + \frac{1}{3}\rho_r\right) = 0$

$$\frac{\partial \rho_r}{\rho_r} = -4\frac{da}{a} \longrightarrow \boxed{\rho_r \sim a^{-4}}$$

☆ Dark Energy Λ $\omega = -1$

$$\frac{\partial \rho_\Lambda}{\partial t} + 3\frac{\dot a}{a}\left(\rho_\Lambda - \rho_\Lambda\right) = 0 \Rightarrow \boxed{\rho_\Lambda = \text{cts}}$$

☆ For a generic case such that

$$\omega = \frac{P}{\rho} \qquad \omega \neq 0$$

$$\frac{\partial \rho}{\partial t} + \frac{3\dot a}{a}\left(\rho + \omega\rho\right) = 0$$

$$\boxed{\frac{\partial \rho}{\rho} = -3\frac{da}{a}(1+\omega(a))} \longrightarrow \boxed{\rho(a) = e^{-3\int_0^a \frac{da'}{a'}(1+w(a'))}}$$

For $\omega = \text{cts}$

$$\boxed{\rho(a) \sim a^{-3(1+w)}}$$

$$\nabla_\mu T^\mu{}_\nu \equiv \frac{\partial T^\mu{}_\nu}{\partial x^\mu} + \Gamma^\mu{}_{\alpha\mu} T^\alpha{}_\nu - \Gamma^\alpha{}_{\nu\mu} T^\mu{}_\alpha = 0.$$

$$T^\mu{}_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & 0 & \mathcal{P} \end{pmatrix}$$

More about Dynamics of Universe in GR

① Einstein Proposed a tensor Eq. instead of Vector Eqn $\vec{a} = -\vec{\nabla}\phi$ given in Classical Mechanics

2.3 Dynamics

So far, we have used the symmetries of the universe to determine its geometry and studied the propagation of particles in the expanding spacetime. The scale factor $a(t)$ has remained an unspecified function of time. The evolution of the scale factor follows from the **Einstein equation**

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}} \,, \tag{2.84}$$

where $G = 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ is Newton's constant. This relates the **Einstein tensor** $G_{\mu\nu}$ (a measure of the "spacetime curvature" of the universe) to the **energy-momentum tensor** $T_{\mu\nu}$ (a measure of the "matter content" of the universe). We will first discuss the possible forms of cosmological energy-momentum tensors, then compute the Einstein tensor for the FRW background, and finally put them together to solve for the evolution of the scale factor $a(t)$ as a function of the matter content.

شکل لازم برای تانسوری که محتوای انرژی عالم را نشان دهد در ساده‌ترین حالت به صورت زیر است.

2.3.1 Perfect Fluids

We have seen how the spacetime geometry of the universe is constrained by homogeneity and isotropy. Now, we will determine which types of matter are consistent with these symmetries. We will find that the coarse-grained energy-momentum tensor is required to be that of a **perfect fluid**,

$$\boxed{T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U_\mu U_\nu + P\, g_{\mu\nu}} \,, \tag{2.85}$$

where ρc^2 and P are the energy density and the pressure in the rest frame of the fluid, and U^μ is its four-velocity relative to a comoving observer.

$$T_{\mu\nu} = \left(\begin{array}{c|c} T_{00} & T_{0j} \\ \hline T_{i0} & T_{ij} \end{array}\right) = \left(\begin{array}{c|c} \text{energy density} & \text{momentum density} \\ \hline \text{energy flux} & \text{stress tensor} \end{array}\right) . \tag{2.96}$$

① For Homogeneous Universe

Energy density must be independent from posit

$T_{00} = c^2 \rho(t)$

② For Isotropic Universe $T_{0i} = 0$ ← Three-Vectors

and three Tensor must be $T_{ij} \sim \delta_{ij}$

Therefore we have

$$T_{00} \equiv \rho(t)c^2 \,, \qquad T_{i0} \equiv c\pi_i = 0 \,, \qquad T_{ij} \equiv P(t)\, g_{ij}(t, \mathbf{x}) \,. \tag{2.98}$$

Raising one of the indices, we find

$$T^{\mu}{}_{\nu} = g^{\mu\lambda} T_{\lambda\nu} = \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} , \tag{2.99}$$

③ For Conservation, we have Two Eqs.

Ⓐ Continuity Eq. $\underbrace{\dot{\rho} = -\partial_i \pi^i}$

تحول چگالی انرژی به دیورژانس شار انرژی وابسته است.

Ⓑ Euler Eq. $\dot{\pi}_i = \partial_i P$ معادله نیرو است

The compact Form of Conservation Eqs.

$$\nabla_\mu T^\mu{}_\nu \equiv \frac{\partial T^\mu{}_\nu}{\partial x^\mu} + \Gamma^\mu{}_{\alpha\mu} T^\alpha{}_\nu - \Gamma^\alpha{}_{\nu\mu} T^\mu{}_\alpha = 0.$$