بسم الله الرحمن الرحیم   جلسه ۲۱ /۱۲/ ۱۴۰۳

☆ Spatial Part of Metric

$$ds^2 = -c^2dt^2 + a^2(t)\, d\chi^2_{3D}$$

my goal is to derive the general form of $g_{\mu\nu}$ for static and 3D Space Homogeneous Universe.

① First approach: 2D Space (spherical Surface embedded in 3D)

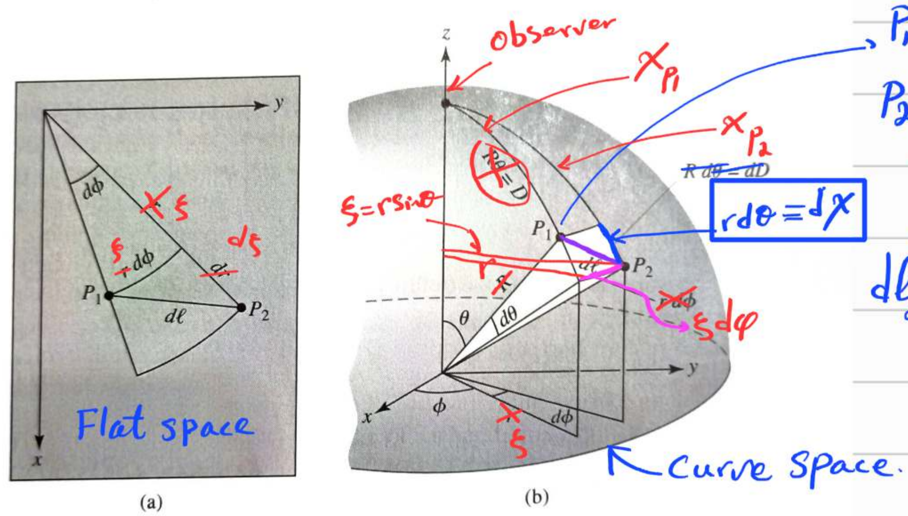


FIGURE 29.18 $d\ell$ as measured for (a) a flat plane and (b) the surface of a sphere.

$P_1(x_1, y_1, z_1)$, $|r_{P_1}| = \sqrt{x_1^2 + y_1^2 + z_1^2} = cts$

$P_2(x_2, y_2, z_2)$, $|r_{P_2}| = \sqrt{x_2^2 + y_2^2 + z_2^2} = cts$

$x^2 + y^2 + z^2 = r^2$

Due to 2D

$d\ell^2_{2D} = ?$

My Notation $R \equiv$ Physical Distance (Proper distance)

$\chi \equiv$ Comoving Distance $\boxed{R(t) = a(t)\, \chi}$ Scale Factor

$\chi = r\theta$, $\xi = r\sin\theta$

$$d\ell^2_{2D} = \lim_{P_2 \to P_1} |r_{P_1} - r_{P_2}|^2 = r^2 d\theta^2 + \xi^2 d\varphi^2$$

$\xi = r\sin\theta \longrightarrow \dfrac{d\xi}{\cos\theta} = r\,d\theta = d\chi$

$\sin\theta = \dfrac{\xi}{r}$

$\cos^2\theta = 1 - \sin^2\theta = 1 - \dfrac{\xi^2}{r^2}$

$$\boxed{d\ell^2_{2D} = \frac{d\xi^2}{\left(1 - \frac{\xi^2}{r^2}\right)} + \xi^2 d\varphi^2}$$

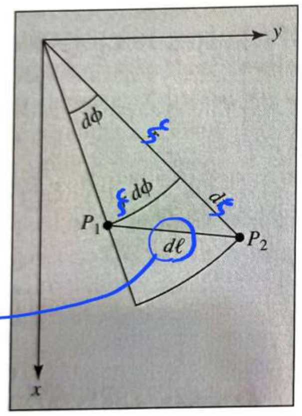
$K \equiv \dfrac{1}{r^2}$: Curvature انحنا

For 2D space $\quad dl_{2D}^2 = \dfrac{d\xi^2}{1-K\xi^2} + \xi^2 d\varphi^2$

$$\boxed{dl_{2D}^2 = dx^2 + S^2(x)\, d\varphi^2}$$

$x \longrightarrow \xi$ for $K=0$ Flat space.

$$\boxed{dl_{2D}^2 = d\xi^2 + \xi^2 d\varphi^2}$$



? $\left\{ \begin{array}{l} \text{Now 3D space (Spherical space embedded in 4D)} \\ dl_{3D}^2 = ? \end{array} \right\}$?

② ☆ Second approach 2D: $\quad dl_{2D}^2 = ?$

$$dl^2 = dx^2 + dy^2 + dz^2$$

$r^2 = x^2 + y^2 + z^2 \longrightarrow$ 2D space , $r =$ fixed

$$0 = x\,dx + y\,dy + z\,dz$$

$$z\,dz = -(x\,dx + y\,dy)$$

sphere

$$* \; dz = -\frac{x\,dx + y\,dy}{z} = \pm \frac{x\,dx + y\,dy}{\sqrt{r^2 - x^2 - y^2}}$$

Hypersphere

$x^2 + y^2 = \xi^2 \rightarrow \left\{ \begin{array}{l} x\,dx + y\,dy = \xi\, d\xi \longrightarrow (x\,dx + y\,dy)^2 = \xi^2 d\xi^2 \\ dx^2 + dy^2 = d\xi^2 + \xi^2 d\varphi^2 \end{array} \right.$

$$dl^2_{2D} = dx^2 + dy^2 + dz^2 = dx^2 + dy^2 + \frac{(x\,dx + y\,dy)^2}{r^2 - (x^2+y^2)}$$

$$= d\xi^2 + \xi^2 d\varphi^2 + \frac{\xi^2 d\xi^2}{r^2 - \xi^2}$$

$$= d\xi^2\left[1 + \frac{\xi^2}{r^2-\xi^2}\right] + \xi^2 d\varphi^2 = d\xi^2\left[\frac{r^2 - \cancel{\xi^2} + \cancel{\xi^2}}{r^2-\xi^2}\right] + \xi^2 d\varphi^2$$

$$= \frac{d\xi^2}{1 - \xi^2/r^2} + \xi^2 d\varphi^2 = \frac{d\xi^2}{1 - K\xi^2} + \xi^2 d\varphi^2$$

$$dl^2_{2D} = d\chi^2 + S^2(\chi)\, d\varphi^2$$

For 3D : $dl^2_{3D} = ?$

$$dl^2_{2D} = d\xi^2 + \xi^2 d\varphi^2 + \frac{\xi^2 d\xi^2}{r^2 - \xi^2}$$

$r^2 = X^2 \pm Z^2 = cts$

$$dl^2_{3D} = d\xi^2 + \xi^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + \frac{\xi^2 d\xi^2}{r^2-\xi^2}$$

$$dl^2_{3D} = \frac{d\xi^2}{1-K\xi^2} + \xi^2 d\Omega^2$$

$$dl^2_{3D} = d\chi^2 + S^2(\chi)\, d\Omega^2$$

$$\boxed{ds^2 = -c^2 dt^2 + a^2(t)\left[d\chi^2 + S^2(\chi)\, d\Omega^2\right]}$$

or $dl^2_{4D} = dX^2 + dZ^2$ $\qquad X^2 \pm Z^2 = r^2$ شعاع کره چهار بعدی

$4D \rightarrow 3D$, $r = \text{fixed}$

$$dl_{3D}^2 = dx^2 + K\frac{(x\cdot dx)^2}{1-Kx^2} \qquad \left(\text{Recall } dl_{2D}^2 = dx^2 + \frac{\xi^2 d\xi^2}{r^2\left(1-\frac{\xi^2}{r^2}\right)} \right.$$

$$ds^2 = -c\,dt^2 + a^2(t)\,dl_{3D}^2$$

What about $S(\chi) = ?$

$$\left.\begin{aligned} d\chi &= \frac{d\xi}{1-K\xi^2} \\ \int d\chi &= \int \frac{d\xi}{1-K\xi^2} \end{aligned}\right\} \longrightarrow \chi = \begin{cases} \xi & K=0 \\ \frac{1}{\sqrt{|K|}} \sinh^{-1}\left(\sqrt{|K|}\,\xi\right) & K<0 \\ \frac{1}{\sqrt{|K|}} \sin^{-1}\left(\sqrt{|K|}\,\xi\right) & K>0 \end{cases}$$

$$\longrightarrow \xi = \begin{cases} \chi & K=0 \\ \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|}\,\chi\right) & K<0 \\ \frac{1}{\sqrt{|K|}} \sin\left(\sqrt{|K|}\,\chi\right) & K>0 \end{cases}$$

$$\boxed{\xi = S(\chi) = \frac{1}{\sqrt{|K|}} \operatorname{Sinn}\left(\sqrt{|K|}\,\chi\right)}$$

$K = 0$ $\quad$ $K > 0$ $\quad$ $K < 0$

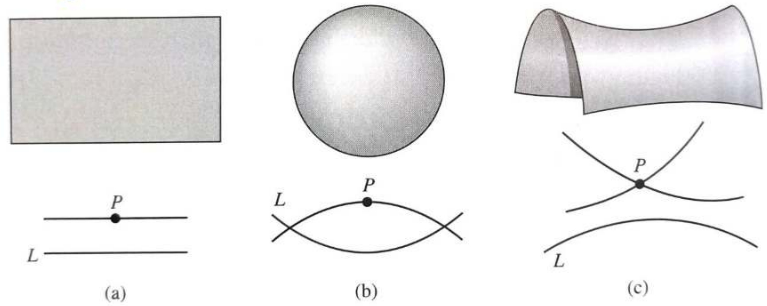


FIGURE 29.15 The parallel postulate, illustrated for three alternative geometries: (a) Euclidean, (b) elliptic, and (c) hyperbolic.

براساس فرضیه اقلیدس
درفضای تخت از نقطه P
تنها یک خط موازی با L است
درفضای خمیده (K>0) هیچ خط
موازی وجود ندارد. درفضای
خمیده (K<0) بی نهایت خط وجود دارد که موازی با L است.

$K = 0$ $\quad$ $K > 0$ $\quad$ $K < 0$

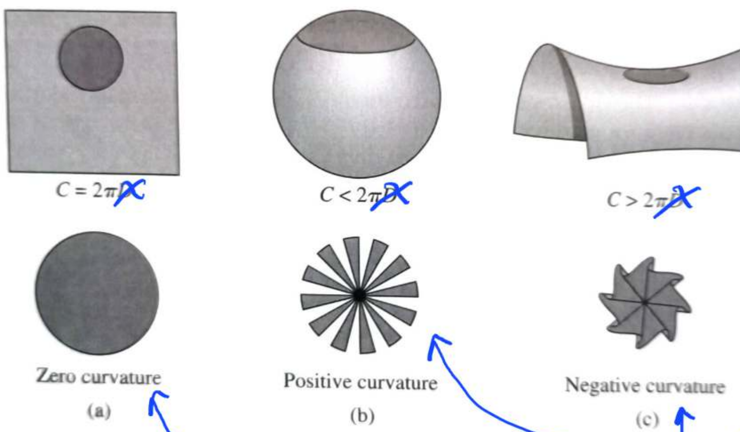


FIGURE 29.17 Calculating the curvature of a surface in three geometries: (a) a flat plane, (b) the surface of a sphere, and (c) the surface of a hyperboloid.

$$C = \int S(X)\, d\varphi$$
$$= 2\pi S(X)$$

$C < 2\pi X$

$C = 2\pi X$

$C > 2\pi X$

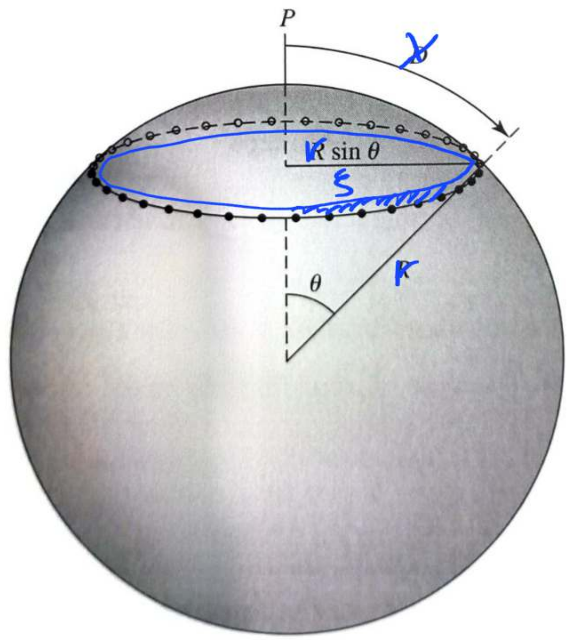
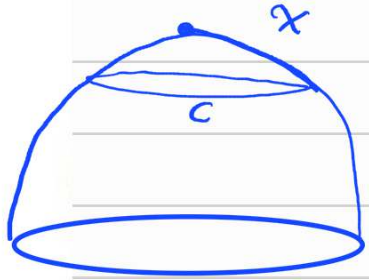


FIGURE 29.16 A local measurement of the curvature of a sphere.



$$C = 2\pi \xi \overset{?}{=} 2\pi X$$

$$dl_{2D}^2 = dx^2 + S^2(x)\, d\varphi \qquad dx = 0 \longrightarrow dl_{2D} = dC = S(x)\,d\varphi$$

$\uparrow$ circumfrence

محیط

$$C = \int_0^{2\pi} d\varphi\, S(x) = S(x)\, 2\pi$$

$$K=0 \quad S(x) = x \longrightarrow C_{K=0} = 2\pi x = 2\pi\xi$$

$$K>0 \quad C_{K>0} = \left[\frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}\, x)\right] 2\pi < 2\pi x$$

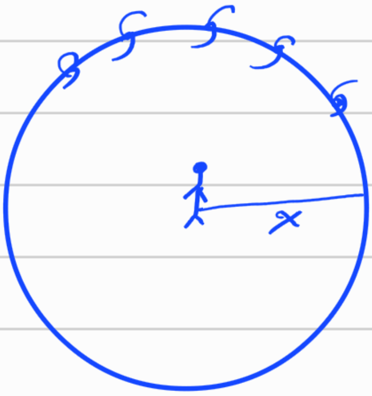
$$K<0 \quad C_{K<0} = \sinh(x)\, 2\pi > 2\pi x$$

$dA_{2D} = ?$

$$dl_{2D}^2 = dx^2 + S(x)^2 d\varphi^2$$

$$dA_{2D} = C_{2D}(x)\, dx$$

$$= \int_0^x dx'\, C_{2D}(x) = \pi x^2 \quad \text{For } K=0$$



یک آزمایش ذهنی برای تعیین انحنای کیهان به شرح زیر است

درصورتی که فاصله بین ستاره‌ها ثابت باشد در آن صورت

$$\underset{\substack{\uparrow \\ K>0}}{N_g} < \underset{\substack{\uparrow \\ K=0}}{N_g} < \underset{\substack{\uparrow \\ K<0}}{N_g}$$

Closed $\leftarrow$ $K>0$, Flat $\nwarrow$ $K=0$, $K<0$ $\nwarrow$ open.

در کیهان بسته تعداد همسایه‌ها کمتر از حالت تخت و کیهان تخت کمتر از کیهان باز است.

Exercise 1: For Static and Homogeneous space and Using Schwarzschild metric show that

$g_{i0} = 0$
$\downarrow$
Time Reversal Symmetry

$$dl^2 = a^2(t)\left[f(r)\, dr^2 + r^2 d\Omega^2\right]$$

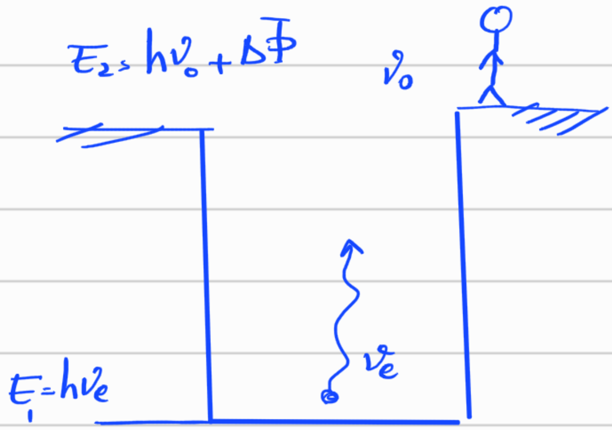
$$\hookrightarrow f(r) = \frac{1}{1-Kr^2}$$

&

Ricci scalare should be Constant

$$R = g^{\mu\nu} R_{\mu\nu} = cts$$

☆ Exercise 2 : Gravitational Red shift



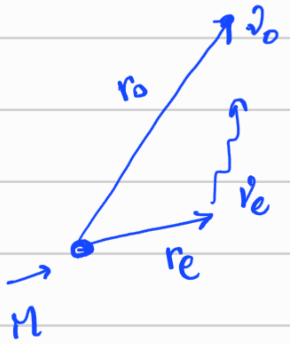
$$\nu_e \xrightarrow{?} \nu_o$$

انرژی پتانسیل

$$E_2 = E_1 - \Delta\Phi \quad \& \quad \Delta\Phi = \Phi_2 - \Phi_1$$

$$h\nu_o = h\nu_e \left(1 - \frac{\Delta V}{c^2}\right) \qquad \Delta V = \frac{\Delta\Phi}{m}$$

$$\frac{\nu_o}{\nu_e} = 1 - \frac{\Delta V}{c^2} \Longrightarrow 1 + z = 1 + \frac{GM}{c^2}\left(\frac{1}{r_e} - \frac{1}{r_o}\right)$$

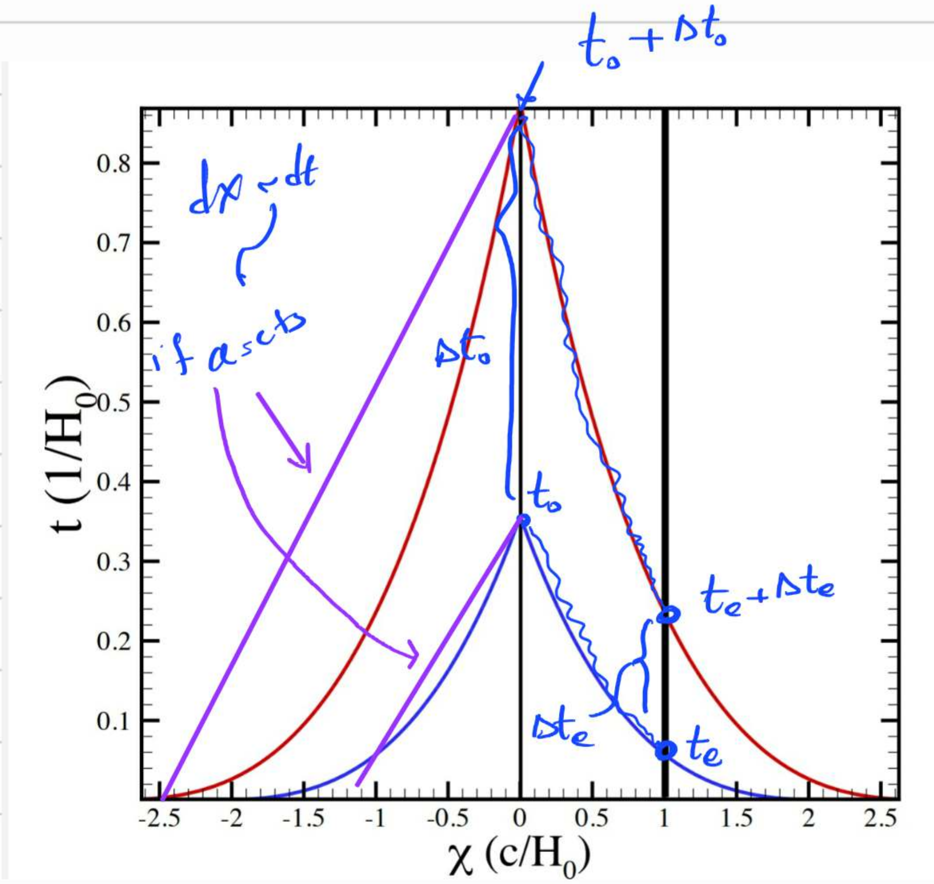


$$r_e \gg \frac{2MG}{c^2}$$
$$r_o \gg \frac{2MG}{c^2}$$

☆ Cosmological Red shift

$$ds^2 = -c^2 dt^2 + a^2(t)\left[dx^2 + S^2(x)\, d\Omega^2\right]$$

$$\begin{pmatrix} \text{Null geodesic} \\ \& \\ d\Omega = 0 \end{pmatrix} \longrightarrow ds^2 = 0 \longrightarrow \boxed{dx = \frac{dt}{c\, a(t)}}$$



$$\chi = c\int_{t_e}^{t_o} \frac{dt}{a(t)} = c\int_{t_e+\Delta t_e}^{t_o+\Delta t_o} \frac{dt}{a(t)}$$

because the relative location of observer and source does not change. χ=fixed

$$\cancel{\int_{t_e}^{t_o} ''} = \int_{t_o+\Delta t_e}^{t_e} '' + \cancel{\int_{t_e}^{t_o} ''} + \int_{t_o}^{t_o+\Delta t_o} ''$$

$$c\int_{t_o}^{t_o+\Delta t_o} \frac{dt}{a(t)} = c\int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a(t)}$$

$$\boxed{\frac{\Delta t_o}{a(t_o)} = \frac{\Delta t_e}{a(t_e)}}$$

$$\Delta t_0 = \frac{a(t_0)}{a(t_e)} \Delta t_e$$

Suppose that $a(t_0) = 1$

$$\nu_0 = a(t_e)\, \nu_e = \frac{1}{1+z} \nu_e$$

$$\lambda_0 = (1+z)\, \lambda_e$$

$$\frac{\Delta\lambda}{\lambda} = z$$

- **Physical length**
- **Comoving length**
- **Angular diameter distance**
- **Luminosity distance**
- **Comoving volume element**
- **Cosmic age**

$$R(t) = a(t)\chi$$

$$ds^2 = c^2dt^2 - a(t)^2\left[d\chi^2 + S(\chi)\, d\Omega^2\right]$$

$$\begin{cases} ds^2 = 0 \\ d\Omega^2 = \end{cases} \rightarrow d\chi = \frac{cdt}{a(t)} \rightarrow \chi = c\int_t^{t_0} \frac{dt'}{a(t')} = c\int_0^z \frac{dz}{H(z)}$$

$$\begin{cases} ds^2 = 0 \\ d\chi^2 = 0 \end{cases} \rightarrow a(t)S(\chi)\theta = \Delta l \rightarrow d_A \equiv a(t)S(\chi)$$

$$d_l \equiv (1+z)S(\chi) = d_A(1+z)^2$$

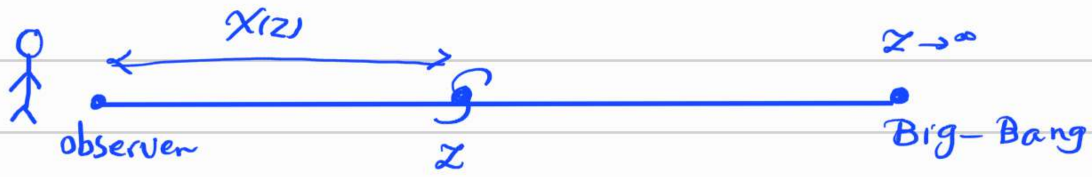
$$f \equiv \frac{dV}{d\Omega dz} = \frac{S(\chi)^2}{H(z)} \rightarrow \frac{dN}{dz} = n(z)\frac{dV}{dz} = 4\pi n(z)\frac{S(\chi)^2}{H(z)}$$

$$\Delta N_{L>L_{\min}} = \int_{L_{\min}}^{\infty} \frac{dN}{dzdl} dl\Delta z$$

$$t_0 = \int_0^{t_0} dt = \int_0^{\infty} \frac{dz}{(1+z)H(z)}$$

Now, I turn to compute some quantities from Background Evolution which are related to Space-Time distance (ds)

① Comoving Distance to a Source at z



$Z_{obs} = 0$

$a_{obs} = 1$

o Horizon

$$d\chi = \frac{c\,dt}{a(t)} \qquad \chi_0 = \int_0^{\chi_0} d\chi = \int_0^{t_0} c\,\frac{dt}{a(t)}$$

$$\boxed{\chi_0 = c\int_0^{\infty} \frac{dz}{H(z)}}$$

$$\begin{cases} Z_0 = 0 \longrightarrow a_0 = 1 \\ Z_{Big\ Bang} = \infty \longrightarrow a_{Big-Bang} = 0 \end{cases}$$

$$\chi(z) = c\int_0^{z} \frac{dz'}{H(z')} = \frac{c}{H_0}\int_0^{z} \frac{dz'}{\mathcal{H}(z')}$$

$$H_0 = h\,100\ \frac{Km/s}{Mpc}$$

$$\chi(z) = \frac{3\times10^5\ Km/s\ \ Mp}{100\,h\ Km/s}\int_0^{z} \frac{dz'}{\mathcal{H}(z')}$$

$$\boxed{\chi(z) = 3000\left(\frac{Mpc}{h}\right)\int_0^{z} \frac{dz'}{\mathcal{H}(z')}}$$

$$\chi_0 = c\int \frac{dt}{a(t)} \qquad \text{if} \quad a = cts \qquad \chi_0 \approx c\,t_0$$

$$a(t_1) < a(t_2) < a(t_3) \ldots < a(t_0)$$

$$\chi_0 > c\,t_0$$

چگونه می‌توان جسمی در فاصله $14\times10^{10}\ yr$ را مشاهده کرد در حالی که سن عالم $13.8\times10^{9}\ yr$ است؟

○ Event Horizon $d_E = \int_{t_0}^{\infty} \frac{c\,dt}{a(t)} = \begin{cases} \infty & \text{No event Horizon} \\ \neq \infty & \text{Event Horizon} \end{cases}$

درخواست می‌کنیم که زمان مورد نیاز برای
دریافت اطلاعات از یک چشمه که در فاصله $x \geqslant d_E$ باشد بی‌نهایت شود. یعنی جسم دیده نمی‌شود.

Ex: For de Sitter Universe $\Lambda > 0$ and $H \propto \sqrt{\Lambda}$

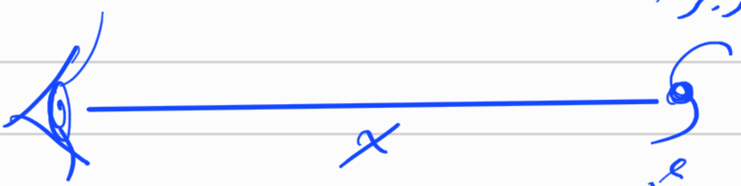
$\underline{a(t) = e^{Ht}}$ $\qquad \dot{a} = H_0 e^{H_0 t}$ $\qquad \frac{\dot{a}}{a} = H_0 \leftarrow$ Cosmological Inflation era

$$d_E = \int_t^{\infty} \frac{c\,dt'}{e^{H_0 t}} = \int_t^{\infty} c\, dt'\, e^{-H_0 t} = \frac{c}{H_0} \neq \infty$$

☆ Maximum Visible age of Cosmological object.

بیشینه سنی که یک جسم کیهانی را می‌بینیم

فرض کنید یک جسم کیهانی در فاصله همراه x قرار گرفته باشد



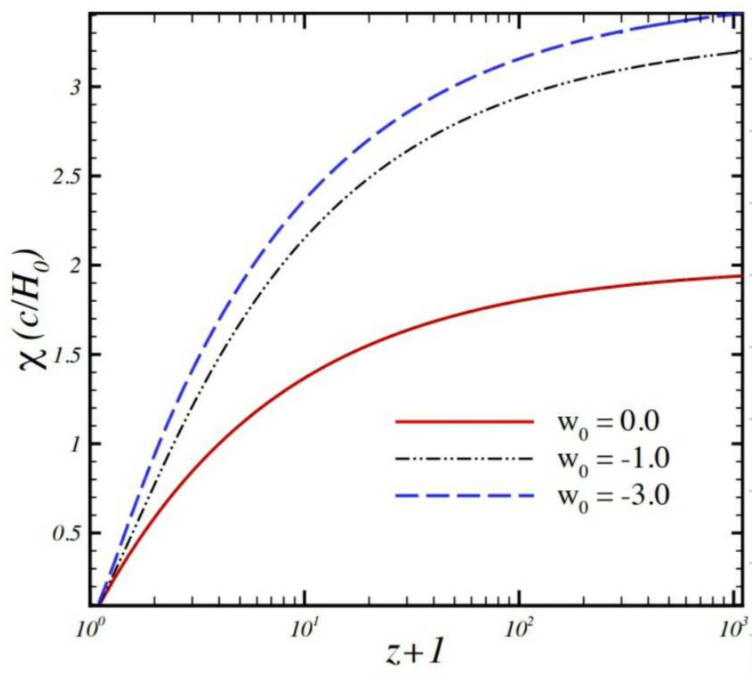
اگر در زمان t_1 یک $t_0 > t_1$
پالس نوری ارسال کند در t_0 دریافت می‌شود
حال اگر در t_2 که $t_0 > t_2 > t_1$ است یک پالس دیگری
ارسال کند دیگر دریافت نشود یعنی زمان دریافت بی‌نهایت باشد

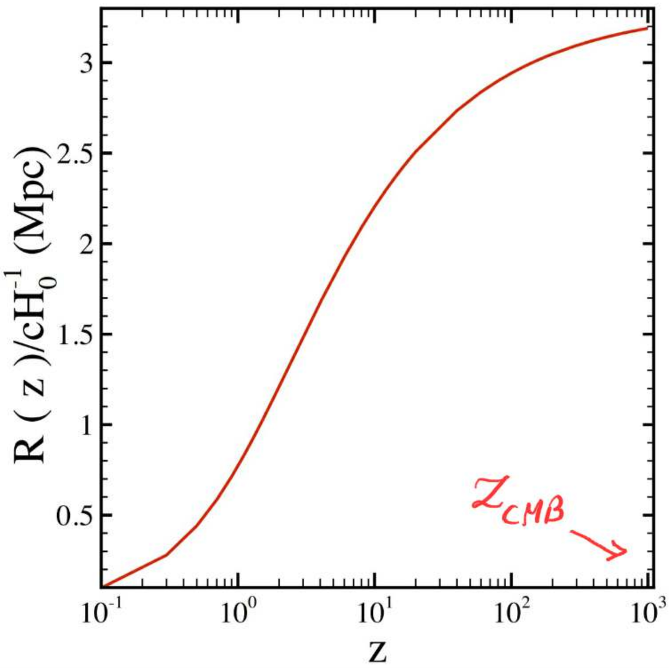
$$x = c\int_0^{z} \frac{dz'}{H(z')}$$

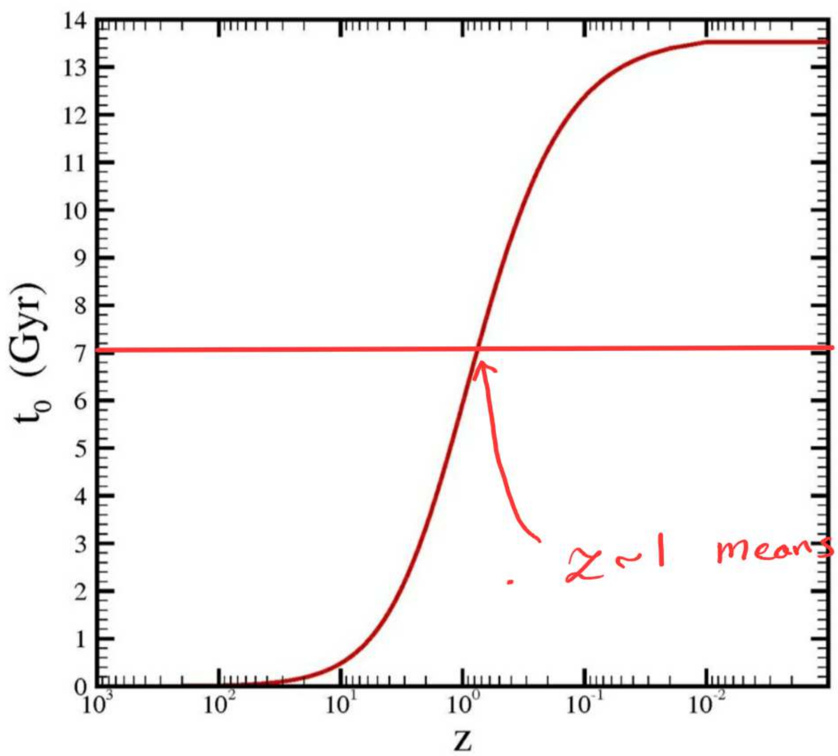
$$= c\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{t_2}^{t_0+\Delta t_0} \frac{dt}{a(t)} \overset{?}{=} \int_{t_{max}}^{\infty} \frac{dt}{a(t)}$$

$t_0 > t_2 > t_1$

$T_{hidden} = t_{max} - t_1$
↓
this means that if we can see an object which is located at z, after passing time about $T_{hidden}(z)$, we are not able to see it.







$z \sim 1$ means that $\boxed{t \simeq \frac{t_0}{2}}$

Maximum visible age of astronomical objects

PRD, 65, 047301 ((2002

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} \geq \int_{t_2}^{\infty} \frac{dt}{a(t)} \rightarrow t_2 = t_{\max} \rightarrow t_{\max} - t_1 = T_{hidden}$$