به نام خدا

Multi-Fractal Analysis

Seyed Mohammad Sadegh Movahed[1,2]
http://faculties.sbu.ac.ir/~movahed

Outline

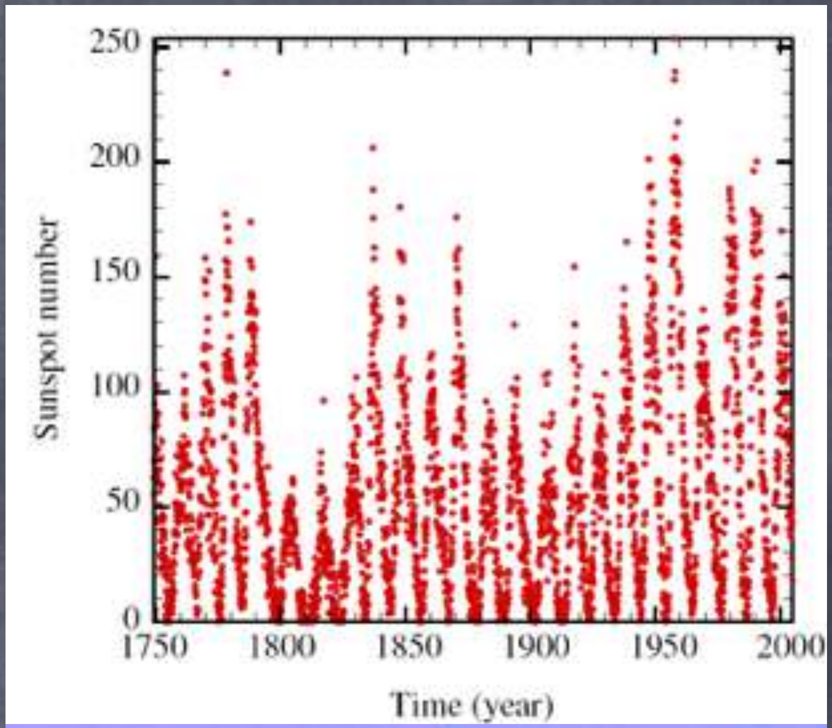
- Motivations
- A brief explanation about Self-similar and self-affine models
- Novel methods in multifractal analysis
- Relation between so-called Hurst exponent and other scaling exponent in 1 and 2 dimensions
- Trends and undesired noise in time series
  1) Polynomial trends
  2) Sinusoidal trends
- Detrending procedures: F-DFA, SVD and EMD
- Summary

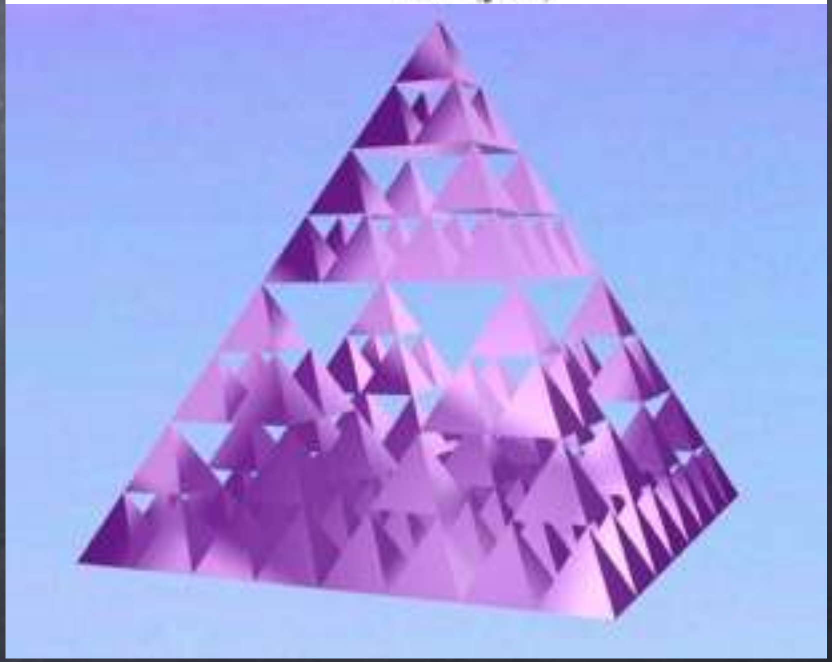
Some relevant references

- http://sharif.ir/~rahimitabar/
- http://faculties.sbu.ac.it/~movahed
- arXiv:0804.0747
- PRE, 74, 061104 (2006)
- Physica A 316, 87-114 (2002)
- PRE 71, 011104 (2005); arXiv:physics/0103018
- Physica A 357, 447-454 (2005); Physica A 354, 182-198 (2005); Chaos, Solitons and fractals 26, 777-784 (2005)
- PNAS, 104, 38 (2007)
- Physiol. Meas. 23 (2002) R1-R38

Glossary

- Complex system: A system consisting of many non-linear components.
- Time series: One dimensional array representing value of an observable based on dynamical variable so-called time.
- Scaling law: A power law function describing the behavior of a typical physical quantity.
- Fractal and multifractal systems: A typical system which characterized by a scaling law with non-integer exponent in all scaling ranges. On the other hand, multifractal has infinite number of different fractal exponents. Each of them are valid in proper scaling range.



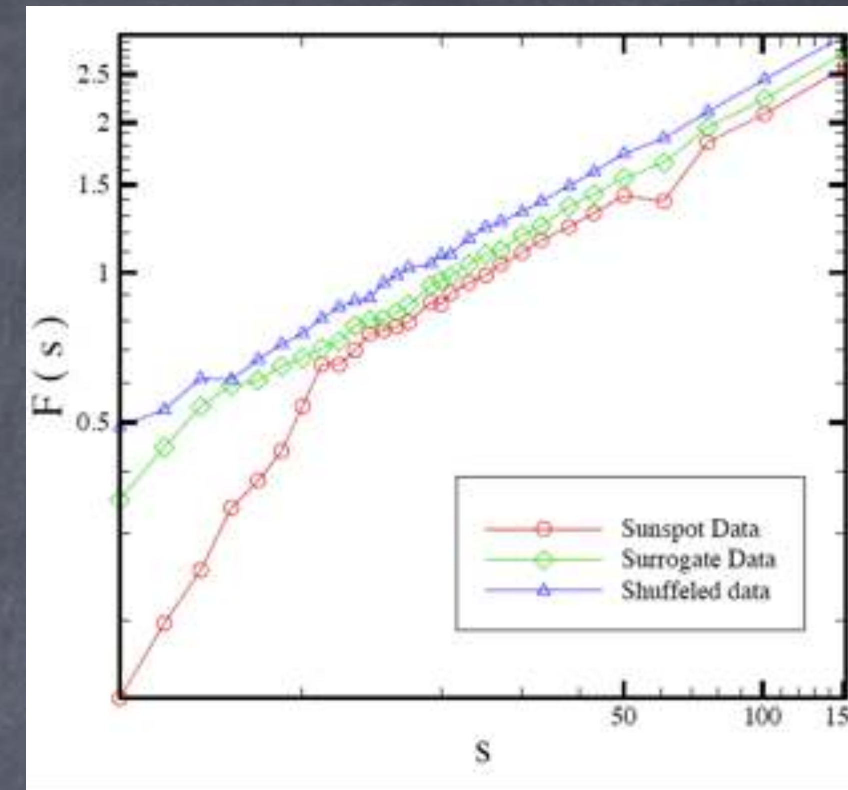


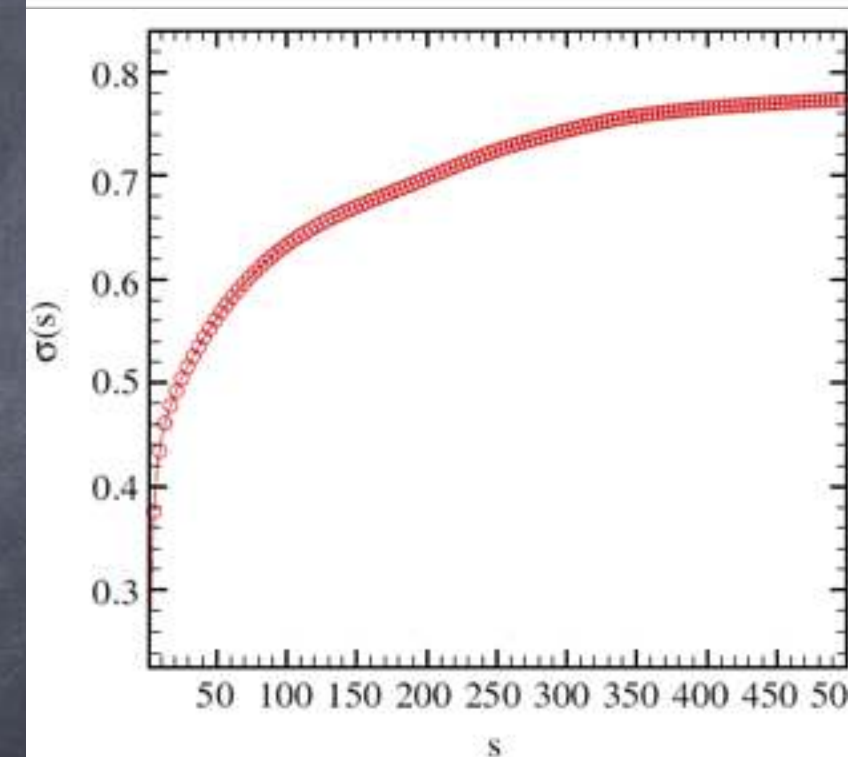
- Self-similar and self-affine: Magnification of system's parts in every directions have same scaling exponent for matching to whole of system. While self-affinity is a generalization for anisotropic scaling behavior.

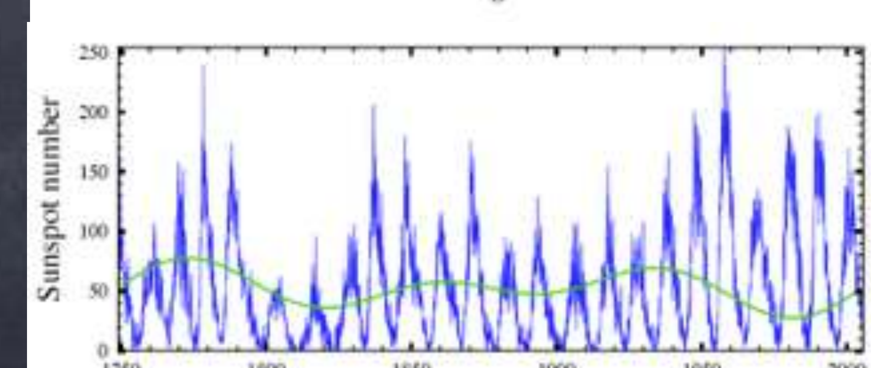
- Cross-over: Changing in the scaling behavior

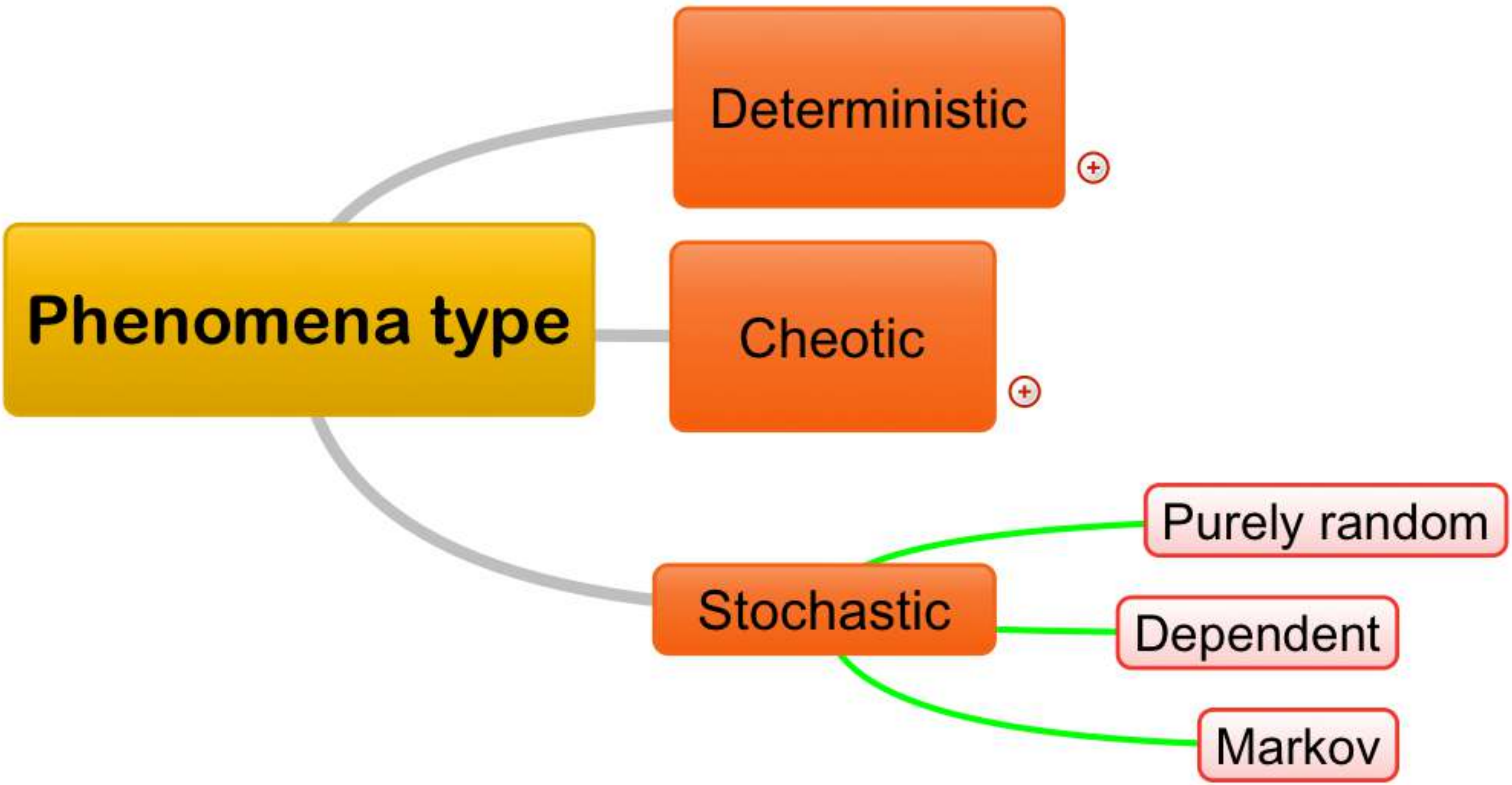
- Non-stationary: The weak definition is concerned to changing the mean standard deviation of time series with time. Strong definition of stationarity requires that all moments remain constant. Usually external affects cause nonstationarity in time series.

- Trend and detrending: It is an intrinsically fitted monotonic function or a function in which there can be at most one extremum within a given data span. Detrending is the operation of removing trend



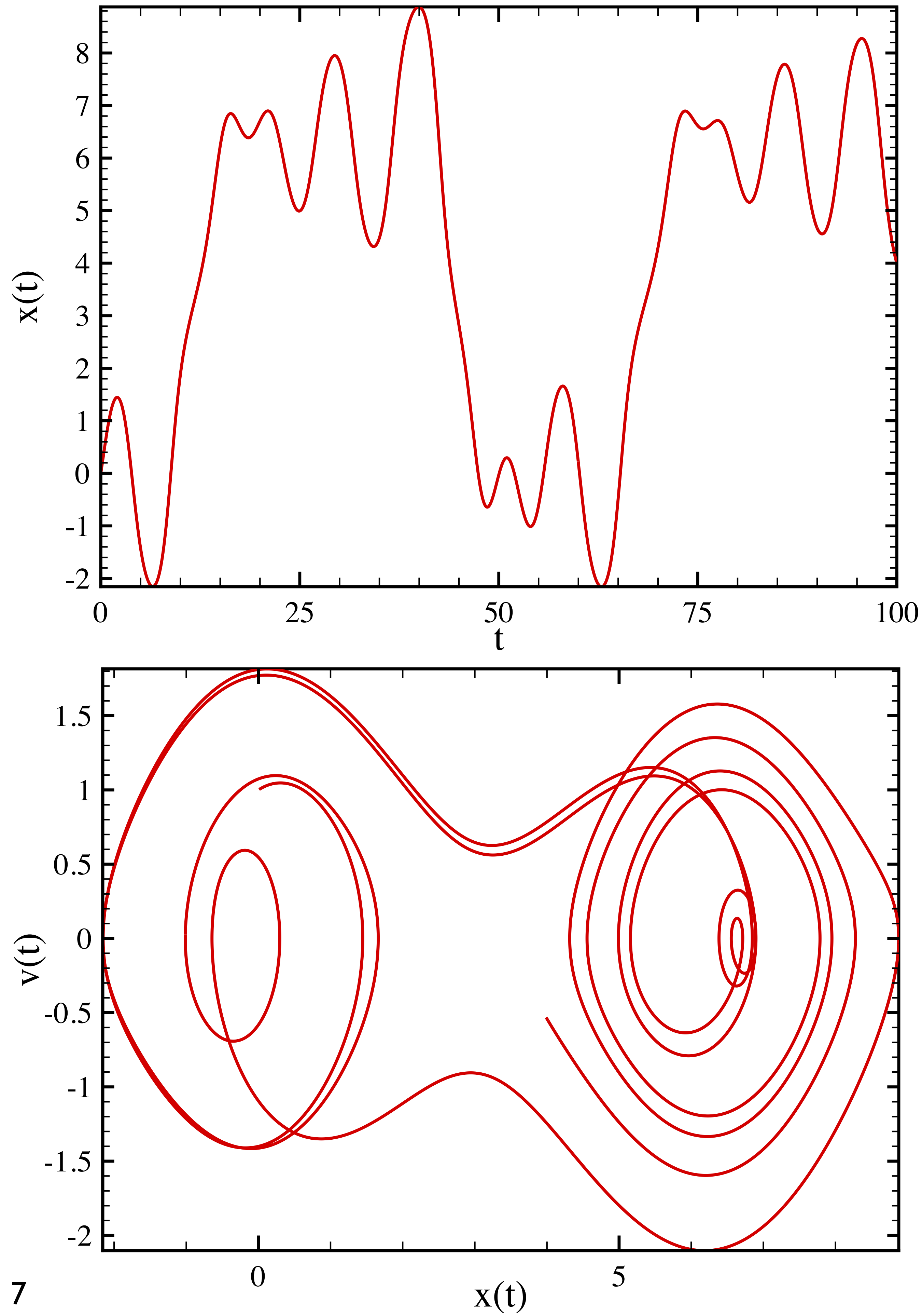





Phenomena type
Deterministic
Cheotic
Stochastic
Purely random
Dependent
Markov

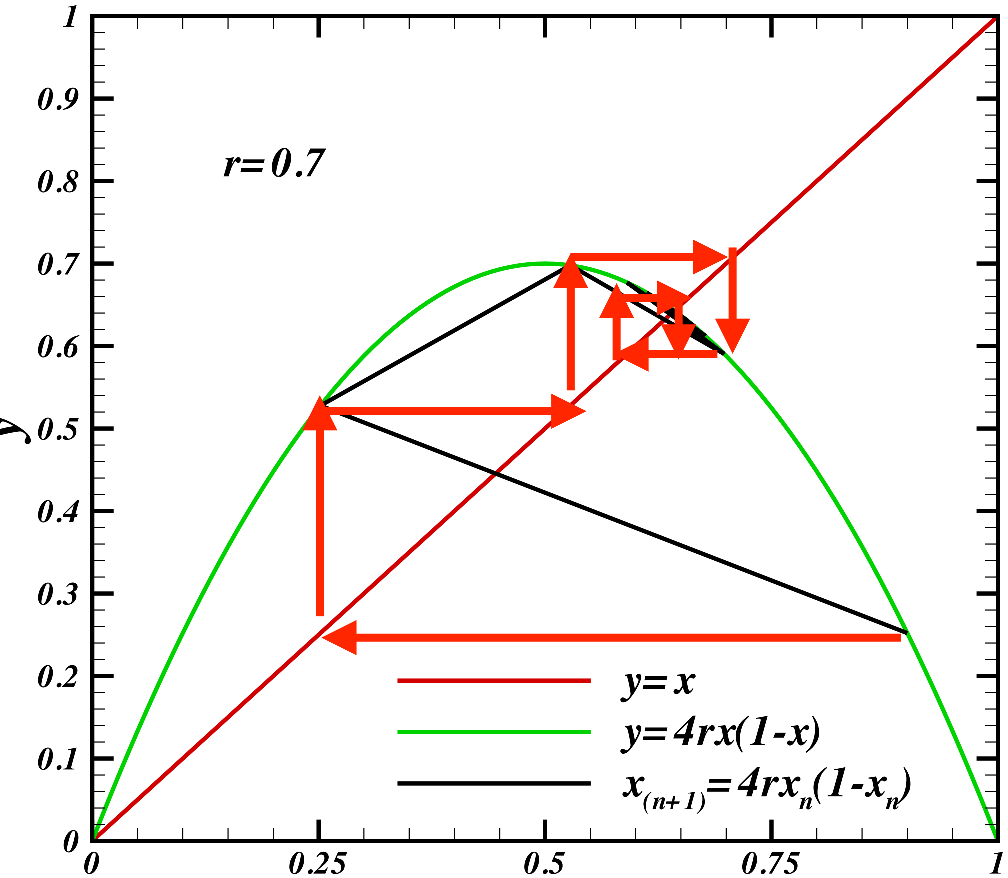
$$\ddot{x}(t) + \alpha\,\dot{x}(t) + \omega_0^2 \sin(x(t)) = f\cos(\omega t)$$

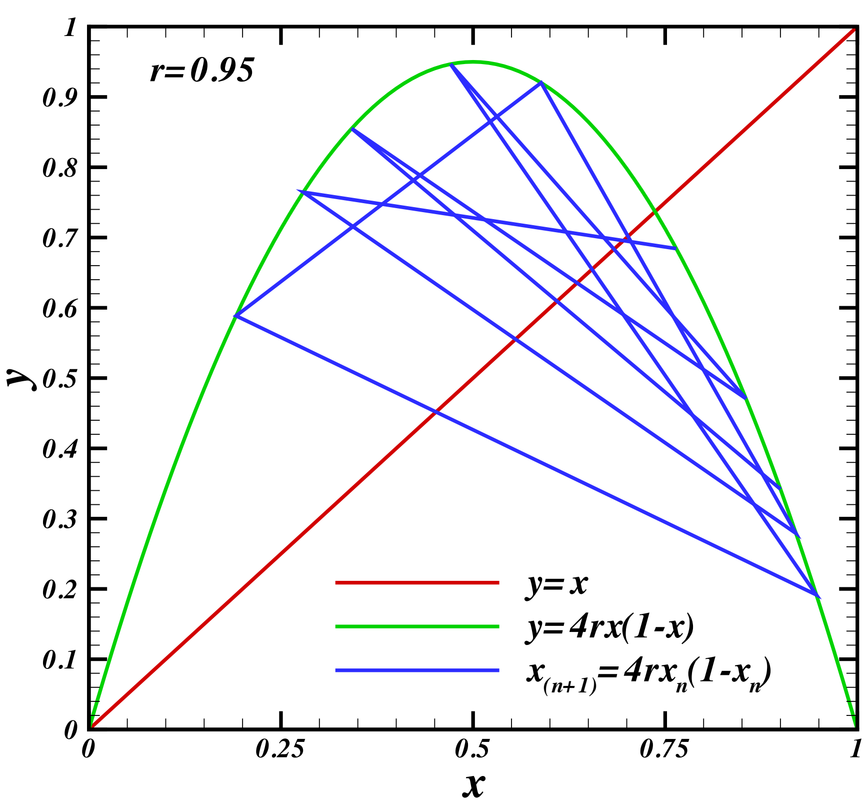
$\alpha = 0.2$, $\omega_0 = 1$, $f = 0.52$, $\omega = 0.666$

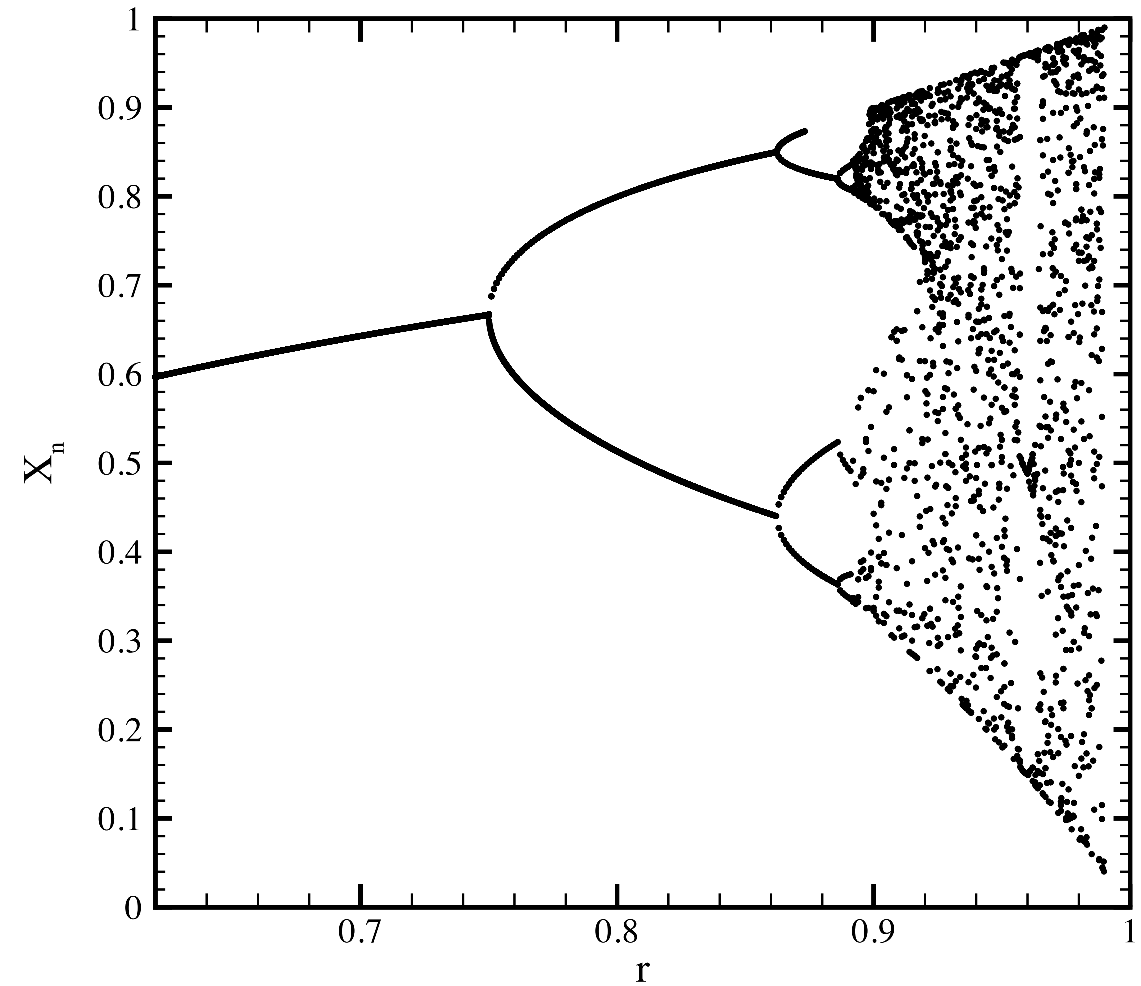


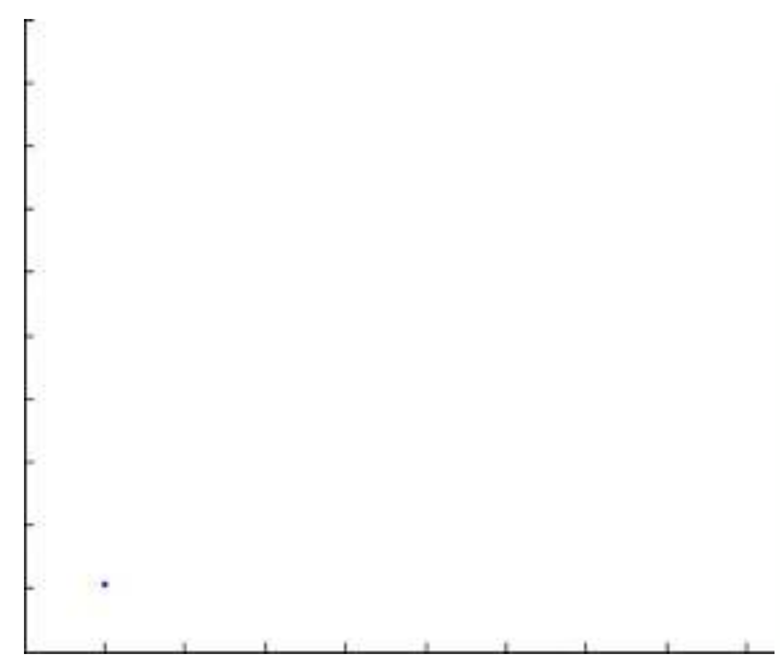
Logistic map

$X_{n+1} = 4r X_n(1-X_n)$







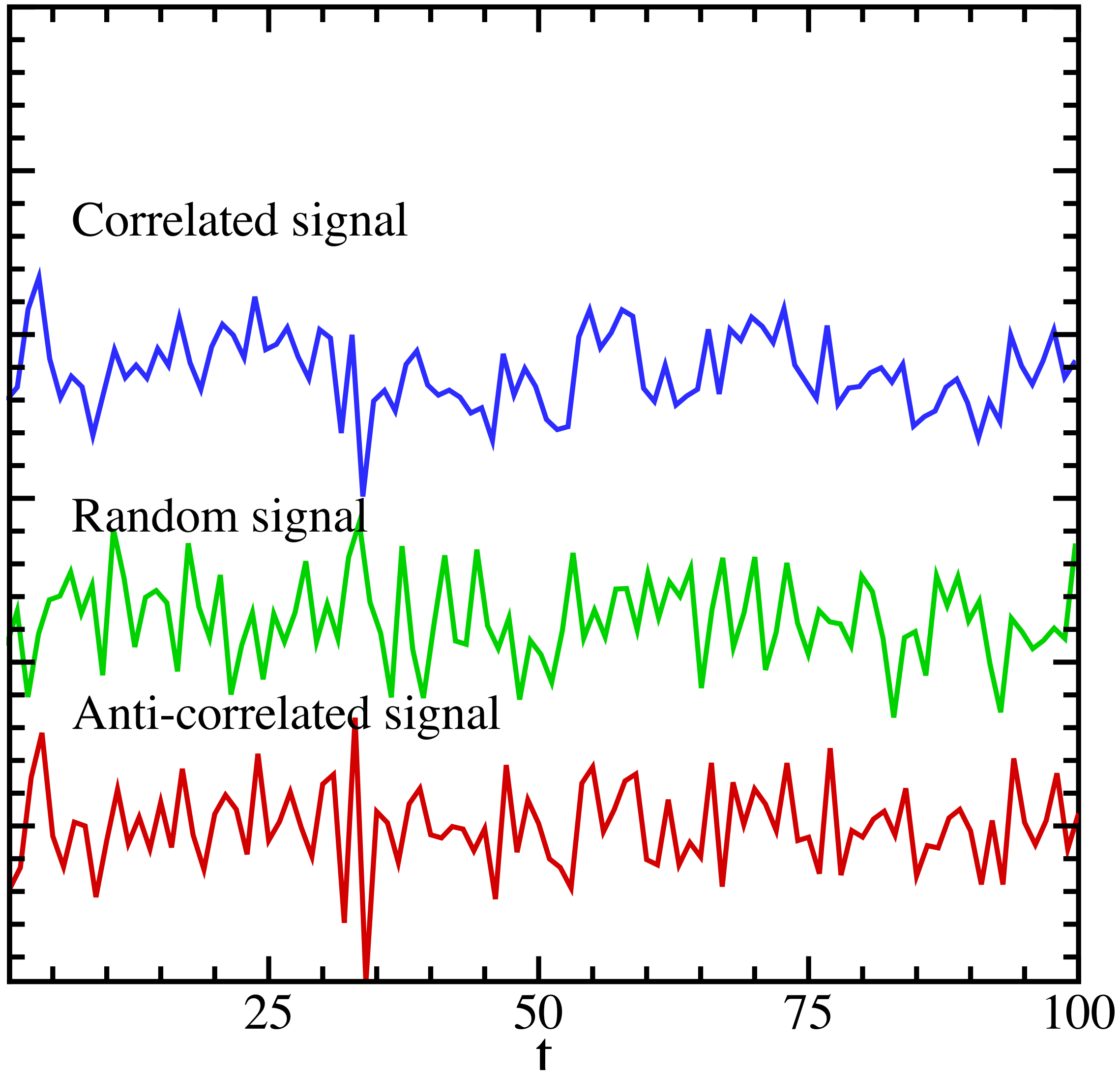


3) Stochastic Processes

- Purely Random processes

-Dependent processes

-Markov processes

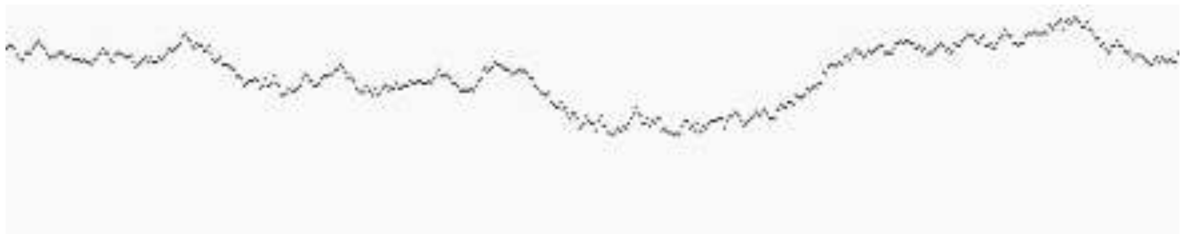


1) Probability distribution function
2) Weighted and Un-weighted Correlation functions

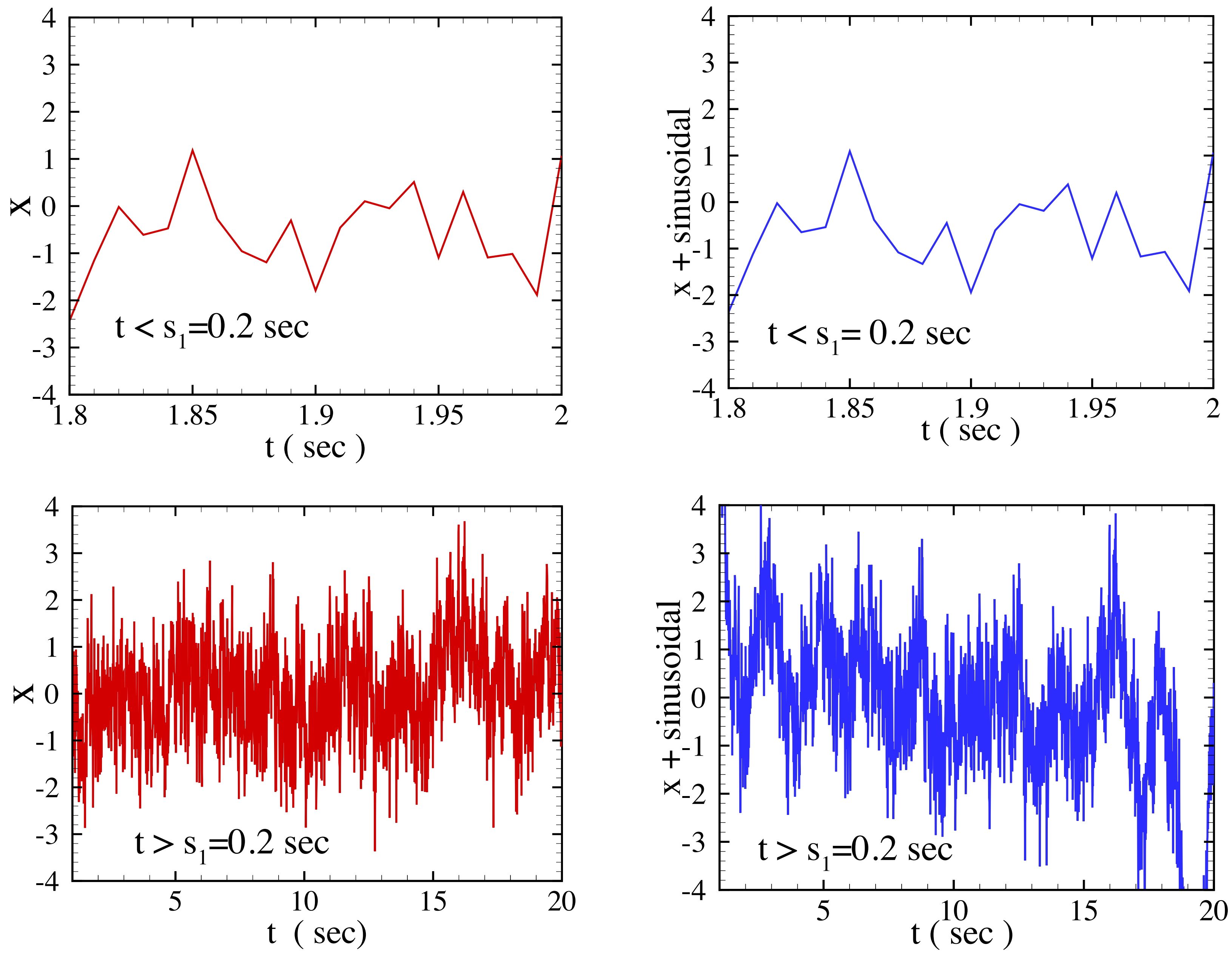
$$\dot{x}_i(t) + \sum_{j=1}^{M} \xi_{ij} \dot{x}_j = \eta_i(t) \qquad \langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} \delta_D(t-t')$$

$$\dot{v}(t) = -\xi v(t) + \eta(t) \longrightarrow \text{Stationary regime} \rightarrow \text{M.B. Distribution}$$

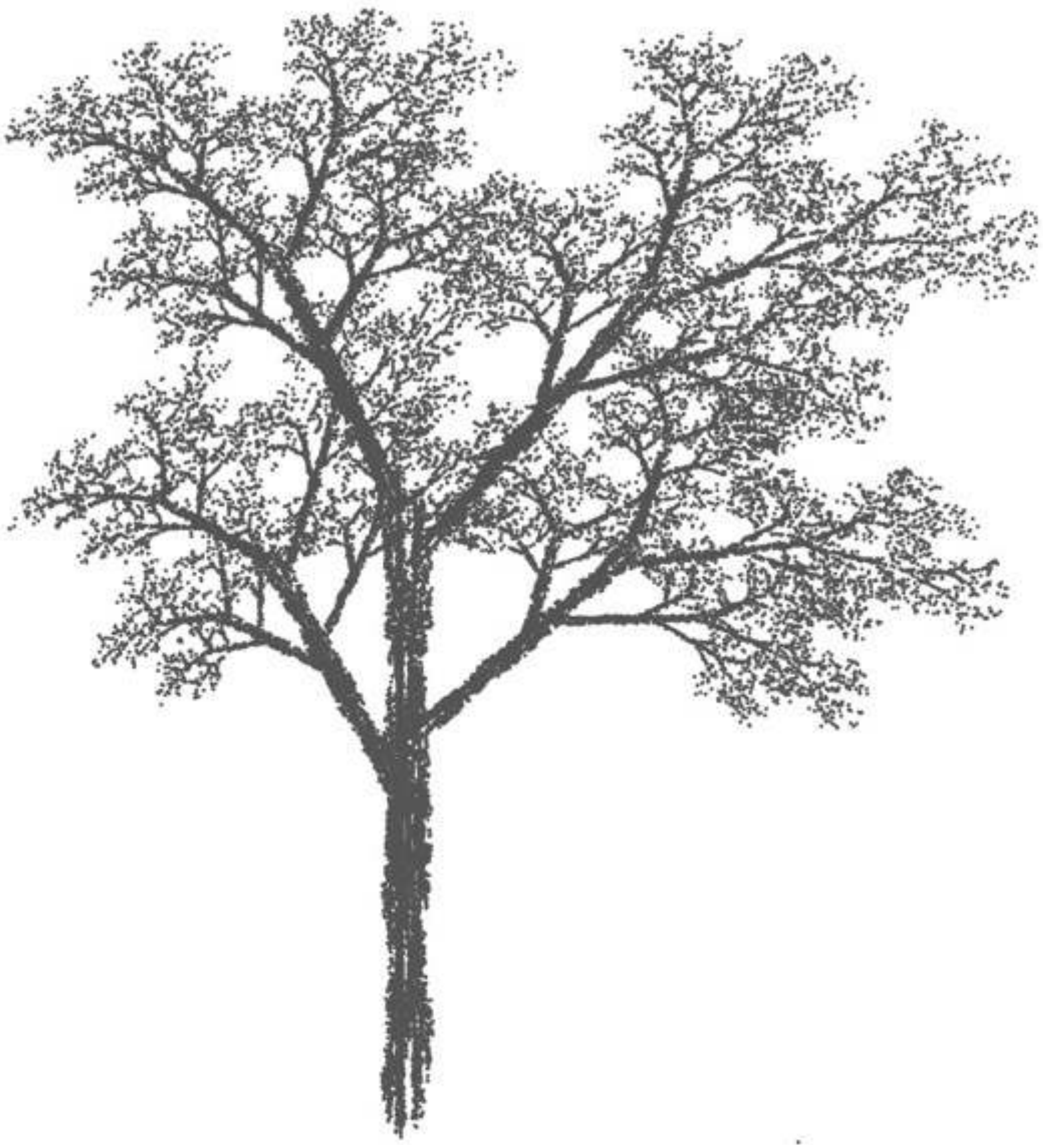
Self-similarity and scale

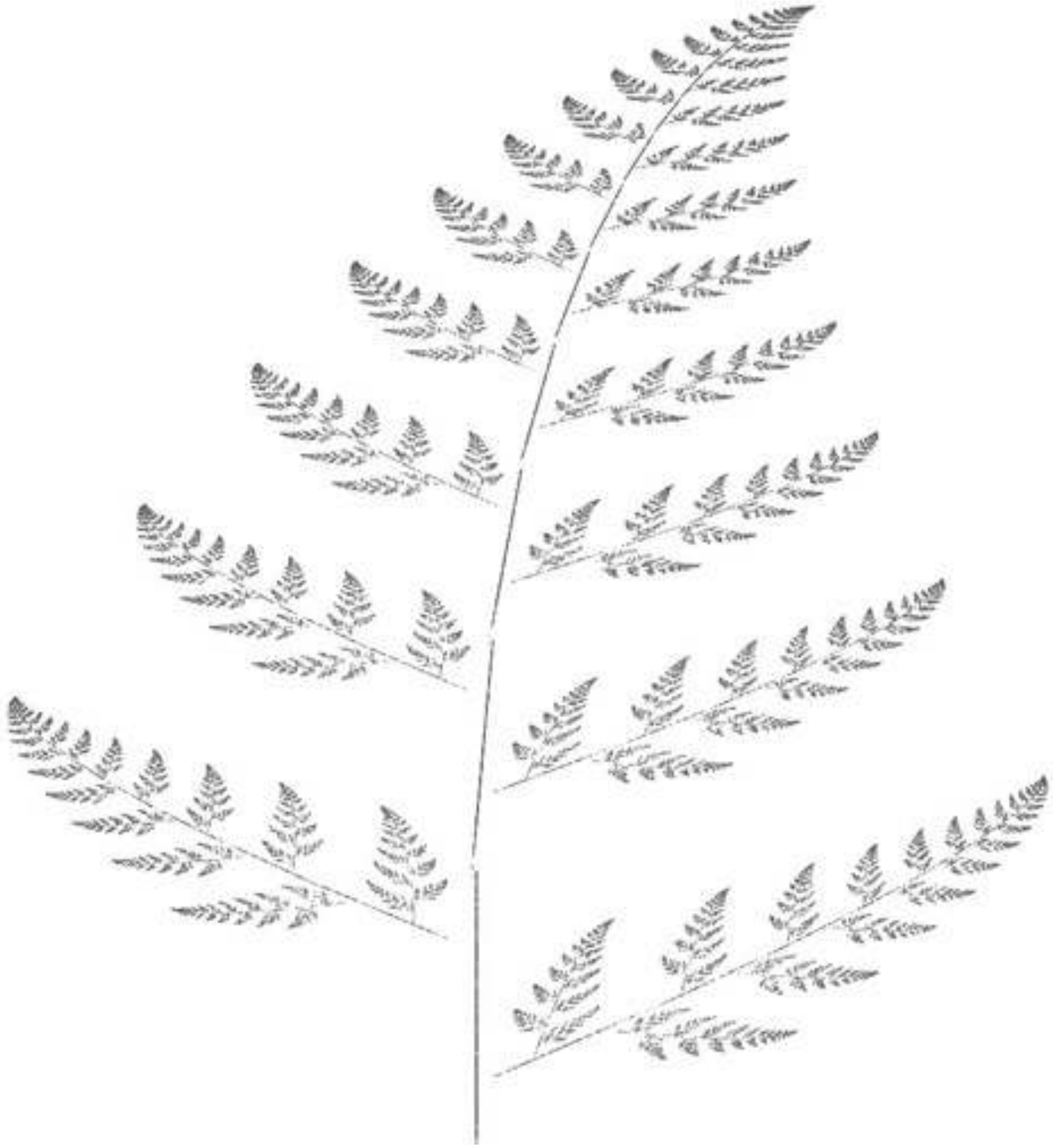


Scale dependency
(Multifractal) Self-similarity and Self-affinity
Complexity



Self-similar process

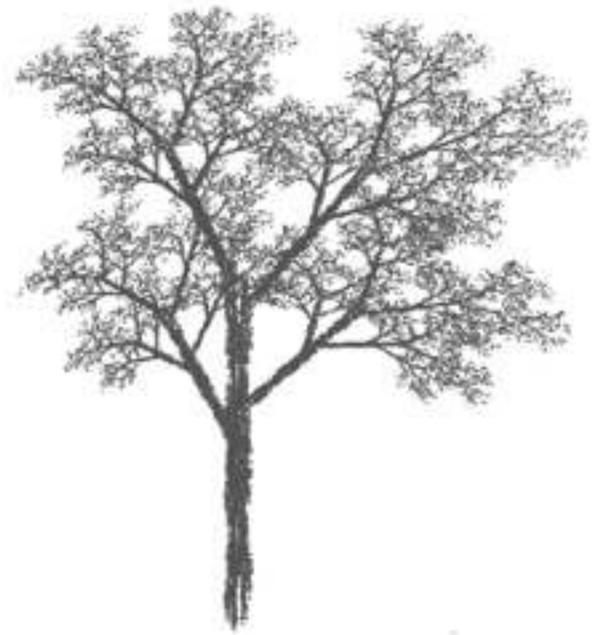




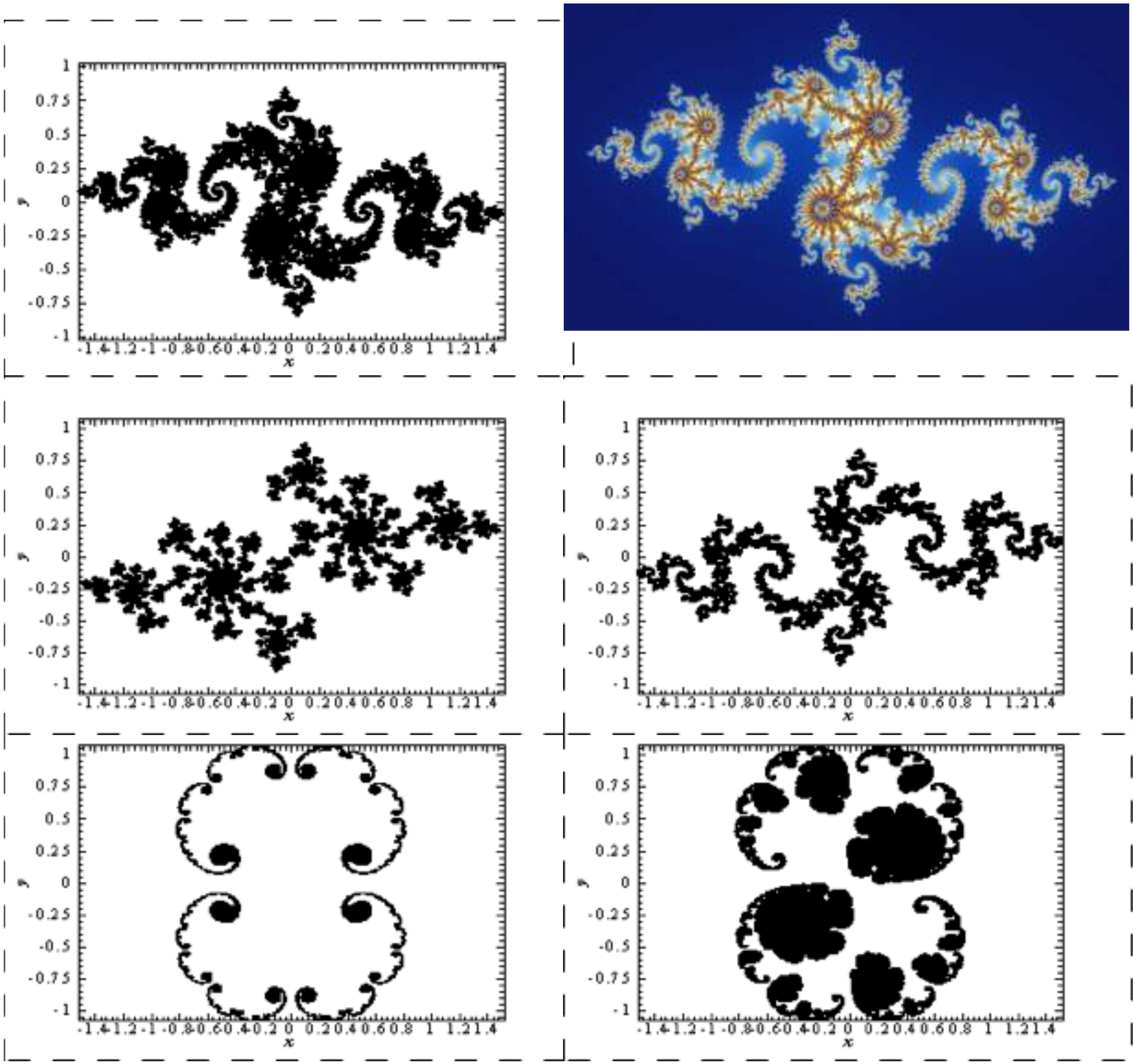
To know more see: http://facultymembers.sbu.ac.ir/movahed/index.php/talks-a-presentations

تاریخچه

- ~1700 A.D. Gottfried leibniz
- ~1872 A.D. Karl Weierstrass
- ~1904 A.D. Helge von Koch
- ~1915 A.D. Waclaw Seirpinski
- ~1951 A.D. H. E. Hurst
- ~1968 A.D. B. B. Mandelbroat



فراکتال (برخال) های هندسی



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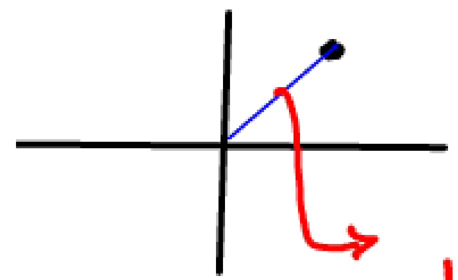
روش کار برای Mandelbrot set

① شروع از ناحیه‌ای

$$-2 \le x \le 2$$
$$-2 \le y \le 2$$

پس $(x_k \,,\, y_k)$ به عنوان ورودی

② محاسبه فاصله این نقطه از مرکز



$$Length = (x_k^2 + y_k^2)^{1/2}$$

③ اگر $Length < 2$ به مرحله بعدی برود

$Z_{old} = x_k + i y_k \qquad n = 0 \,,\quad Length\_New = 0$

④ تا زمانی که شرط خروج ارضاء نشود مراحل زیر انجام شود:

```
Do while ( n < iteration and
              Length_New < 2 )
    n = n+1
    Z_New = Z_old^2 + x_k + i y_k
    Z_old = Z_New
    Length_New = |Z_old|^2
End do
```

۵ـ بررسی شود که آیا (x_k و y_k) انتخاب شده شرط عضویت در شکل فراکتالی را دارد؟

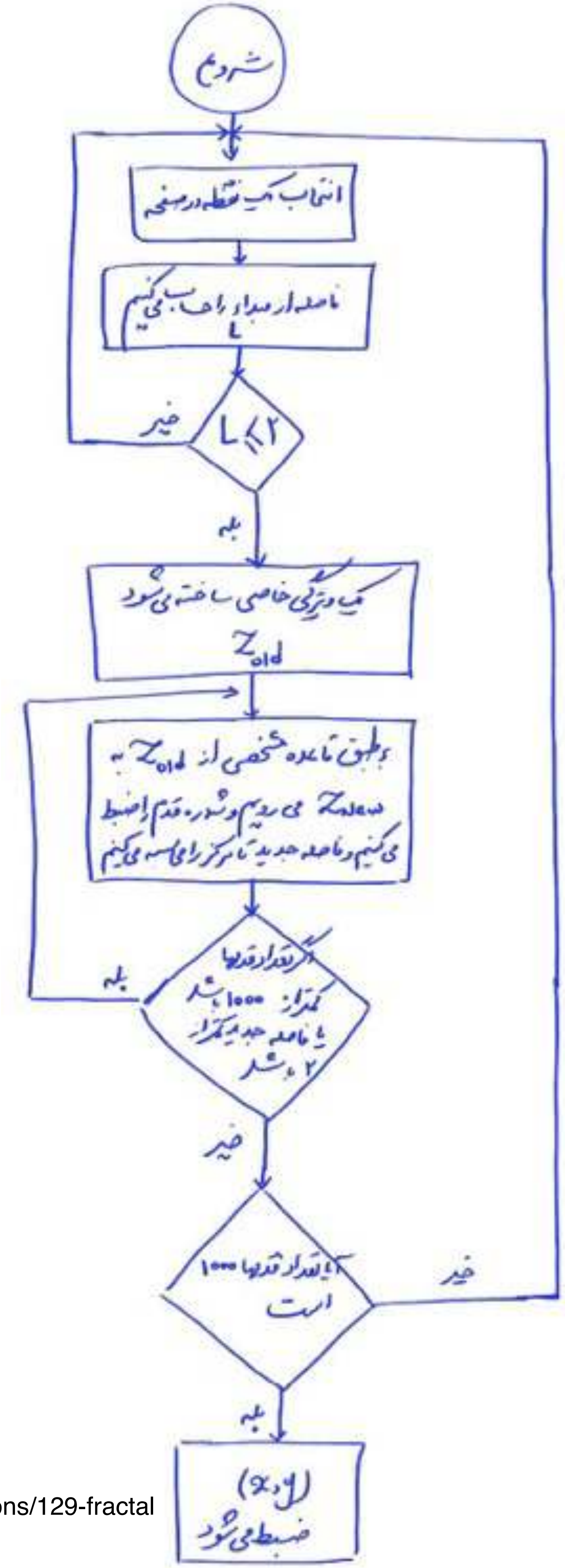
این بررسی شامل این است که دلیل خروج از مرحله ۴ به سبب بیرون افتادن از دایره‌ای به شعاع ۲ است؟ یا به دلیل اینکه n از کمیت Iteration تکرار شده است؟

درصورتی که دلیل خروج به سبب دلیل دوم بوده باشد پس نقطه (x_k و y_k) عضو شکل است

پس

```
if ( Length_New < 2 ) Then
Write(*,*) xk , yk
```

۶ دوباره به ابتدای برنامه برویم و جفت جدیدی انتخاب کنیم و همه مراحل بالا را انجام دهیم. نهایتاً باید نرم افزار x_k, y_k را رسم کنیم



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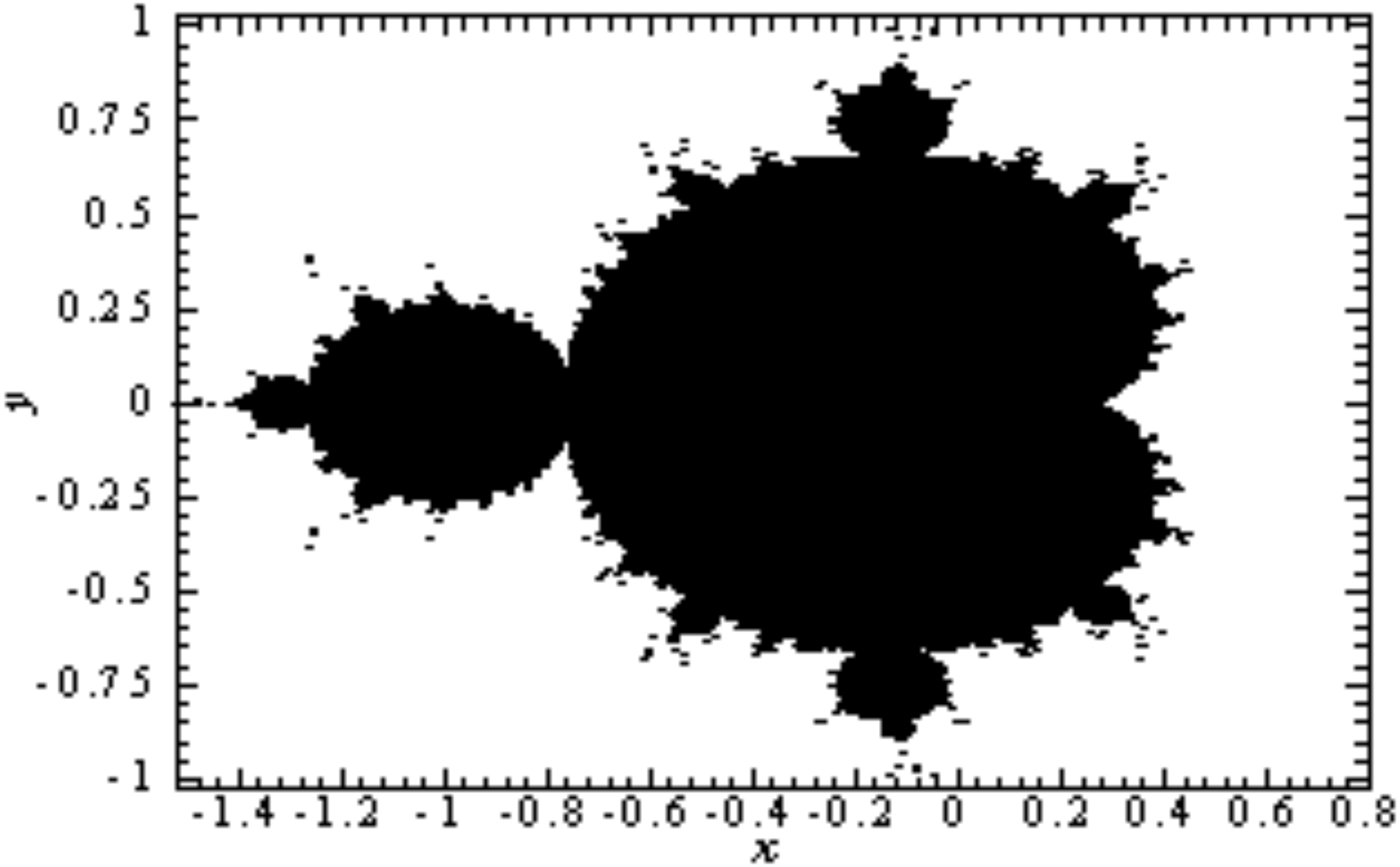
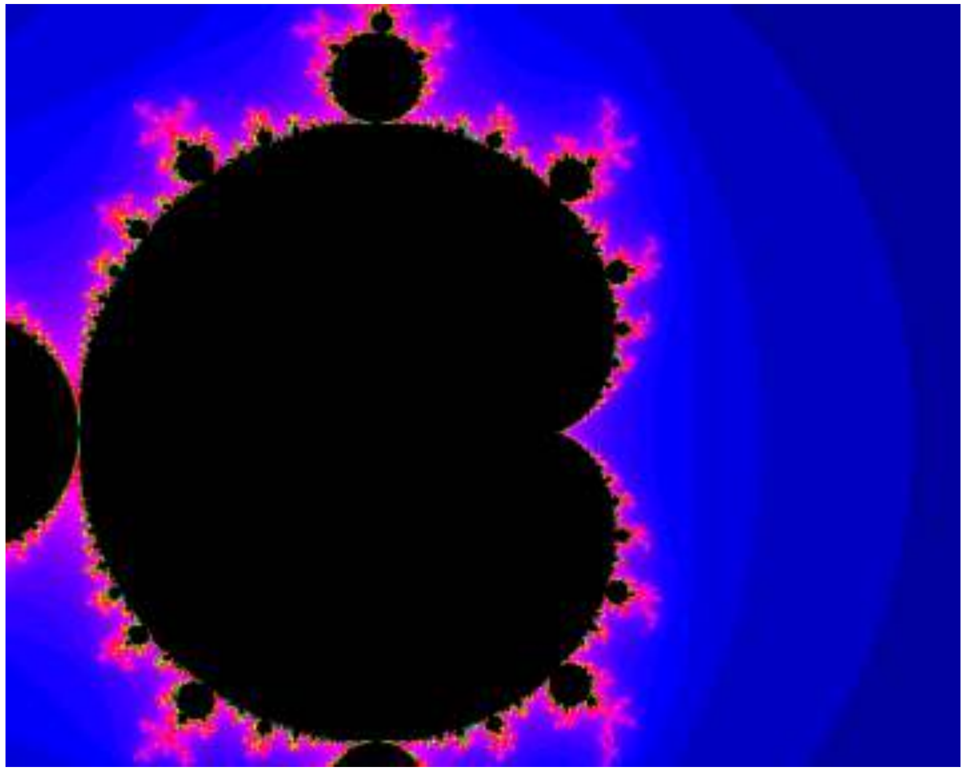
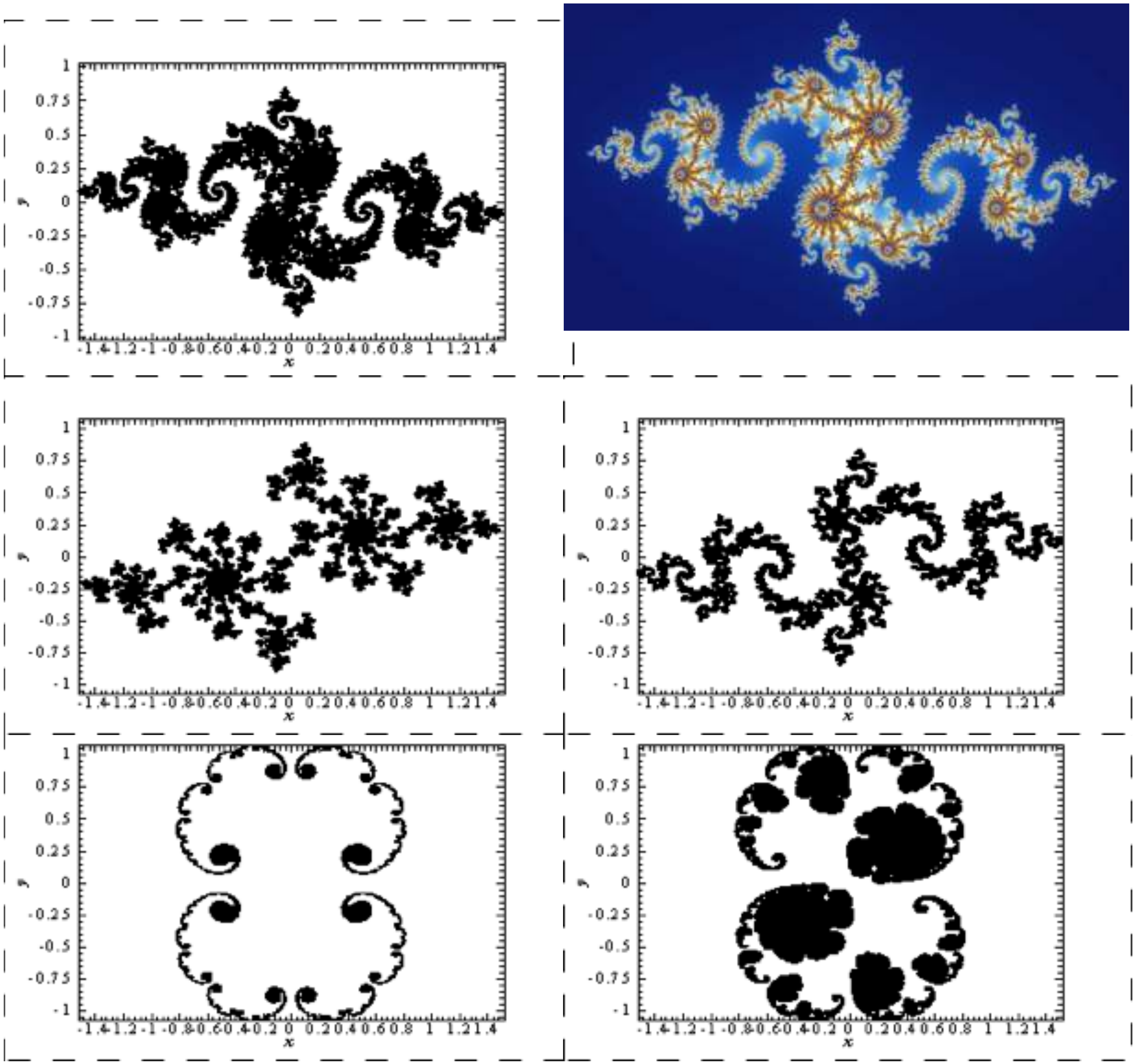

1
0.75
0.5
0.25
0
-0.25
-0.5
-0.75
-1
y
-1.4 -1.2 -1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8
x



فراکتال (برخال) های هندسی



برای Julian Set

همه چیز مانند قبل است جز اینکه

$$Z_{New} = Z_{old}^2 + C$$

است که در آن به ازای c های مختلف شکلهای مختلف

بدست می آید

$C = -0.8 + 0.156\,i$

$C = -0.4 + 0.6\,i$

$C = 0.285$

$C = 0.285 + 0.01\,i$

$C = -0.70176 - 0.3842\,i$

$\vdots$

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Fractal Tree

۱ انتخاب نقطه شروع

$x_{old} = 0.5$

$y_{old} = 0.0$

Writ(*,*) x_{old} , y_{old}

۲ به تعداد نقاطی که می خواهیم انتخاب کنیم یک حلقه ایجاد می کنیم

۳ یک عدد تصادفی تولید می کنیم

Call random_number(u)

۴

if ($u \leq 0.1$) Then

$x_{New} = 0.05\, x_{old}$

$y_{New} = 0.6\, y_{old}$

End if

۵

if ($0.1 < u \leq 0.2$) then

$x_{New} = 0.05\, x_{old}$

$y_{New} = -0.5\, y_{old} + 1.0$

End if

۶

if ($0.2 < u \leq 0.4$) Then

$x_{New} = 0.46\, x_{old} - 0.15\, y_{old}$

$y_{New} = 0.39\, x_{old} + 0.38\, y_{old} + 0.6$

End if

if($0.4 < u \leq 0.6$) Then ٧

$$x_{New} = 0.47\,x_{old} - 0.15\,y_{old}$$

$$y_{New} = 0.17\,x_{old} + 0.42\,y_{old} + 1.1$$

End if

if($0.6 < u \leq 0.8$) Then ٨

$$x_{New} = 0.43\,x_{old} + 0.28\,y_{old}$$

$$y_{New} = -0.25\,x_{old} + 0.45\,y_{old} + 1.0$$

End if

if($0.8 < u \leq 1.0$) Then ٩

$$x_{New} = 0.42\,x_{old} + 0.26\,y_{old}$$

$$y_{New} = -0.35\,x_{old} + 0.31\,y_{old} + 0.7$$

End if

$$x_{old} = x_{New}$$

$$y_{old} = y_{New}$$

Write(*,*) x_{old} , y_{old}

Fractal Tree

۱ انتخاب نقطه شروع

$x_{old} = 0.5$

$y_{old} = 0.0$

Writ(*,*) x_{old} , y_{old}

۲ به تعداد نقاطی که می خواهیم انتخاب کنیم یک حلقه ایجاد می کنیم

۳ یک عدد تصادفی تولید می کنیم

Call random_number(u)

۴

if ($u \leq 0.1$) Then

$x_{New} = 0.05\, x_{old}$

$y_{New} = 0.6\, y_{old}$

End if

۵

if ($0.1 < u \leq 0.2$) Then

$x_{New} = 0.05\, x_{old}$

$y_{New} = -0.5\, y_{old} + 1.0$

End if

۶

if ($0.2 < u \leq 0.4$) Then

$x_{New} = 0.46\, x_{old} - 0.15\, y_{old}$

$y_{New} = 0.39\, x_{old} + 0.38\, y_{old} + 0.6$

End if

٩

if ($0.8 < u \leq 1.0$) Then

$x_{New} = 0.42\, x_{old} + 0.26\, y_{old}$

$y_{New} = -0.35\, x_{old} + 0.31\, y_{old} + 0.7$

End if

$x_{old} = x_{New}$

$y_{old} = y_{New}$

Write(*,*) x_{old} , y_{old}

٧

if ($0.4 < u \leq 0.6$) Then

$x_{New} = 0.47\, x_{old} - 0.15\, y_{old}$

$y_{New} = 0.17\, x_{old} + 0.42\, y_{old} + 1.1$

End if

٨

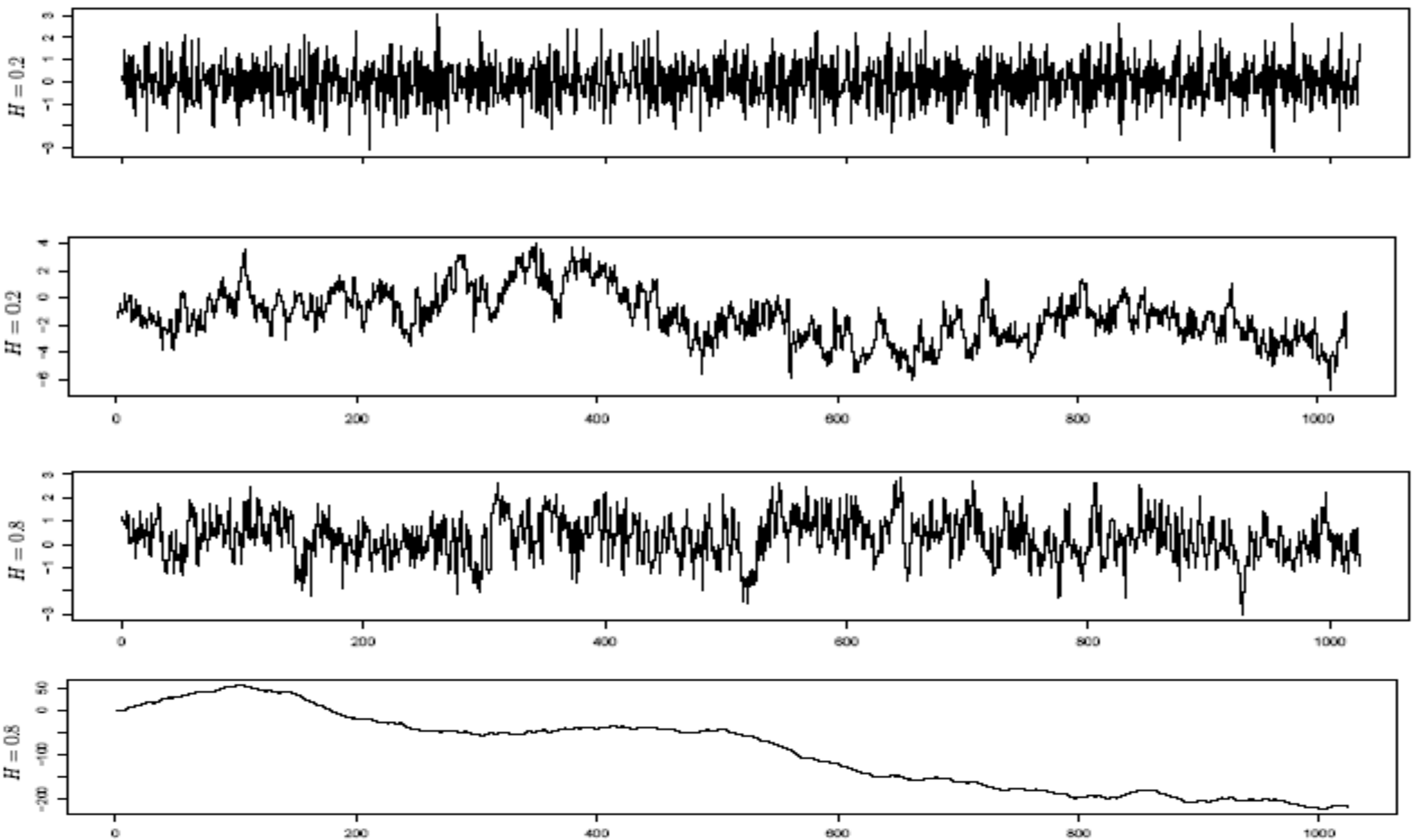
if ($0.6 < u \leq 0.8$) Then

$x_{New} = 0.43\, x_{old} + 0.28\, y_{old}$

$y_{New} = -0.25\, x_{old} + 0.45\, y_{old} + 1.0$

End if

Fractional Gaussian Noise (fGn)
&
Fractional Brownian Motion (fBm)



fGn

fBm

fGn

fBm

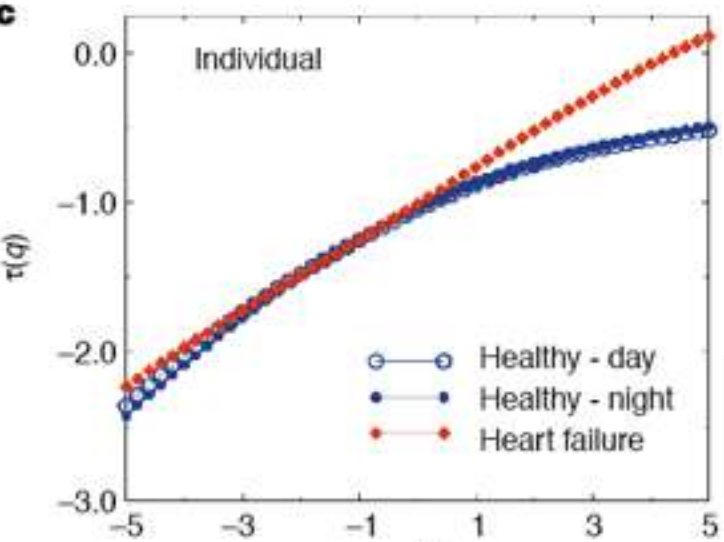
Why Fractal and multifractal ?Analysis

- Prediction of the future behavior of the systems
- Classification of various systems from complex systems point of view
- Find the universality properties of underlying systems

Multifractality in human heartbeat dynamics

Plamen Ch. Ivanov*†, Luís A. Nunes Amaral
Ary L. Goldberger†, Shlomo Havlin‡,
Michael G. Rosenblum§, Zbigniew R. Struzik

c



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Long range correlations in the heart rate variability following the injury of cardiac arrest

Shanbao Tong[a,*], Dineng Jiang[a], Ziming Wang[a], Yisheng Zhu[a], Romeryko G. Geocadin[b], Nitish V. Thakor[c]

[a] Department of Biomedical Engineering, Shanghai Jiao Tong University, Shanghai 200240, PR China
[b] Department of Neurology, Johns Hopkins School of Medicine, Baltimore, MD 21287, USA
[c] Department of Biomedical Engineering, Johns Hopkins School of Medicine, Baltimore, MD 21205, USA

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Scaling and correlation in financial time series

P. Gopikrishnan[a,*], V. Plerou[a], Y. Liu[a], L.A.N. Amaral[a], X. Gabaix[b,c], H.E. Stanley[a]

[a] Center for Polymer Studies and Department of Physics Boston University, Boston, MA 02215, USA
[b] Department of Economics, Massachusetts Institute of Technology, Cambridge, MA 02142, USA
[c] Department of Economics, The University of Chicago, Chicago, IL 60637, USA

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Statistical physics and physiology: Monofractal and multifractal approaches

H.E. Stanley[a,*], L.A.N. Amaral[a,b], A.L. Goldberger[b], S. Havlin[a,c], P.Ch. Ivanov[a,b], C.-K. Peng[b]

[a] Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215, USA
[b] Harvard Medical School, Beth Israel Deaconess Medical Center, Boston, MA 02215, USA
[c] Gonda Goldschmid Center and Department of Physics, Bar-Ilan University, Ramat Gan, Israel

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Multifractal detrended fluctuation analysis of pressure fluctuation signals in an impinging entrained-flow gasifier

Miaoren Niu*, Fuchen Wang, Qinfeng Liang, Guangsuo Yu, Zunhong Yu

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Music walk, fractal geometry in music

Zhi-Yuan Su[a], Tzuyin Wu[b,*]

[a] Department of Information Management, Chia Nan University of Pharmacy & Science, Tainan 717, Taiwan, ROC
[b] Department of Mechanical Engineering, National Taiwan University, Taipei 106, Taiwan, ROC

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Statistical physics and physiology: Monofractal and multifractal approaches

H.E. Stanley[a,*], L.A.N. Amaral[a,b], A.L. Goldberger[b], S. Havlin[a,c], P.Ch. Ivanov[a,b], C.-K. Peng[b]

[a] Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215, USA
[b] Harvard Medical School, Beth Israel Deaconess Medical Center, Boston, MA 02215, USA
[c] Gonda Goldschmid Center and Department of Physics, Bar-Ilan University, Ramat Gan, Israel

Problems and Discrepancies regarding to Observations and Models

- ~~Direct computation and determination~~

Trend and unknown noise

- Indirect computation and determination

Self-affinity in time series

- Suppose a time series as:

$$y:\{y(i)\} \quad i = 1,\ldots,N$$

$$i \rightarrow a \times i$$

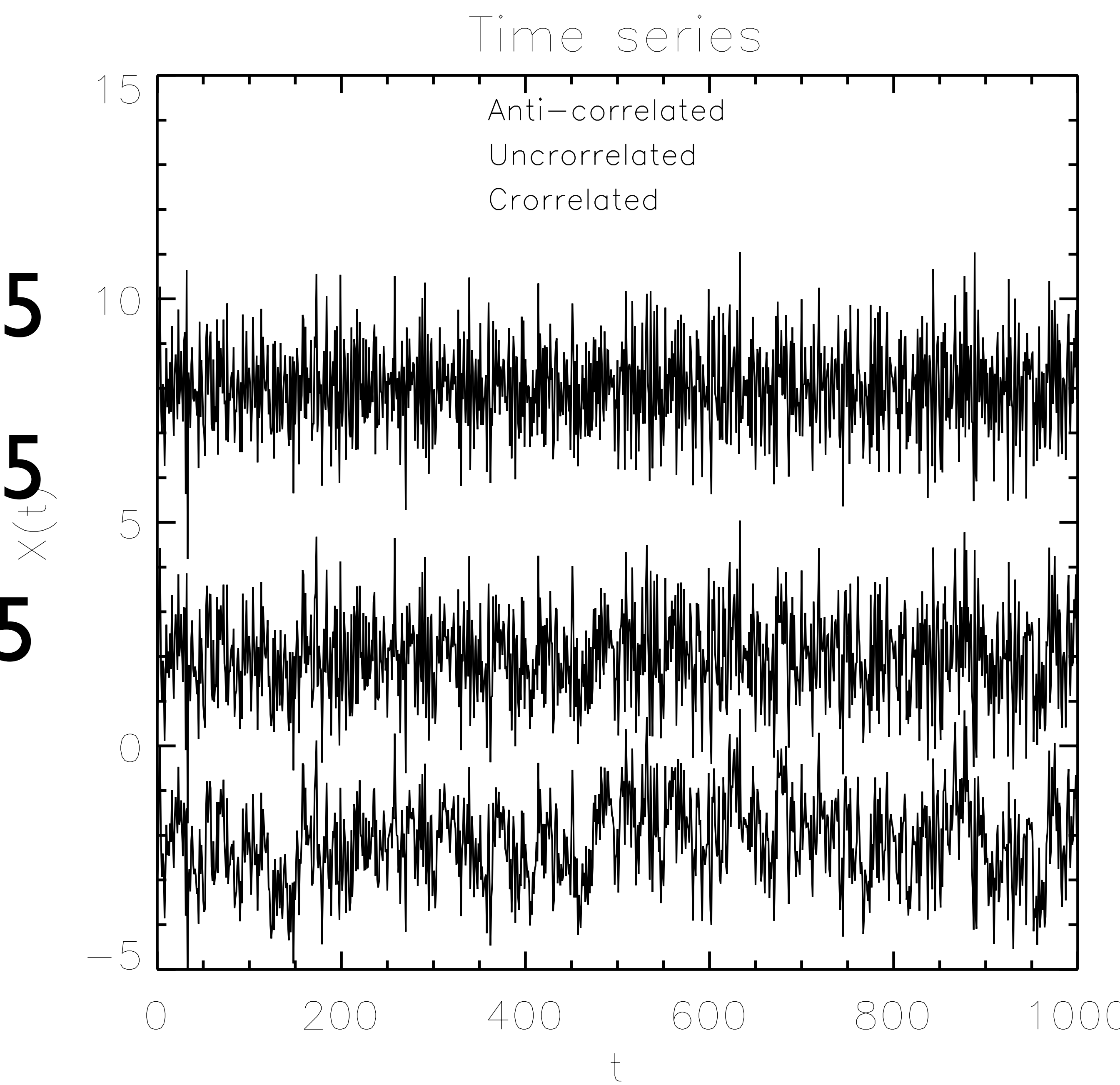
So-called Hurst exponent

$$y(a \times i) = a^H y(i)$$

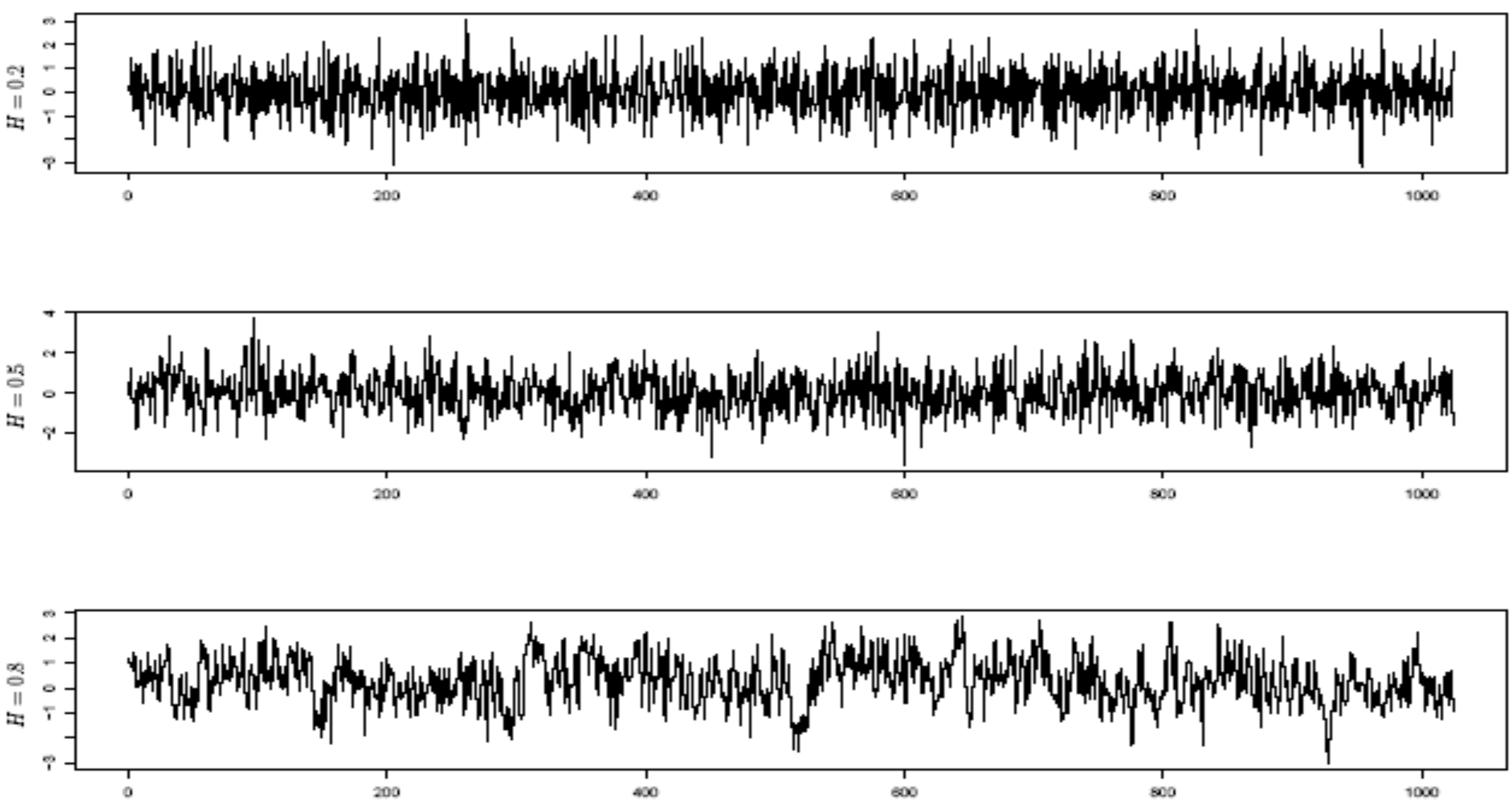
$$y(i) = x(1) + x(2) + x(3) + \ldots + x(i) = i^H x(1)$$

Classification of time series based on Hurst exponent

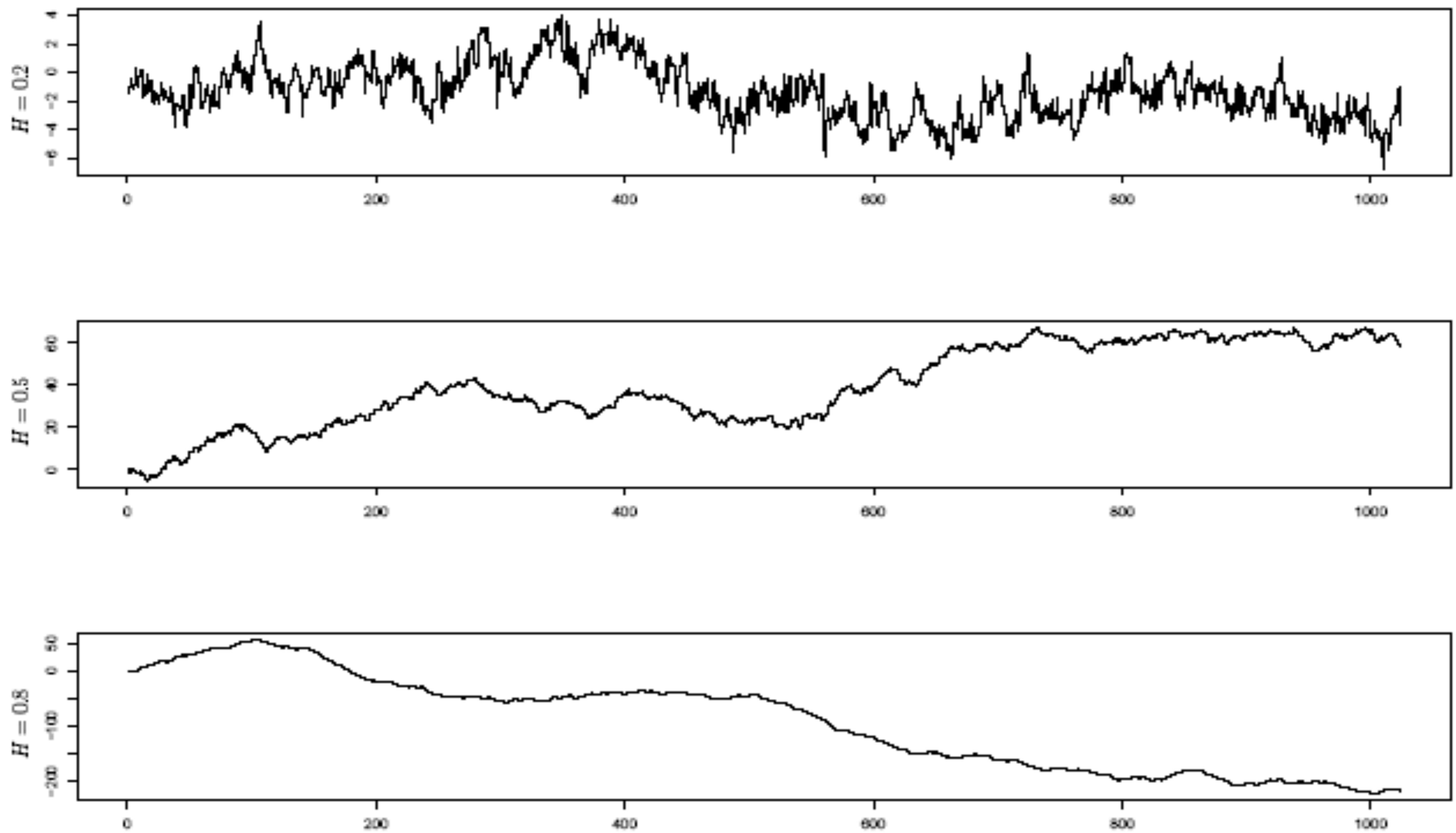
- Anti-correlated : H<0.5
- Uncorrelated: H=0.5
- Correlated: H>0.5

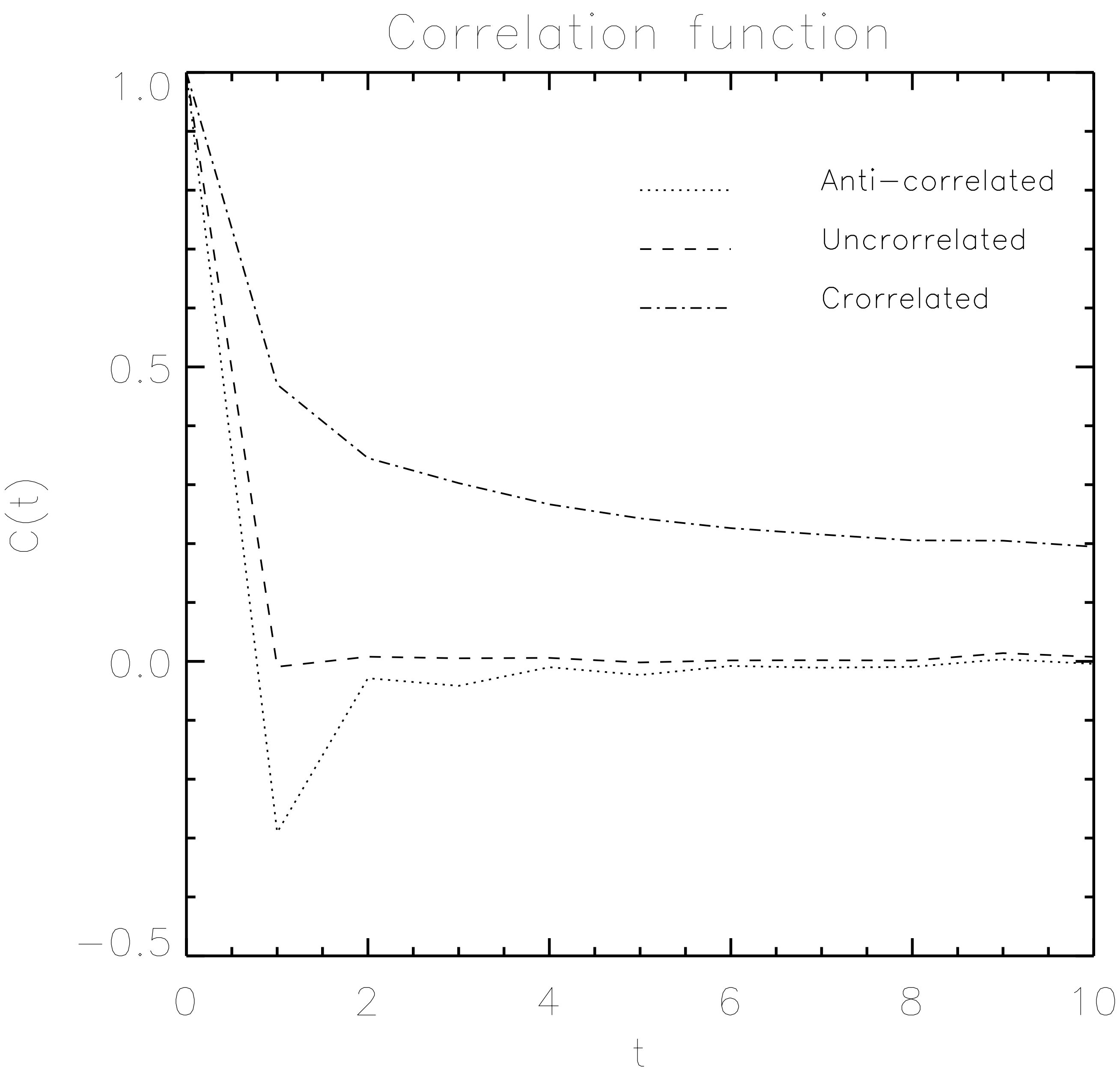


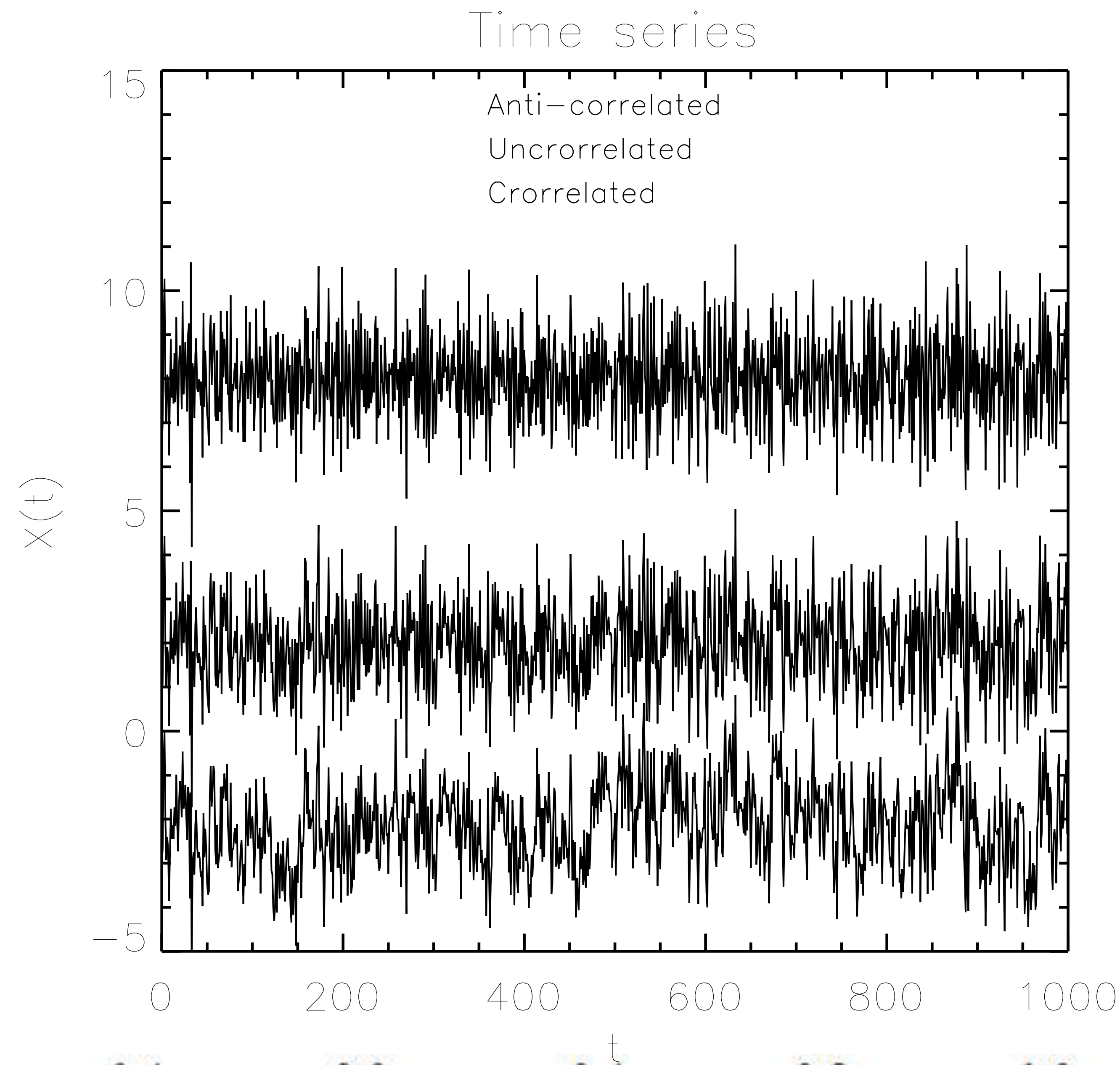
Fractional Gaussian Noise

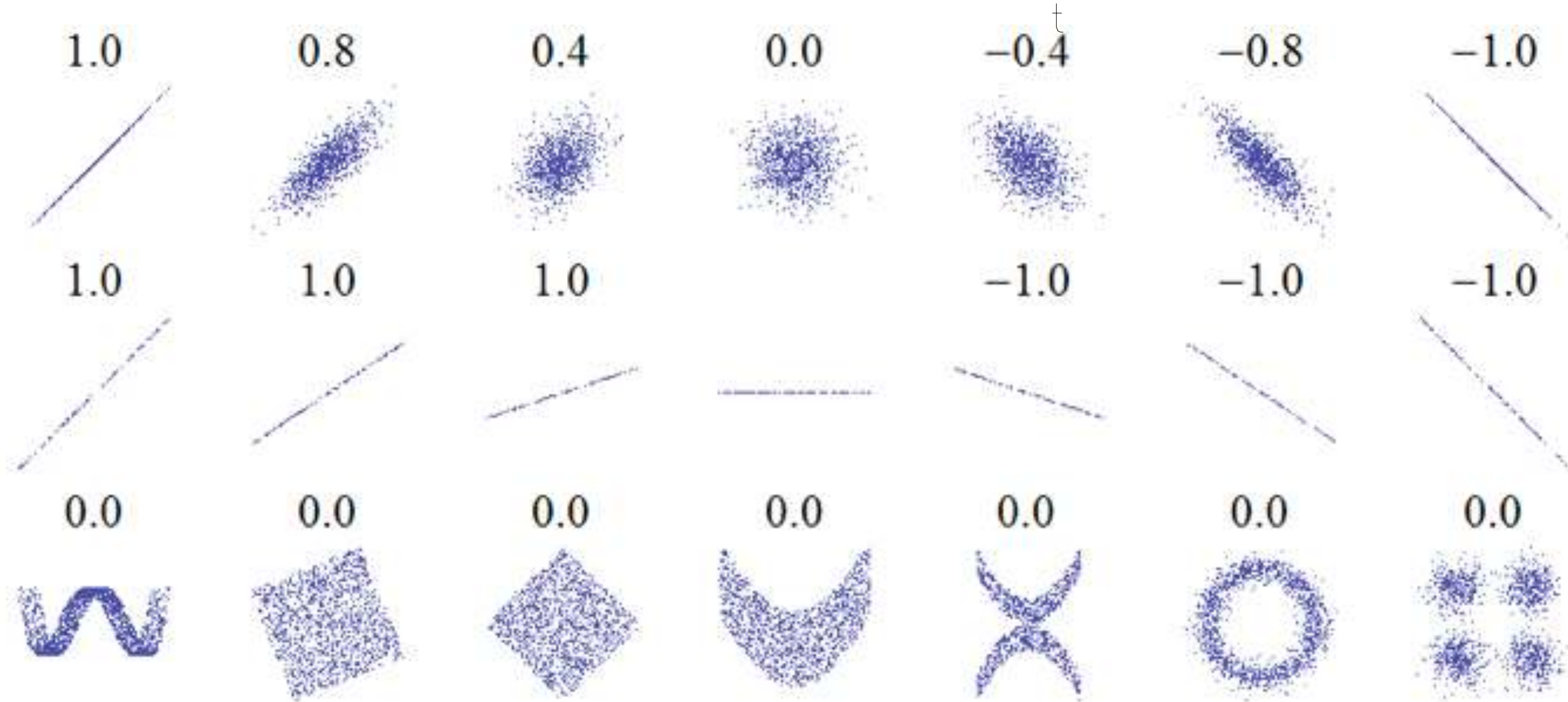


Fractional Brownian Motion









S. Kimiagar, M. Sadegh Movahed et. al., arXiv:0710.5270

Novel Fractal Analysis methods

- Hurst' rescaled range (R/S) analysis : By Hurst (1951)
- Scaled windowed variance analysis (SWA) : By Mandelbort (1985)
- Dispersional analysis (Disp) : By Bassingthwaighte (1988)
- Detrended fluctuation analysis (DFA) : By Peng (1994)
- Some state-of-the-art algorithm based on previous idea such as: MF-DFA, MF-DCCA, MF-TWDFA, DMA (BDMA & CDMA), WTMM

Detrending methods

- Parametric: Done in DFA
- Non-parametric: Empirical mode decomposition (EMD)

I must point out that now a days there are some challenge regarding to Detrending methods in multifractal analyses

Description and Application of mentioned methods

- Part A: For stationary case without trends
- Part B: For non-stationary case with trends

SWV method

- *Step 1.* Determine the 'profile'

$$Y(i) \equiv \sum_{k=1}^{i} [x_k - \langle x \rangle], \qquad i = 1, \dots, N.$$

- *Step 2.* Divide the profile $Y(i)$ into $N_s \equiv \mathrm{int}\,(N/s)$ non-overlapping segments of equal lengths s.

$$SWV(s) = \left(\frac{\text{١}}{s} \sum_{i=\text{١}}^{s} [Y(i) - \langle Y(s) \rangle]^{\text{٢}} \right)^{\text{١}/\text{٢}}$$

$$SWV(s) \sim s^H$$

R/S method

- *Step 1.* Determine the 'profile'

$$Y(i) \equiv \sum_{k=1}^{i} [x_k - \langle x \rangle], \qquad i = 1, \ldots, N.$$

- *Step 2.* Divide the profile $Y(i)$ into $N_s \equiv \mathrm{int}\,(N/s)$ non-overlapping segments of equal lengths s.

$$R(s) = Max\{Y(s)\} - Min\{Y(s)\}$$

$$S(s) = \left(\frac{۱}{s} \sum_{i=۱}^{s} [x(i) - \langle X \rangle]^{۲} \right)^{۱/۲}, \; s = ۱, \ldots, N$$

$$R(s)/S \sim s^H$$

Dispersional method

- *Step 1.* Determine the 'profile'

$$Y(i) \equiv \sum_{k=1}^{i} \left[x_k - \langle x \rangle\right], \qquad i = 1, \ldots, N.$$

- *Step 2.* Divide the profile $Y(i)$ into $N_s \equiv \mathrm{int}(N/s)$ non-overlapping segments of equal lengths s.

$$\mu(\nu, s) = \frac{1}{s} \sum_{i=1}^{s} Y[(\nu - 1)s + i]$$

$$\langle \mu(s) \rangle = \frac{1}{N_s} \sum_{\nu=1}^{N_s} \mu(\nu, s)$$

$$M(s) = \frac{1}{N_s} \sum_{\nu=1}^{N_s} [\mu(\nu, s) - \langle \mu(s) \rangle]$$

$$M(s) \sim s^{2H}$$

Multi-Fractal Detrended Fluctuation in 1D

DFAm remove trend of order m in profile or trend of order m-1 in original seris

- *Step 1.* Determine the 'profile'

$$Y(i) \equiv \sum_{k=1}^{i} \left[x_k - \langle x \rangle\right], \qquad i = 1, \ldots, N.$$

- *Step 2.* Divide the profile $Y(i)$ into $N_s \equiv \mathrm{int}\,(N/s)$ non-overlapping segments of equal lengths s.

- *Step 3.* Calculate the local trend for each of the $2N_s$ segments by a least squares fit of the series. Then determine the variance

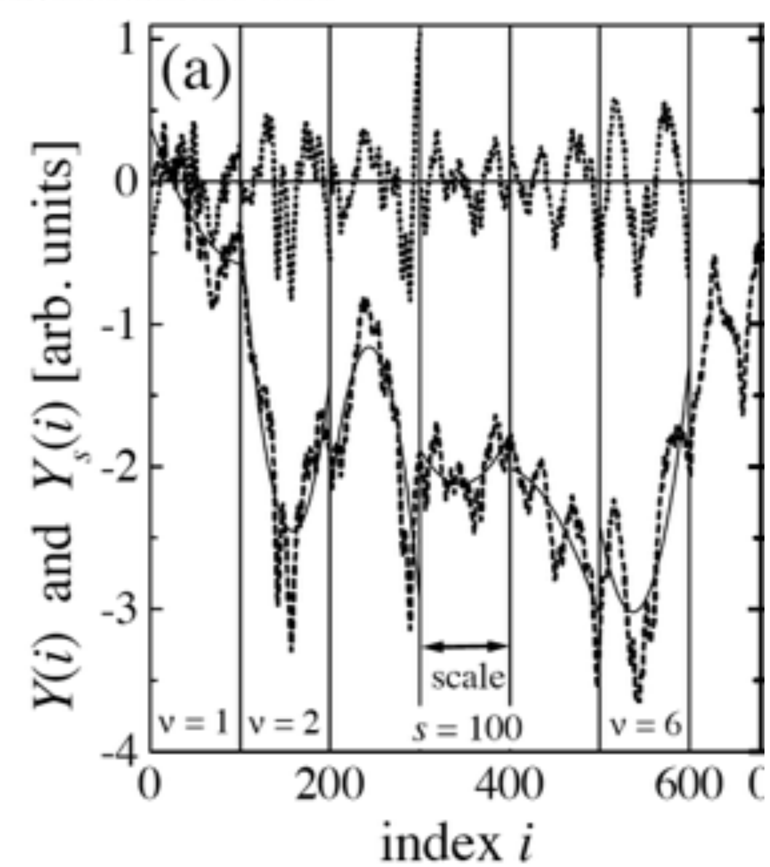
$$F^2(s,\nu) \equiv \frac{1}{s}\sum_{i=1}^{s} \left\{Y[(\nu-1)s+i] - y_\nu(i)\right\}^2$$

for each segment ν, $\nu = 1, \ldots, N_s$, and

$$F^2(s,\nu) \equiv \frac{1}{s}\sum_{i=1}^{s} \left\{Y[N-(\nu-N_s)s+i] - y_\nu(i)\right\}^2,$$

for $\nu = N_s + 1, \ldots, 2N_s$.

Here, $y_\nu(i)$ is the fitting polynomial in segment ν.



- *Step 4*. Average over all segments to obtain the qth-order fluctuation function, defined as

$$F_q(s) \equiv \left\{ \frac{1}{2N_s} \sum_{\nu=1}^{2N_s} \left[F^2(s,\nu) \right]^{q/2} \right\}^{1/q}$$

$F_q(s)$ is only defined for $s \geq m+2$.

- *Step 5*. Determine the scaling behaviour of the fluctuation functions by analysing log–log plots of $F_q(s)$ versus s for each value of q. If the series x_i are long range power law correlated, $F_q(s)$ increases, for large values of s, as a power law,

$$F_q(s) \sim s^{h(q)}.$$

$\{X\} : \{F_q(s)\}$ $\qquad$ $\{\Theta\} : \{h(q)\}$

$$P(h(q)|X) = \frac{\mathcal{L}(X|h(q))P(h(q))}{\int \mathcal{L}(X|h(q))dh(q)} \qquad \mathcal{L}(X|h(q)) \sim \exp\left(\frac{-\chi^2(h(q))}{2}\right)$$

$$\chi^2(h(q)) = \int ds \frac{[F_{\mathrm{obs.}}(s) - F_{\mathrm{The.}}(s;h(q))]^2}{\sigma^2_{\mathrm{obs.}}(s)}$$

$$68.3\% = \int_{-\sigma^-}^{+\sigma^+} \mathcal{L}(X|h(q))dh(q) \qquad h^{+\sigma^+}_{-\sigma^-}$$

Jan W. Kantelhart, et. al., arXiv:physics/0202070; M. Sadegh Movahed et. al., arXiv:physics/050814
S. Hajian and M. Sadegh Movahed, arXiv:0908.013

h(2) and Hurst exponent in DFA1 for fGn

$$\begin{aligned} F^2(s) &\equiv \frac{1}{N_s}\sum_{\nu=1}^{N_s}[F^2(s;\nu)], \\ &= \left\langle [F^2(s;\nu)] \right\rangle_\nu , \\ &\equiv \mathcal{C}_H s^{2H}, \end{aligned} \quad \text{(1}$$

$$F^2(s;\nu) = \frac{1}{s}\sum_{i=1}^{s}[Y_\nu(i) - y_\nu(i)]^2 \quad \text{(2}$$

$$Y(i) = \sum_{k=1}^{i} x(k) - \langle x \rangle \quad \text{(3}$$

$$y_\nu(i) = a_\nu + b_\nu i \quad \text{(4}$$

$$\left\langle[F^2(s;v)]\right\rangle = \left\langle\frac{1}{s}\sum_{i=1}^{s}[Y(i)-a-bi]^2\right\rangle$$

$$\simeq \left\langle\frac{1}{s}\sum_{i=1}^{s}Y(i)^2\right\rangle + \langle a^2\rangle + \frac{s^2}{3}\langle b^2\rangle + s\,\langle ab\rangle - 2\left\langle\frac{a}{s}\sum_{i=1}^{s}Y(i)\right\rangle - 2\left\langle\frac{b}{s}\sum_{i=1}^{s}iY(i)\right\rangle$$

$$Y(i) = i^H x \longrightarrow Y(i) - Y(k) = |i-k|^H x$$

$$\langle[Y(i)-Y(k)]^2\rangle = \sigma^2|i-k|^{2H} \qquad \sigma^2 = \langle x(i)^2\rangle \qquad \langle Y(i)^2\rangle = \sigma^2 i^{2H}$$

$$\langle Y(i)Y(k)\rangle = \frac{\sigma^2}{2}[i^{2H}+k^{2H}-|i-k|^{2H}]$$

$$\left\langle[F^2(s;v)]\right\rangle_v = C_{\mathcal{H}}(s)^{2H}$$

$$\sum_{i,j=1}^{s}\langle iY(i)Y(j)\rangle = \frac{\sigma^2}{2}\sum_{i,j=1}^{s}\left(i^{2H+1}+ij^{2H}-i|i-j|^{2H}\right),$$

$$= \frac{\sigma^2}{2}\sum_{i,j=1}^{s}\left(i^{2H+1}+ij^{2H}\right) - \frac{\sigma^2}{2}\sum_{i=1}^{s}\sum_{j=1}^{i}i(i-j)^{2H} - \frac{\sigma^2}{2}\sum_{i=1}^{s}\sum_{j=i}^{s}i(j-i)^{2H},$$

$$\sim \frac{\sigma^2}{2}\left(\frac{s^{2H+3}}{2H+2}+\frac{s^{2H+3}}{2(2H+1)}\right) - \frac{\sigma^2}{2}\sum_{i=1}^{s}i^{2H+2}\left(\int_0^1(1-x)^{2H}\mathrm{d}x - \int_0^1 x(1-x)^{2H}\mathrm{d}x\right)$$

$$= \frac{\sigma^2 s^{2H+3}}{4}\left(\frac{2}{H+1}-\frac{1}{2H+1}\right)$$

$$\sum_{i,j=1}^{s} \langle Y(i)Y(j)\rangle = \frac{\sigma^2}{2}\sum_{i,j=1}^{s}\left(i^{2H} + j^{2H} - |i-j|^{2H}\right),$$

$$= \frac{\sigma^2}{2}\sum_{i,j=1}^{s}\left(i^{2H} + j^{2H}\right) - \sigma^2 \sum_{i=1}^{s}\sum_{j=1}^{i}(i-j)^{2H},$$

$$\sim \sigma^2\left(\frac{s^{2H+2}}{2H+1} - \sum_{i=1}^{s} i^{2H+1}\int_0^1 (1-x)^{2H}\right),$$

$$\sim \sigma^2 s^{2H+2}\left(\frac{1}{2H+1} - \frac{1}{(2H+2)(2H+1)}\right).$$

$$F^2(s) \equiv \frac{1}{N_s}\sum_{\nu=1}^{N_s}[F^2(s;\nu)],$$

$$= \left\langle [F^2(s;\nu)]\right\rangle_{\nu},$$

$$\equiv \mathcal{C}_H s^{2H},$$

$$\mathcal{C}_{\mathcal{H}} = \frac{\sigma^2}{(2H+1)} - \frac{4\sigma^2}{2H+2} + 3\sigma^2\left(\frac{2}{H+1} - \frac{1}{2H+1}\right) - \frac{3\sigma^2}{(H+1)}\left(1 - \frac{1}{(H+1)(2H+1)}\right)$$

$$h(q=2) = H$$

M. S. Taqqu et. al., Fractals, Vol. 3, No. 4 (1995)M. Sadegh Movahed et. al., arXiv:physics/0608056

h(2) and Hurst exponent in DFA1 for fBm

$$
\begin{aligned}
\langle [F^2(s,\nu)] \rangle &= \left\langle \frac{1}{s} \sum_{i=1}^{s} (Y(i) - a - bi)^2 \right\rangle \\
&\simeq \left\langle \frac{1}{s} \sum_{i=1}^{s} Y(i)^2 \right\rangle + \langle a^2 \rangle + \frac{s^2}{3} \langle b^2 \rangle \\
&\quad - 2 \left\langle \frac{a}{s} \sum_{i=1}^{s} y(i) \right\rangle - 2 \left\langle \frac{b}{s} \sum_{i=1}^{s} iY(i) \right\rangle + s \langle ab \rangle , \\
&= \left\langle \frac{1}{s} \sum_{i=1}^{s} Y(i)^2 \right\rangle - \frac{4}{s^2} \left\langle \left[\sum_{i=1}^{s} Y(i) \right]^2 \right\rangle \\
&\quad - \frac{12}{s^4} \left\langle \left[\sum_{i=1}^{s} iY(i) \right]^2 \right\rangle + \frac{12}{s^3} \left\langle \sum_{i=1}^{s} iY(i) \sum_{i=1}^{s} Y(i) \right\rangle \\
&= \frac{A}{s} - \frac{4}{s^2} B - \frac{12}{s^4} D + \frac{12}{s^3} C
\end{aligned}
$$

$$
\begin{aligned}
x(i) &= Y(i) - Y(i-1) \\
u(i) &= x(i) - x(i-1),
\end{aligned}
$$

$$
\langle x(i) x(j) \rangle = \frac{\sigma^2}{2} \left[i^{2H} + j^{2H} - |i-j|^{2H} \right],
$$

$$
\langle Y(i) Y(j) \rangle = \frac{\sigma^2}{(H+1)^2} (ij)^{H+1},
$$

For fBm series

$$\left\langle \left[F^2(s,\nu)\right]\right\rangle_\nu = \mathcal{C}_H s^{2(H+1)},$$

$$\mathcal{C}_H = \frac{\sigma^2}{(2H+3)(H+1)^2} - \frac{4\sigma^2}{[(H+1)(H+2)]^2} - \frac{12\sigma^2}{[(H+1)(H+3)]^2} + \frac{12\sigma^2}{(H+1)^2(H+2)(H+3)}.$$

For fGn series

$$\left\langle [F^2(s;\nu)]\right\rangle_\nu = \mathcal{C}_{\mathcal{H}}(s)^{2H}$$

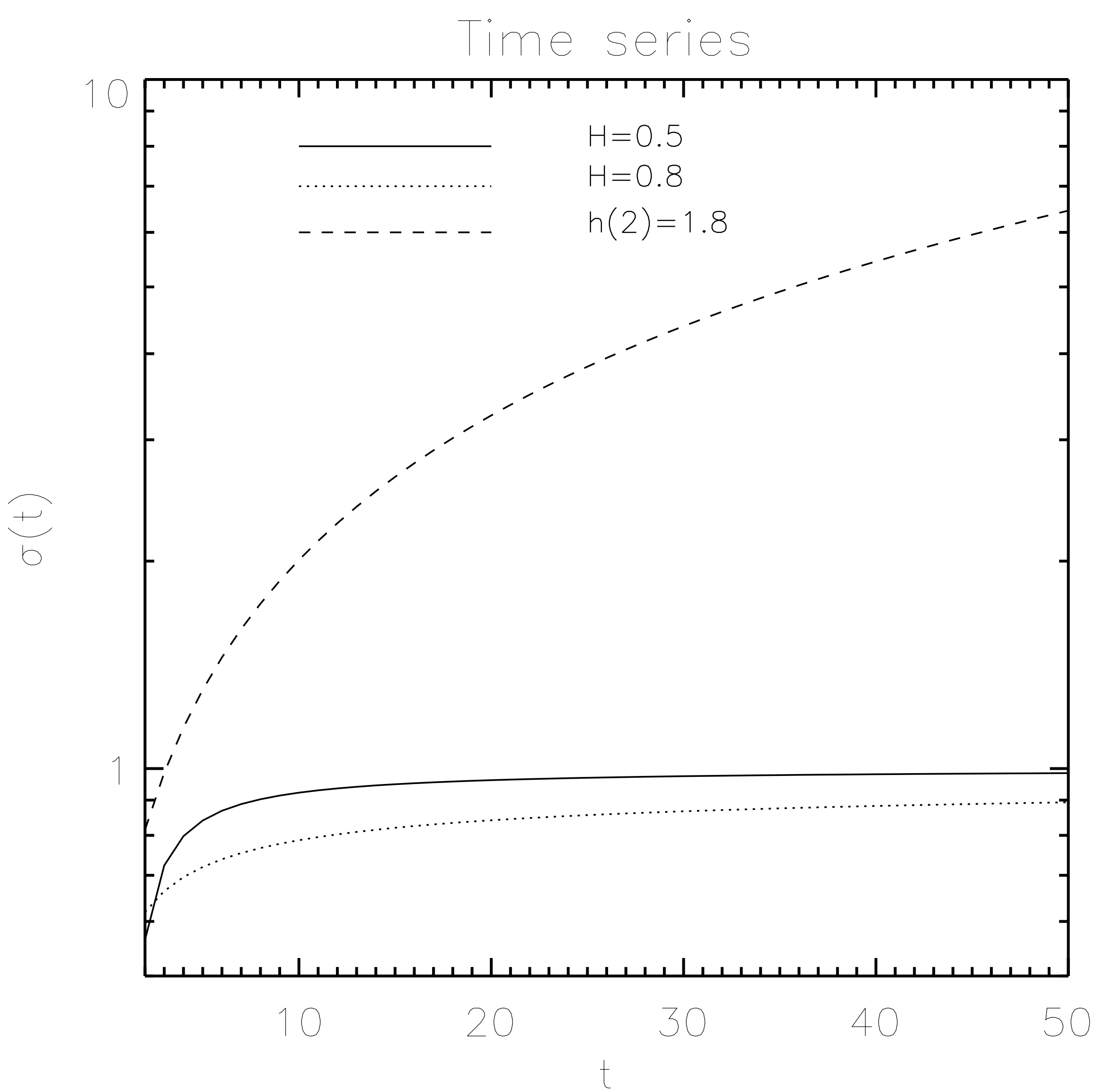
$$\mathcal{C}_{\mathcal{H}} = \frac{\sigma^2}{(2H+1)} - \frac{4\sigma^2}{2H+2} + 3\sigma^2\left(\frac{2}{H+1} - \frac{1}{2H+1}\right) - \frac{3\sigma^2}{(H+1)}\left(1 - \frac{1}{(H+1)(2H+1)}\right)$$

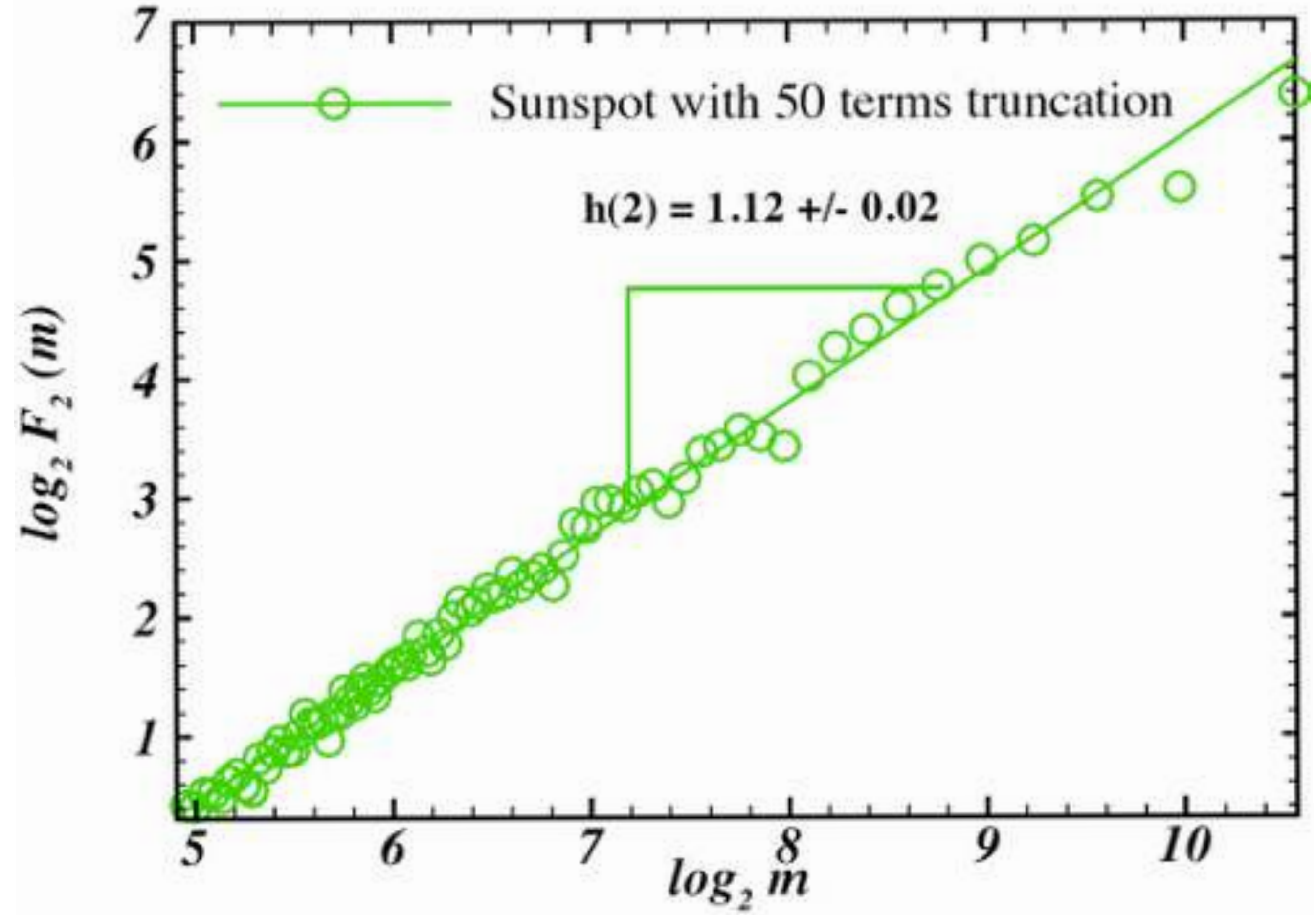
$$h(q=2) = H$$

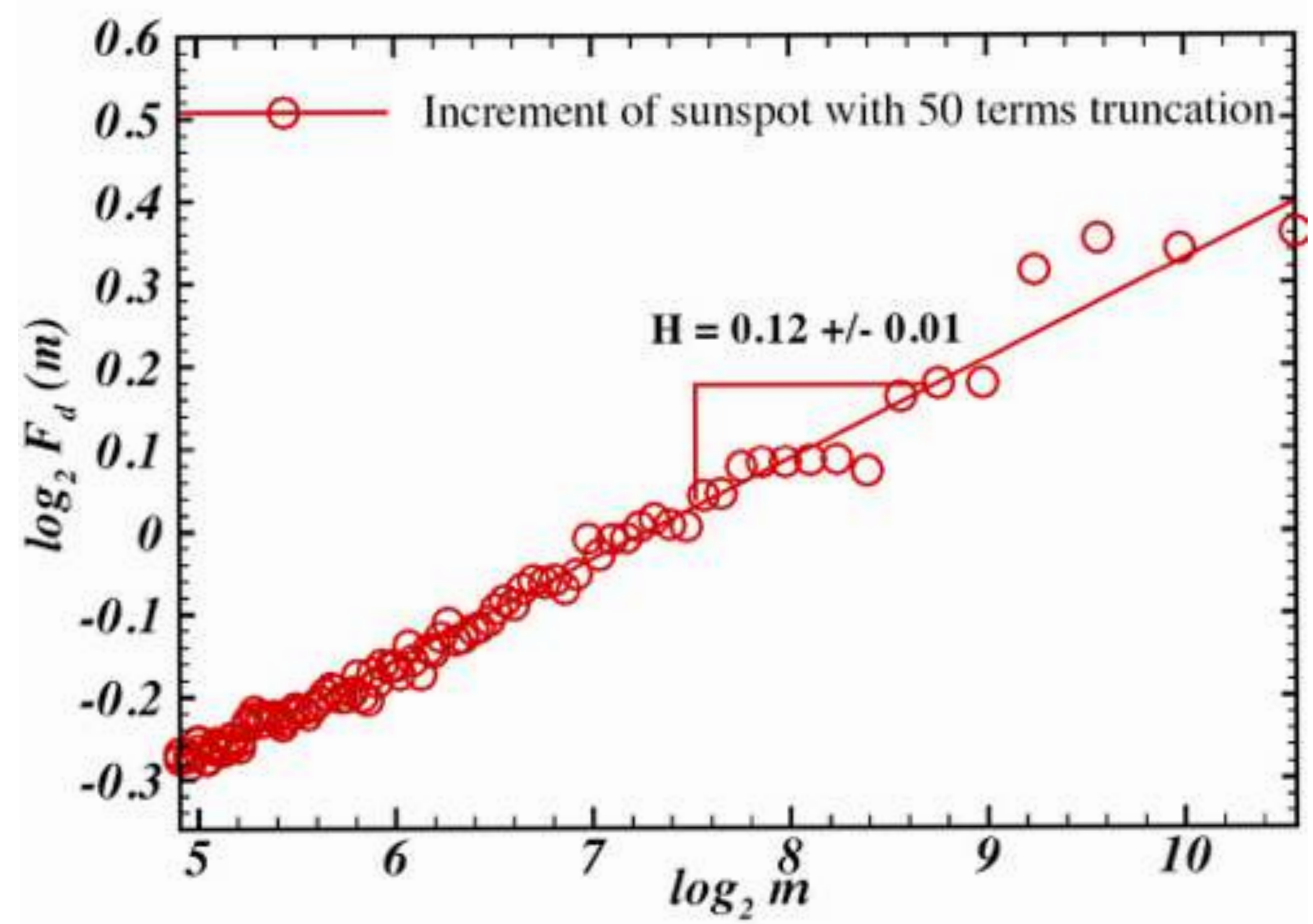
M. Sadegh Movahed et. al., arXiv:physics/050814

For fBm: $h(q=2)>1$

For fGn: $h(q=2)<1$







Scaling exponents

- Multifractal scaling exponent $\tau(q) = qh(q) - 1$
- Generalized multifractal dimension $D(q) = \dfrac{\tau(q)}{q-1}$
- Autocorrelation exponent $\begin{cases} C(s) : s^{-\gamma} \\ C(i,j) : i^{-\gamma} + j^{-\gamma} - |i-j|^{-\gamma} \end{cases}$
- Power spectrum scaling exponent $S(\omega) : \omega^{-\beta}$
- Holder exponent $\alpha = \tau'(q)$

  $\alpha = h(q) + qh'(q)$
- Singularity spectrum $f(\alpha) = q\left[\alpha - h(q)\right] + 1$

Correlation and Hurst exponents

$$C(s)=\frac{\langle x(i+\tau)x(i)\rangle}{\sigma^2} : \tau^{-\gamma}$$

$$Y(s)=\sum_{k=1}^{s}x(k)=x(1)\times s^{H}$$

$$\langle Y(s)^2\rangle=\sigma^2\times s^{2H}$$

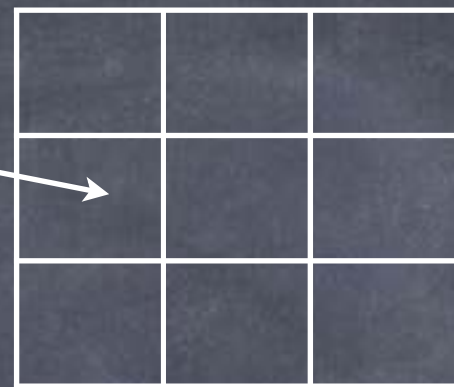
$$=\left\langle\left(\sum_{k=1}^{s}x(k)\right)^2\right\rangle=\left\langle\sum_{k=1}^{s}x(k)^2\right\rangle+\left\langle\sum_{k\neq j}^{s}x(k)x(j)\right\rangle$$

$$=i\sigma^2+2\sum_{j=1}^{s-1}(s-j)C(j) : s^{2-\gamma}=s^{2H}\rightarrow\gamma=2-2H$$

$$\mathit{for}\qquad 0.5<H<1$$

Generalized fractal dimension based on partition function

$p(\nu, s)$



$$\sum_{\nu=1}^{N_s} p(\nu, s) = 1$$

$$p^2(\nu, s) = \frac{1}{s}\sum_{i=1}^{s}\left\{Y[(\nu-1)s+i] - y_\nu(i)\right\}^2$$

$$F_q(s) = \left(\frac{1}{N_s}\sum_{\nu=1}^{N_s}|p(\nu, s)|^q\right)^{1/q}$$

$$Z_q(s) \equiv \sum_{\nu=1}^{N_s}|p(\nu, s)|^q \sim s^{\tau(q)}$$

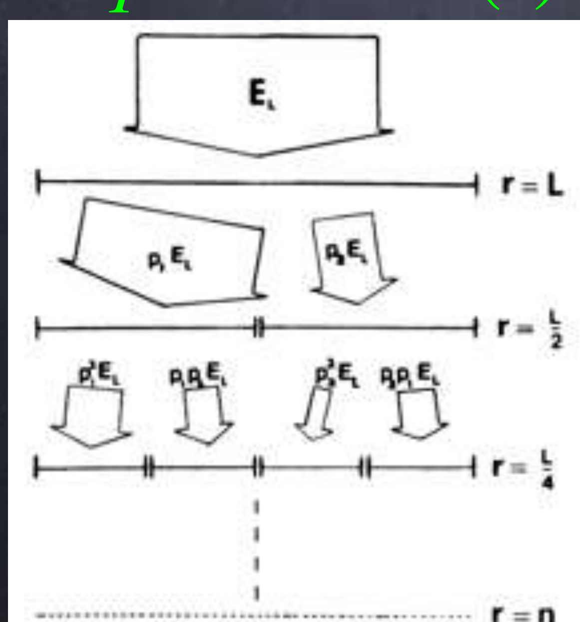
$$Z_q(s) \equiv \sum_{\nu=1}^{N_s}|p(\nu, s)|^q = N_s F_q^q(s)$$

$$\sim \frac{N}{s} s^{qh(q)} \sim s^{qh(q)-1} \sim s^{\tau(q)}$$

$$D(q) \equiv \frac{1}{q-1}\lim_{s\to 0}\frac{\ln Z_q(s)}{\ln s} = \frac{\tau(q)}{1-q}$$

for $q = 0$ $\quad D(0) = D_f$

for $q = 1$ $\quad D(1) \sim \sum p \ln p$



$\tau(q) = qh(q) - 1$

Free energy and T^{-1}

Singularity spectrum

criterion for scaling behavior of
easure at each subinterval of time
eries

$$p(v,s):\ s^{\alpha_v} \qquad for \quad s \to 0$$

$$PDF \to \mu(\alpha):\ l^{-f(\alpha)}$$

$$\alpha = \tau'(q)$$

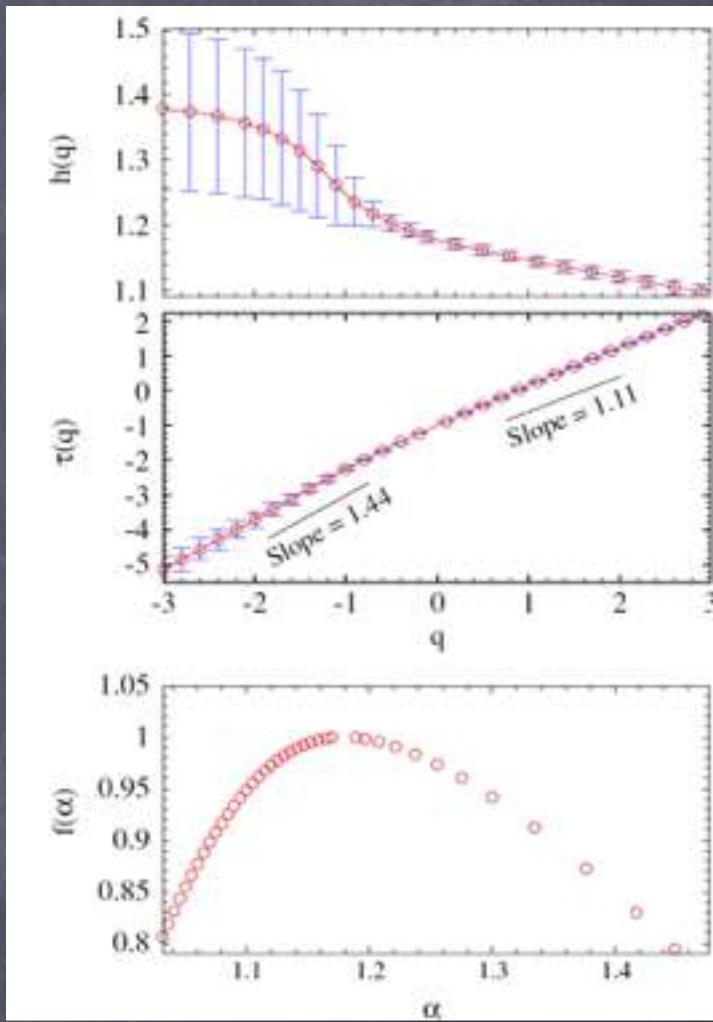
$$\alpha = h(q) + qh'(q)$$

$$f(\alpha) = q\left[\alpha - h(q)\right] + 1$$

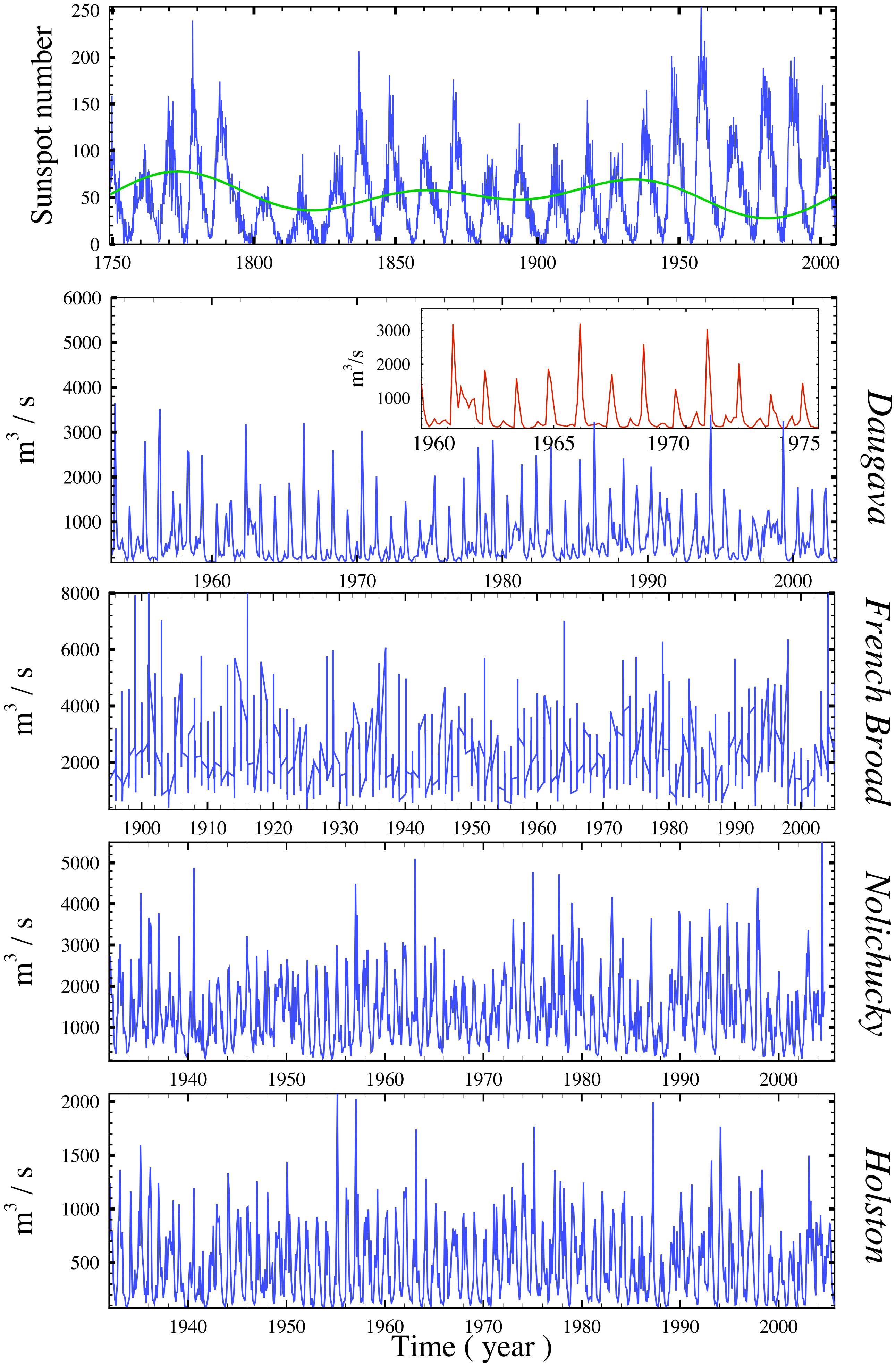
$$\Delta\alpha \equiv \alpha(q_{\min}) - \alpha(q_{\max})$$

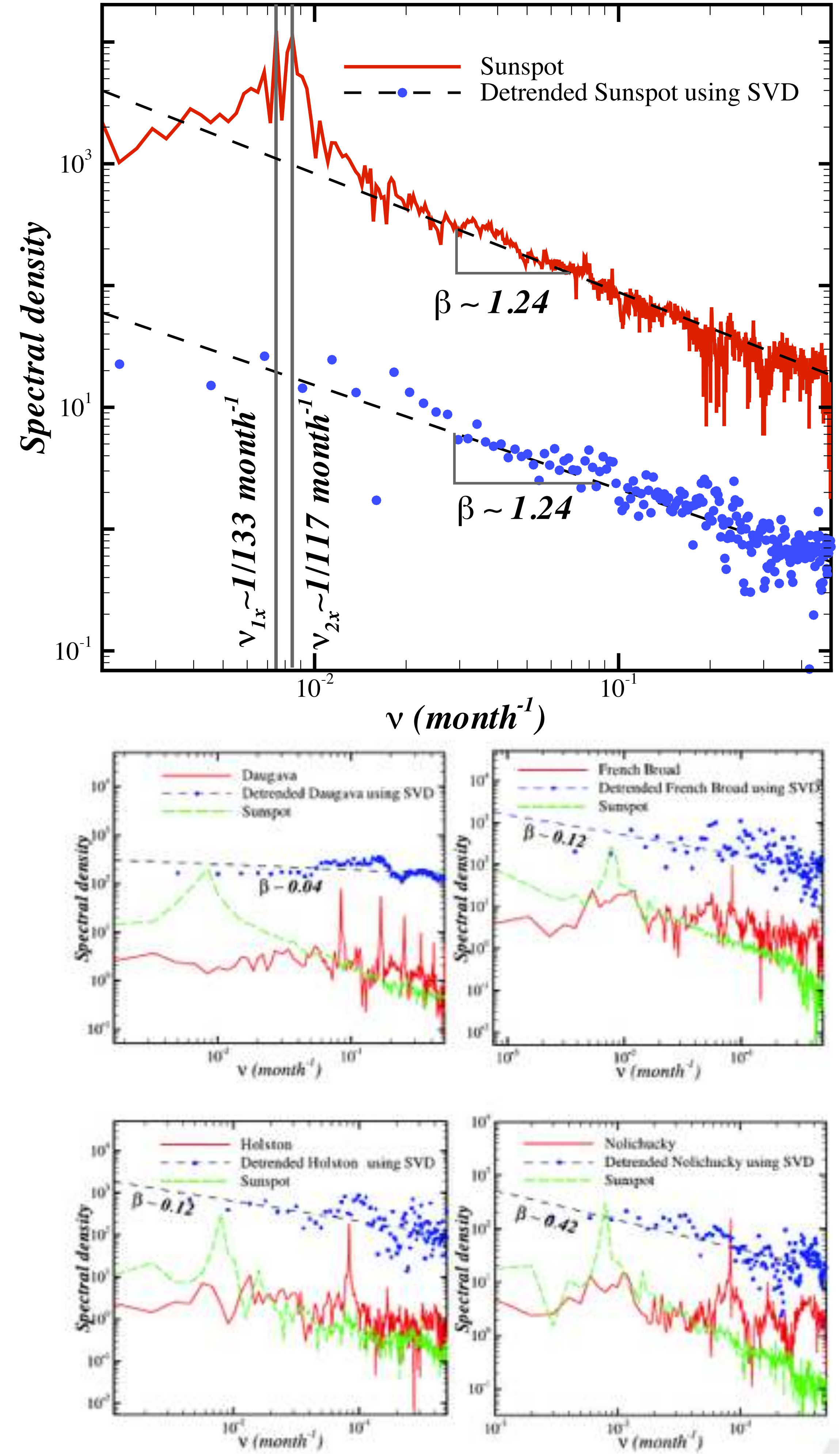
$$\Delta\alpha \to 0$$

$$f(\alpha = H) = 1$$



A Holder exponent represents monofractal process while the existence of spectrum for Holder exponent demonstrates multifractality nature of time series


Sunspot number
Daugava
French Broad
Nolichucky
Holston
m³ / s
Time (year)


Sunspot
Detrended Sunspot using SVD
Spectral density
β ~ 1.24
β ~ 1.24
ν1x~1/133 month-1
ν2x~1/117 month-1
ν (month-1)
Daugava
Detrended Daugava using SVD
Sunspot
β ~ 0.04
French Broad
Detrended French Broad using SVD
β ~ 0.12
Holston
Detrended Holston using SVD
β ~ 0.12
Nolichucky
Detrended Nolichucky using SVD
β ~ 0.42

Power spectrum exponent

$$S(\nu) \sim \nu^{-\beta}$$

$$S(\nu) : \nu^{-\beta} : \nu^{-1+\gamma}$$

$$\gamma = 2 - 2H$$

$\beta = 2H - 1$ For fGn

$\beta = 2H + 1$ For fBm

$D(fractal-dimention)$	H_{fBm}	H_{fGn}	β	$h(q=۲)$	
—	$۲-H_{fBm}$	—	$\frac{۵-\beta}{۲}$	$۳-h(q=۲)$	D
$۲-D$	—	—	$\frac{\beta-۱}{۲}$	$h(q=۲)-۱$	H_{fBm}
—	—	—	$\frac{\beta+۱}{۲}$	$h(q=۲)$	H_{fGn}
$۵-۲D$	$۲H_{fBm}+۱$	$۲H_{fGn}-۱$	—	$۲h(q=۲)-۱$	β
$۳-D$	$H_{fBm}+۱$	H_{fGn}	$\frac{\beta+۱}{۲}$	—	$h(q=۲)$

q	$\tau(q)$	$\alpha=-\frac{d\tau(q)}{dq}$	$f=q\alpha+\tau(q)$
$q\to-\infty$	$-q\alpha_{\max}$	$\alpha_{\max}=-\ln\mu_{-}/\ln\delta$	0
$q=0$	D	α_0	D
$q=1$	0	$\alpha_1=-S(\delta)/\ln\delta$	α_1
$q\to+\infty$	$-q\alpha_{\min}$	$\alpha_{\min}=-\ln\mu_{+}/\ln\delta$	0

Fractals: Jens Feder 1988

Multifractality

A: $h(q)$ depends on "q"

B: There is a spectrum for holder exponent

C: There are various slopes for $\tau(q)$ in different scales

What are the sources?

1) Multifractality due to a fatness of PDF

2) Multifractality due to different correlations in small and large scales

$$F_q(s)/F_q^{\text{shuf}}(s) \sim s^{h(q)-h_{\text{shuf}}(q)} = s^{h_{\text{cor}}(q)},$$

$$F_q(s)/F_q^{\text{sur}}(s) \sim s^{h(q)-h_{\text{sur}}(q)} = s^{h_{\text{PDF}}(q)}.$$

$h_{\text{cor}}(q) = 0$ For Fatness

$h_{\text{PDF}}(q) = 0$ For correlation

Surrogate method

(i) Computing the discrete Fourier transform (DFT) coefficients of the series

$$\mathcal{F}^2\{x(t)\} \equiv |X(\nu)|^2 = |X(k)|^2 = \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(t_n) \mathrm{e}^{\mathrm{i}2\pi nk/N} \right|^2 \tag{9}$$

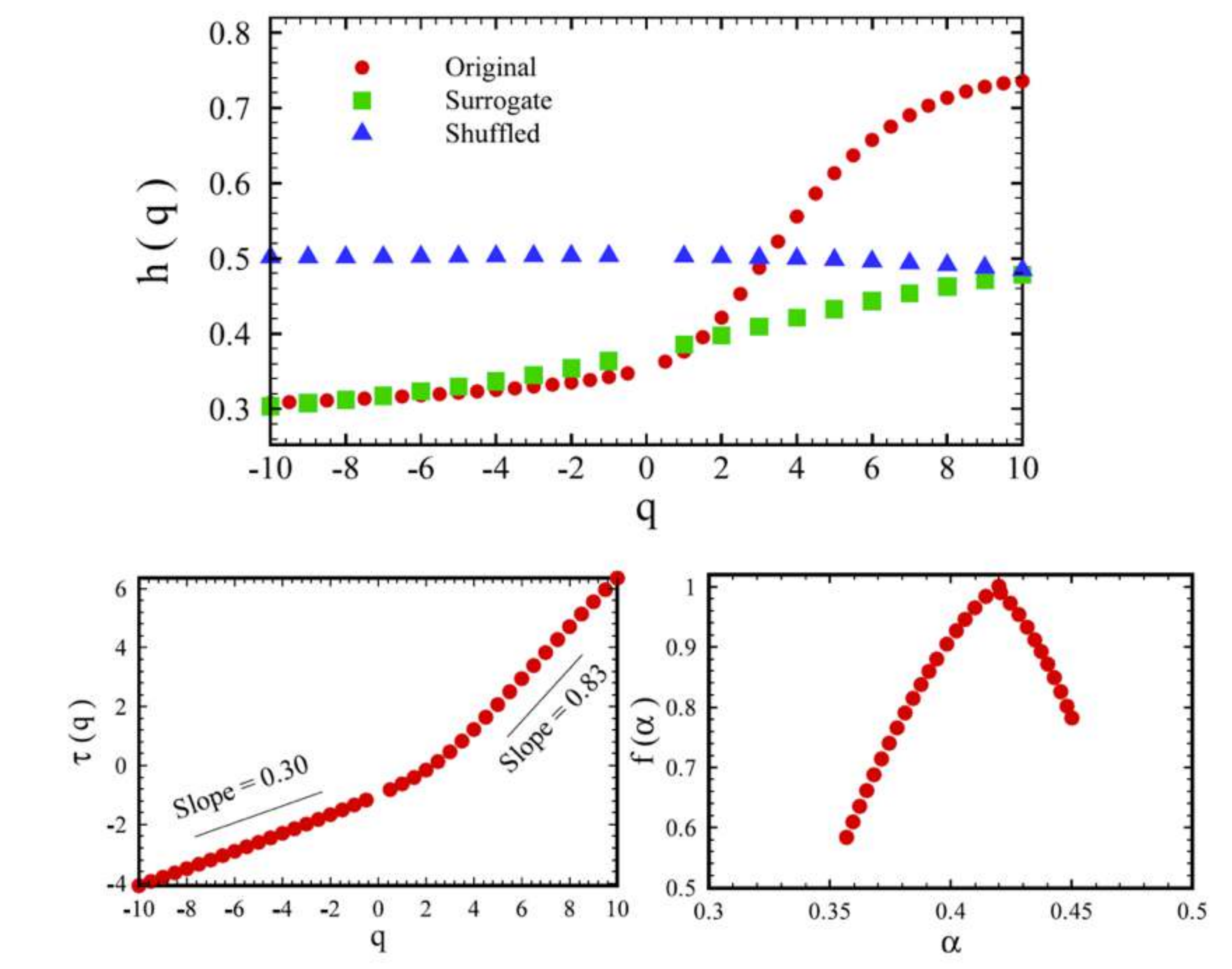
where $\nu = k/N\Delta t$ and Δt is the step of digitization in the experimental setup.

(ii) Multiplying the DFT coefficients of the series by a set of pseudo-independent, uniformly distributed $\phi(\nu)$ quantities in the range $[0, 2\pi)$:

$$\tilde{X}(\nu) = X(\nu)\mathrm{e}^{\mathrm{i}\phi(\nu)}. \tag{10}$$

(iii) The surrogate data set is given by the inverse DFT as

$$\mathcal{F}^{-1}\{\tilde{X}(\nu)\} \equiv \tilde{x}(t_n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |X_k| \, \mathrm{e}^{\mathrm{i}\phi(k)} \mathrm{e}^{-\mathrm{i}2\pi nk/N}. \tag{11}$$



(S. Kimiagar, M. Sadegh Movahed et. al., JSTAT P03020 (2009

MF-DFA in higher dimension

In many cases, one encounters with self-similar of self-affine surface which is denoted by a two dimensional array X(i,j). For this case the MF-DFA has the following steps:

Step I: Suppose

$$x(i,j), \quad \begin{cases} i = 1,\dots,M \\ j = 1,\dots,N \end{cases}$$

$$M_s = \mathrm{int}\left(\frac{N}{s}\right)$$

$$N_s = \mathrm{int}\left(\frac{M}{s}\right)$$

$$x_{v,w}(i,j) = x(l_1 + i, l_2 + j) \quad 1 \le i,j \le s \qquad \begin{cases} l_1 = (v-1)s \\ l_2 = (w-1)s \end{cases}$$

Step II: For each non-overlapping segment, the cumulative sum is calculated by:

$$Y_{v,w}(i,j) = \sum_{k=1}^{i}\sum_{l=1}^{j} x_{v,w}(k,l) \qquad 1 \le i,j \le s$$

Step III: The trend of constructed cumulative arrays such as:

$$u_{v,w}(i,j) = a_{v,w}i + b_{v,w}j + c_{v,w}$$

$$u_{v,w}(i,j) = a_{v,w}i^2 + b_{v,w}j^2 + c_{v,w}$$

$$u_{v,w}(i,j) = a_{v,w}ij + b_{v,w}i + c_{v,w}j + d_{v,w}$$

$$u_{v,w}(i,j) = a_{v,w}i^2 + b_{v,w}j^2 + c_{v,w}i + d_{v,w}j + e$$

$$u_{v,w}(i,j) = a_{v,w}i^2 + b_{v,w}j^2 + c_{v,w}ij + d_{v,w}i + e_{v,w}j + f_{v,w}$$

Step IV: For each non-overlapping segment, the cumulative sum is calculated by:

$$\varepsilon_{v,w}(i,j) = Y_{v,w}(i,j) - u_{v,w}(i,j)$$

$$F^2_{v,w}(s) = \frac{1}{s^2}\sum_{i=1}^{s}\sum_{j}^{s}\varepsilon_{v,w}(i,j)^2$$

Step V: By averaging over all segments as:

$$F_q(s) = \left[\frac{1}{N_s M_s}\sum_{v=1}^{N_s}\sum_{w=1}^{M_s}\left\{F^2_{v,w}(s)\right\}^{q/2}\right]^{1/q}$$

$$F_q(s) = \mathrm{A} \times s^{h(q)} \qquad \begin{cases} s_{\min} \approx 6 \\ s_{\max} \approx \min(M,N)/4 \end{cases}$$

$$Y(i) = \sum_{k=۱}^{i} \sum_{l=۱}^{j} \left[x(k,l) - \langle x \rangle\right]^{۲}$$

$$.y_\nu(i,j) = a_\nu + b_\nu i + c_\nu j$$

$$\begin{aligned} b_\nu &= \frac{\sum_{i,j=۱}^{s,m} Y(i,j)i - \frac{۱}{s\times m}\sum_{i,j=۱}^{s,m} Y(i,j)\sum_{i,j=۱}^{sm} i}{\sum_{i,j=۱}^{s,m} i^{۲} - \frac{۱}{s\times m}\left[\sum_{i,j=۱}^{s,m} i\right]^{۲}}, \\ &\simeq \frac{\sum_{i,j=۱}^{s,m} Y(i,j)i - \frac{s}{۲}\sum_{i,j=۱}^{s,m} Y(i,j)}{m \times s^{۳}/۱۲} \end{aligned}$$

$$\begin{aligned} c_\nu &= \frac{\sum_{i,j=۱}^{s,m} Y(i,j)j - \frac{۱}{s\times m}\sum_{i,j=۱}^{s,m} Y(i,j)\sum_{i,j=۱}^{sm} j}{\sum_{i,j=۱}^{s,m} j^{۲} - \frac{۱}{s\times m}\left[\sum_{i,j=۱}^{s,m} j\right]^{۲}}, \\ &\simeq \frac{\sum_{i,j=۱}^{s,m} Y(i,j)j - \frac{m}{۲}\sum_{i,j=۱}^{s,m} Y(i,j)}{s \times m^{۳}/۱۲} \end{aligned}$$

$$\begin{aligned} a_\nu &= \frac{۱}{s \times m}\sum_{i=۱}^{s}\sum_{j=۱}^{m} Y(i,j) - \frac{b_\nu}{s \times m}\sum_{i=۱}^{s}\sum_{j=۱}^{m} i - \frac{c_\nu}{s \times m}\sum_{i=۱}^{s}\sum_{j=۱}^{m} j \\ &\simeq \frac{۱}{s \times m}\sum_{i,j=۱}^{s,m} Y(i,j) - \frac{b_\nu s}{۲} - \frac{c_\nu m}{۲}, \end{aligned}$$

$$F^{۲}(s;\nu) = \frac{۱}{sm}\sum_{i=۱}^{s}\sum_{j=۱}^{m}\left[Y_\nu(i,j) - y_\nu(i,j)\right]^{۲}$$

$$
\begin{aligned}
\left\langle\left[F^{۲}(s,m;\nu)\right]\right\rangle &= \left\langle\frac{۱}{s\times m}\sum_{i,j=۱}^{s,m}\left[Y(i,j)-a-bi-cj\right]^{۲}\right\rangle \\
&\simeq \left\langle\frac{۱}{s\times m}\sum_{i,j=۱}^{s,m}Y(i,j)^{۲}\right\rangle+\left\langle a^{۲}\right\rangle+\frac{s^{۲}}{۳}\left\langle b^{۲}\right\rangle+\frac{m^{۲}}{۳}\left\langle c^{۲}\right\rangle \\
&-۲\left\langle\frac{a}{s\times m}\sum_{i,j=۱}^{s,m}Y(i,j)\right\rangle-۲\left\langle\frac{b}{s\times m}\sum_{i,j=۱}^{sm}iY(i,j)\right\rangle \\
&-۲\left\langle\frac{c}{s\times m}\sum_{i,j=۱}^{sm}jY(i,j)\right\rangle+s\left\langle ab\right\rangle+m\left\langle ac\right\rangle+\frac{s\times m}{۲}\left\langle bc\right\rangle
\end{aligned}
$$

$$
Y(i,j)=(ij)^{H}x
$$

$$
\begin{aligned}
Y(i,j)-Y(k,l) &= Y(i,l)+Y(k,j)+|i-k|^{H}|j-l|^{H}x \\
&= \left[(il)^{H}+(kj)^{H}+|i-k|^{H}|j-l|^{H}\right]x,
\end{aligned}
$$

$$\langle [Y(i,j) - Y(k,l)]^{۲} \rangle = \sigma^{۲} \left[(il)^{H} + (kj)^{H} + |i-k|^{H} |j-l|^{H} \right]^{۲}$$

$$\sigma^{۲} = \left\langle x(i,j)^{۲} \right\rangle$$

$$\left\langle Y(i,j)^{۲} \right\rangle = \sigma^{۲} (ij)^{۲H}$$

$$\begin{aligned}
\langle Y(i,j) Y(k,l) \rangle &= \frac{\sigma^{۲}}{۲} [(ij)^{۲H} + (kl)^{۲H} - (ik)^{۲H} - (jl)^{۲H} \\
&\quad - ۲|i-k|^{H} |j-l|^{H} \left[(il)^{H} + (kj)^{H} \right] \\
&\quad - |i-k|^{۲H} |j-l|^{۲H} - ۲(ijkl)^{H}],
\end{aligned}$$

$$\left\langle \left[F^{۲}(s, m; \nu) \right] \right\rangle_{\nu} = C_H (s \times m)^{۲H}$$

$$
\begin{aligned}
C_H \;=\; & \frac{۱}{۲}\sigma^{۲}\Big[\frac{۲۰ + H\,(۸۰ + H\,[۱۴۳ + H(۲ + H)(۷۹ + ۲H(۶ + H))])}{(۱ + H)^{۴}(۲ + H)^{۲}(۱ + ۲H)^{۲}} \\
& + \Big\{-\frac{۱۲(۲\Gamma[۲ + H]^{۲} + \Gamma[۳ + ۲H])}{(۱ + H)^{۳}(۲ + H)\Gamma[۳ + ۲H]\Gamma[۴ + ۲H]^{۲}} \\
& \times \{۲\Gamma[۴ + ۲H]\left(\Gamma[۲ + H]\left\{۲H^{۳}\Gamma[H] + (۳ + ۵H)\Gamma[۱ + H] + \Gamma[۲ + H]\right\} + (۳ + H)\Gamma[۳ + ۲H]\right) \\
& + \Gamma[۲ + H]^{۲}\Gamma[۵ + ۲H]\} + \left(\frac{۱}{(۱ + H)^{۲}} + \frac{۲\Gamma[۱ + H]^{۲}}{\Gamma[۳ + ۲H]}\right) \\
& \times \left(۷\left[\frac{۱}{(۱ + H)^{۲}} + \frac{۲\Gamma[۱ + H]^{۲}}{\Gamma[۳ + ۲H]}\right] + \frac{۴۸}{(۱ + H)(۲ + H)\Gamma[۵ + ۲H]^{۲}}\right. \\
& \times \left(۲\Gamma[۳ + H]^{۲}\Gamma[۴ + ۲H] + \left(۲(۱ + H)(۲ + H)\Gamma[۲ + H]^{۲} + \Gamma[۴ + ۲H]\right)\Gamma[۵ + ۲H]\right)\Big)\Big\}\Big]
\end{aligned}
$$

$$
\Gamma(x) \equiv (x - ۱)! = \int_{۰}^{\infty} t^{x-۱} e^{-t}\, dt
$$

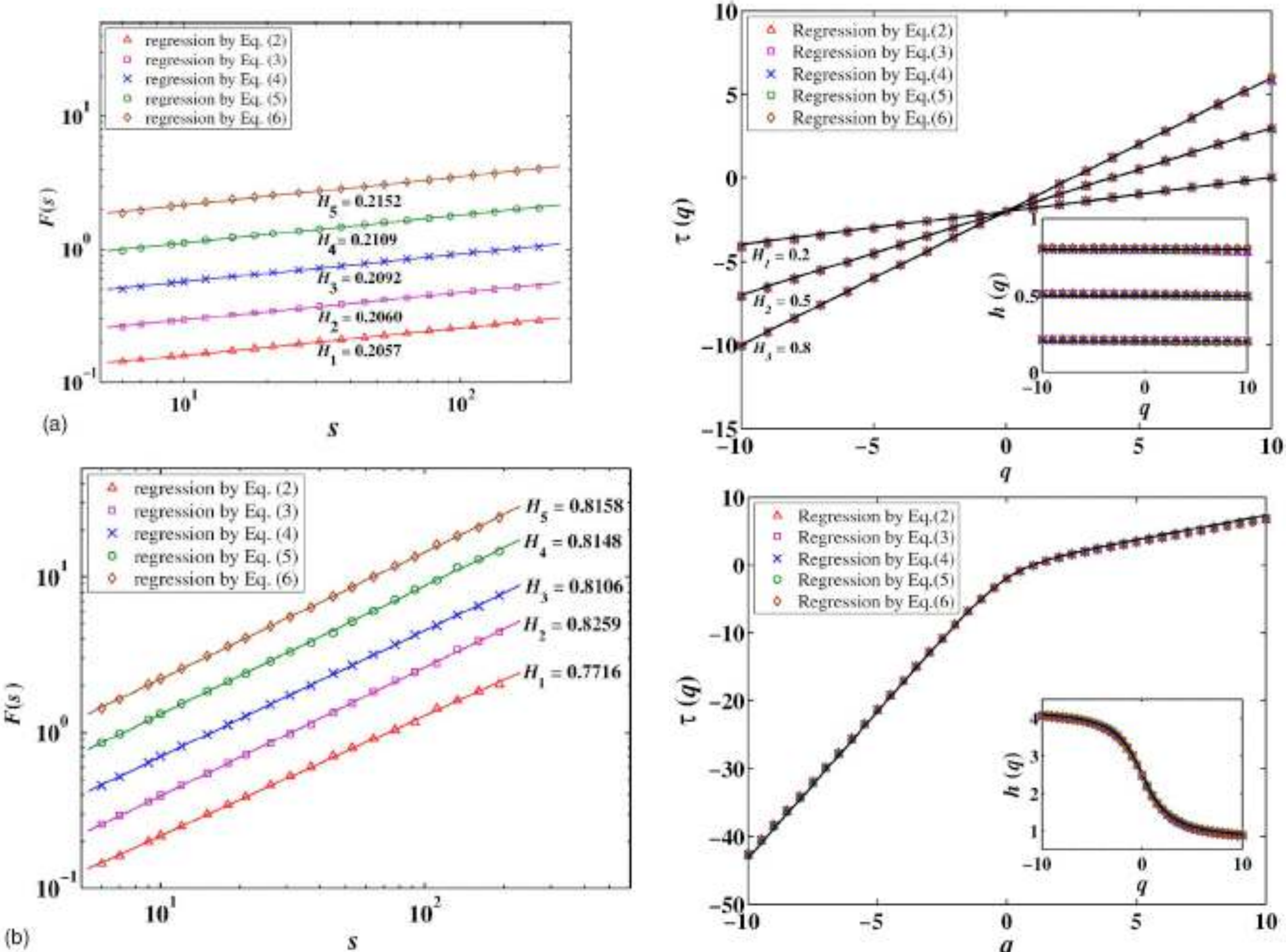
Some important exponents

$$\tau(q) = qh(q) - d_f$$

$$D_f = 3 - H$$

$$f(x) = \mu(x) \otimes |x|^{-(1-H)} \qquad H \in (0,1)$$

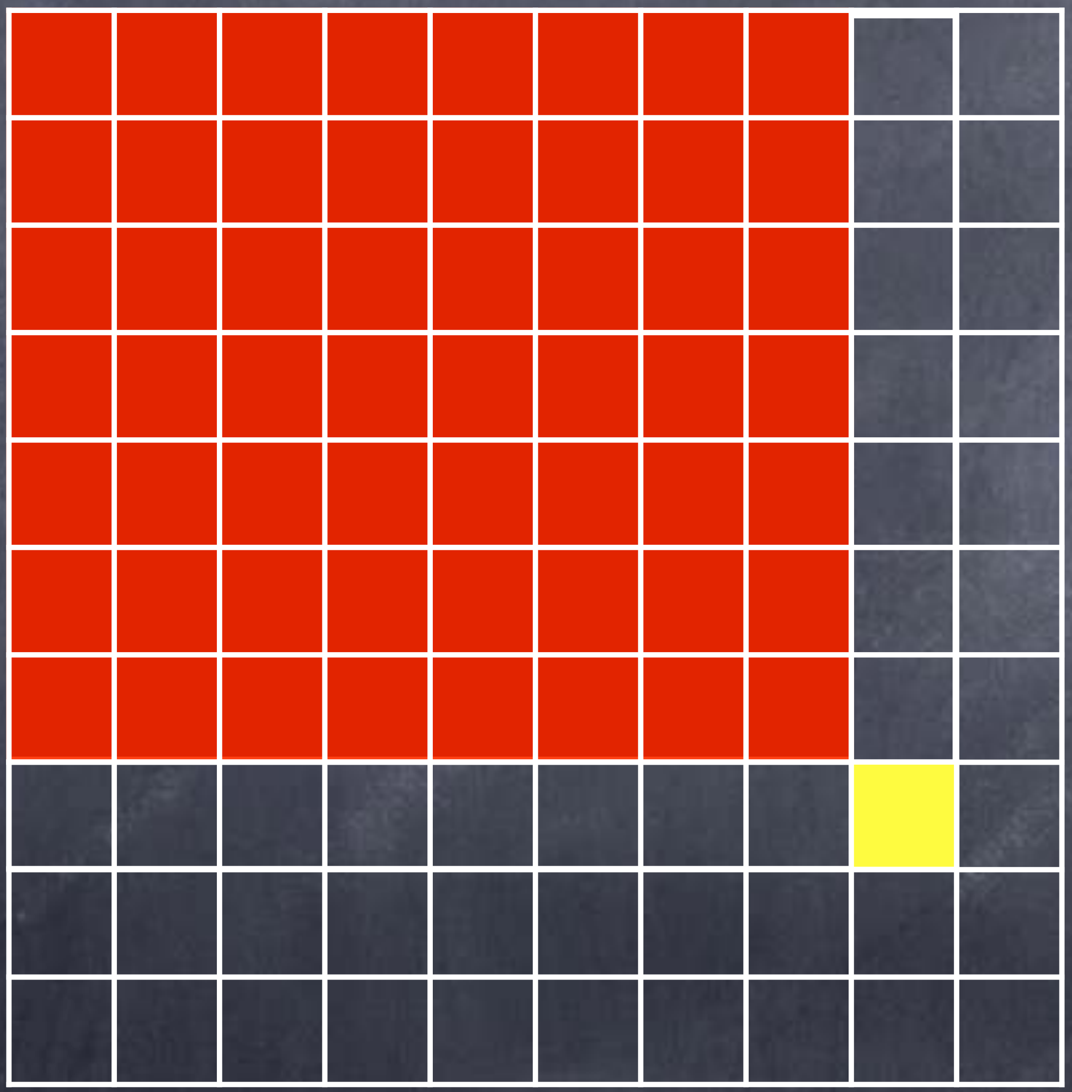
$$\tau(q) = q(1 + H) - 1 - \log_2 \left[p^q + (1 - p)^q \right]$$



(Gao-Feng Gu and Wei-Xing Zhou, PHYSICAL REVIEW E **74**, 061104 (2006

More about cumulative sum

$$X(i,j) = X_{v,w}(i-1,j-1) + \sum_{k=1}^{i-1} x(k,j) + \sum_{l=1}^{j-1} x(i,l) + x(i,j)$$



More about cumulative sum

$$X(l_v + i, l_w + j) = X_{v,w}(i,j) + \sum_{k=1}^{l_v} \sum_{l=1}^{l_w} x(k,l) + \sum_{k=1}^{l_v} \sum_{l=l_w+1}^{l_w+j} x(k,l) + \sum_{k=l_v+1}^{l_v+i} \sum_{l=1}^{l_w} x(k,l)$$

$s = 4$

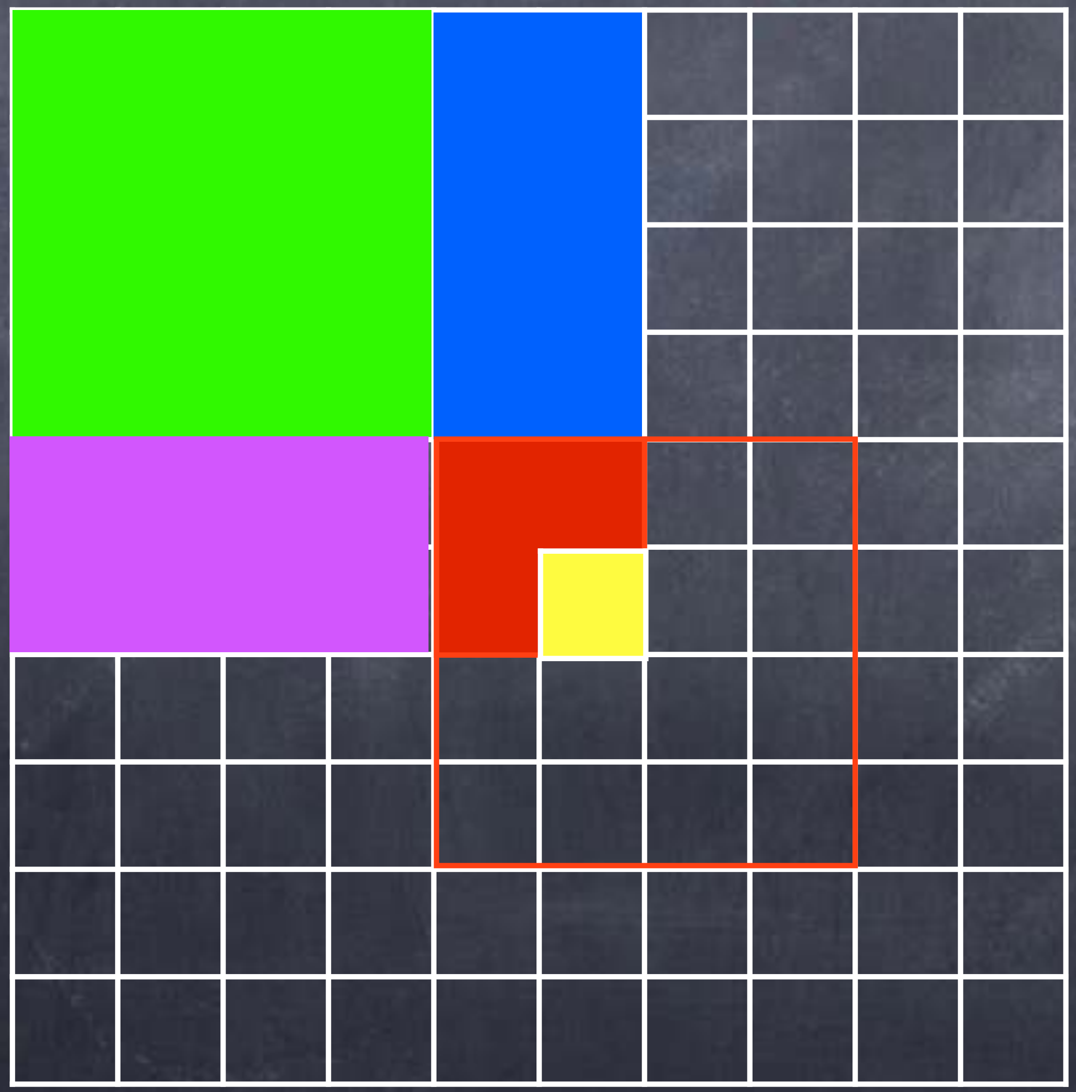
$i = j = 2$

$v = 2$

$w = 2$

$l_2 = 4$

$l_2 = 4$



Multifractal Detrended cross-correlation ((MF-DCCA

- Step I: Consider two time series as:

$$\{x(i)\} \quad \{y(i)\} \quad i = 1,2,\ldots,N$$

$$M_s = \mathrm{int}\left(\frac{N}{s}\right)$$

- Step II: Construct profile and trend functions. Polynomials or based on empirical mode decomposition (EMD, non-parametric)

$$X_v(k) = \sum_{i=1}^{k} x(l_v + i) \qquad l_v = (v-1)s$$

$$Y_v(k) = \sum_{i=1}^{k} y(l_v + i)$$

$$F(s,v) = \frac{1}{s}\sum_{i}^{s}\left\{Y[(v-1)s + i - y_v(i)\right\} \times \left\{X[(v-1)s + i - x_v(i)\right\} \qquad v = 1,\ldots,M_s$$

$$F(s,v) = \frac{1}{s}\sum_{i}^{s}\left\{Y[N-(v-1)s + i - y_v(i)\right\} \times \left\{X[N-(v-1)s + i - x_v(i)\right\} \qquad v = M_s + 1,\ldots,2M_s$$

(B. Podobnik and H. Euge
(W

- Step IV: Averaging over all segments as:

$$F_q(s) = \left\{ \frac{1}{M_s} \sum_{v=1}^{M_s} \left[F(s,v) \right]^{q/2} \right\}^{1/q}$$

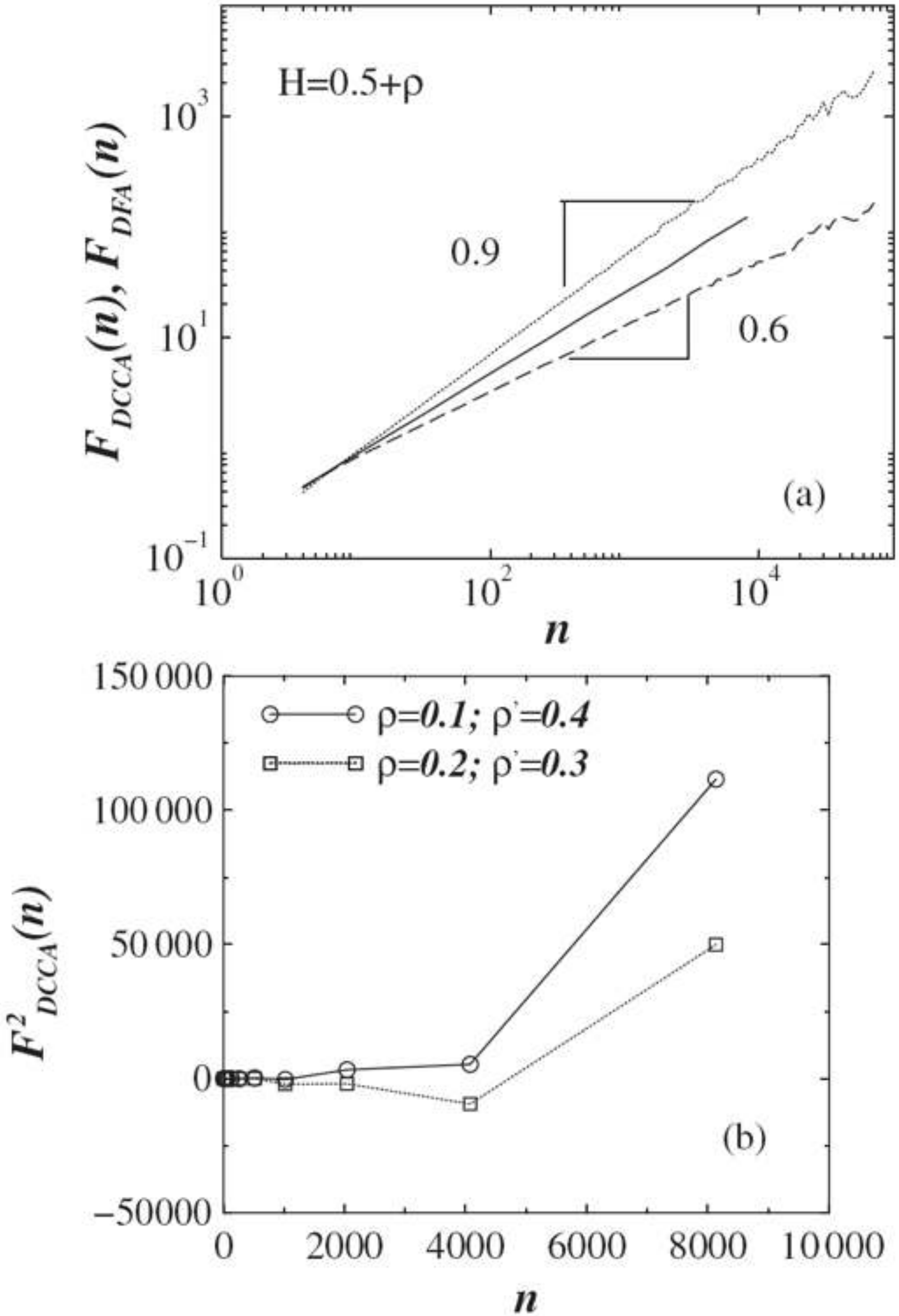
$$F_0(s) = \exp\left(\frac{1}{2M_s} \sum_{v=1}^{M_s} \ln\left[F(s,v) \right] \right)$$

- Step V: Demanding a scaling relation according to:

$$F_q(s) \sim s^{\lambda(q)}$$

- If two underlying series to be equal so one finds nothing except the Hurst exponent:

$$F_q(s) \sim s^{h(q)}$$



,B. Podobnik and H. Eugene Stanley
(PRL 100, 084102 (2008

2D version of MF-DCCA

$x(i,j) \qquad y(i,j) \qquad i = 1,\dots,M \qquad j = 1,\dots,N$

$$M_s = \mathrm{int}\left(\frac{M}{s}\right) \qquad N_s = \mathrm{int}\left(\frac{N}{s}\right)$$

$$X_{v,w}(i,j) = \sum_{k=1}^{i}\sum_{l=1}^{j} x_{v.w}(k,l)$$

$$Y_{v,w}(i,j) = \sum_{k=1}^{i}\sum_{l=1}^{j} y_{v.w}(k,l)$$

$$F_{v,w}(s) = \frac{1}{s^2}\sum_{i=1}^{s}\sum_{j=1}^{s}\left[X_{v.w}(i,j) - \tilde{X}_{v,w}(i,j)\right]\left[Y_{v.w}(i,j) - \tilde{Y}_{v,w}(i,j)\right]$$

$$F_q(s) = \left(\frac{1}{M_s N_s}\sum_{v=1}^{M_s}\sum_{w=1}^{N_s}\left[F_{v,w}(s)\right]^{q/2}\right)^{1/q}$$

$$F_0(s) = \exp\left(\frac{1}{2M_s N_s}\sum_{v=1}^{M_s}\sum_{w=1}^{N_s}\ln\left[F_{v,w}(s)\right]\right)$$

$$F_q(s) \sim s^{-\lambda(q)}$$

Wei-Xing Zhou, PRE **77**, 066211 2008

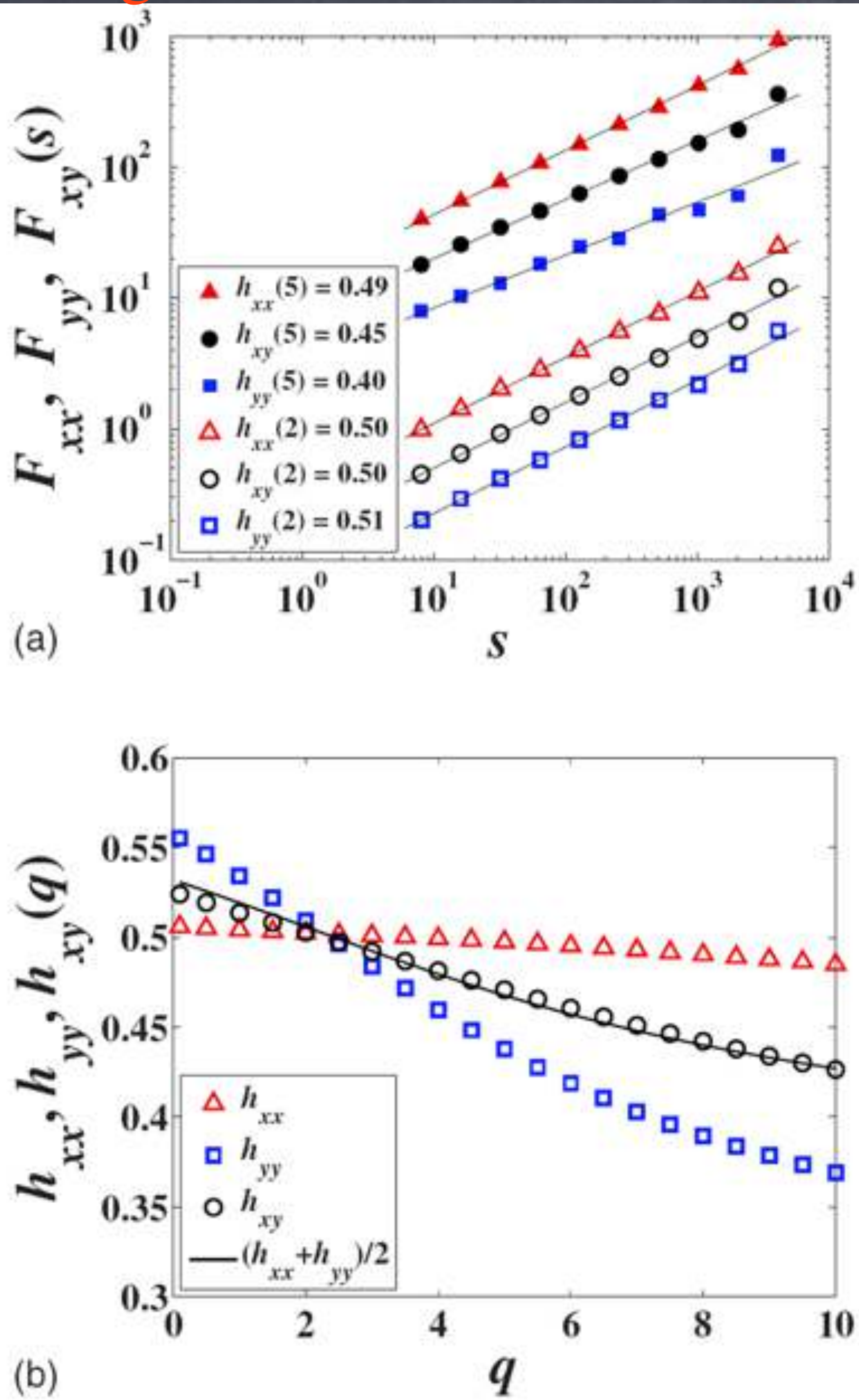


FIG. 2. (Color online) Multifractal nature of the power-law cross correlations of two MRWs. (a) Power-law scaling in F_{xy}, F_{xx}, and F_{yy} with respect to s for q=2 and 5; (b) power-law exponents h_{xy}, h_{xx}, and h_{yy}.

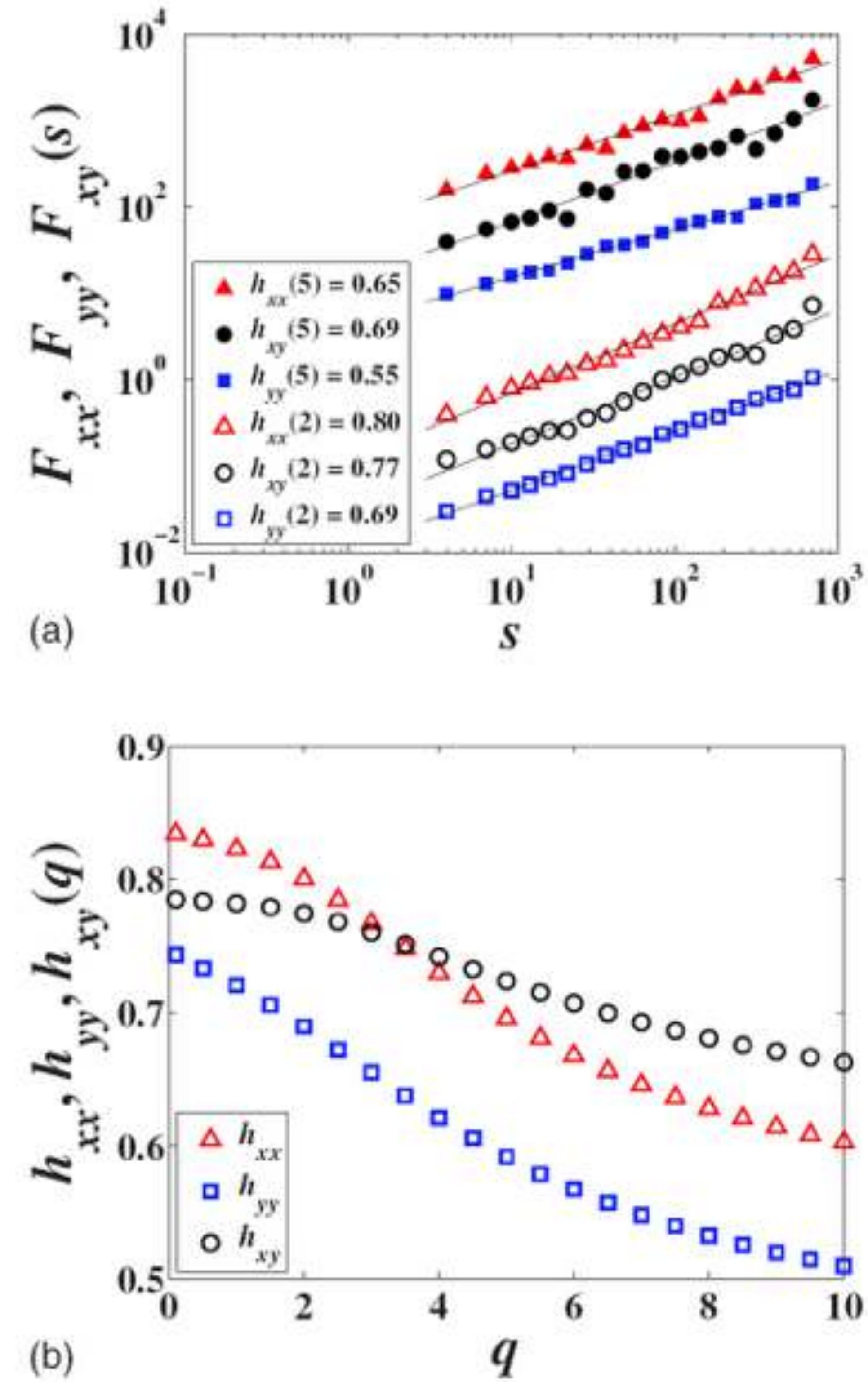


FIG. 3. (Color online) Multifractal nature of the power-law cross correlations of the absolute values of daily price changes for DJIA and NASDAQ indices in the period from July 1993 to November 2003. (a) Power-law scaling in F_{xy}, F_{xx}, and F_{yy} with respect to s for q=2 and 5. The scaling range is the same as in Ref. [15]. (b) Dependence of the power-law exponents h_{xy}, h_{xx}, and h_{yy} as nonlinear functions of q, indicating the presence of multifractality. There is no clear relation between these exponents.

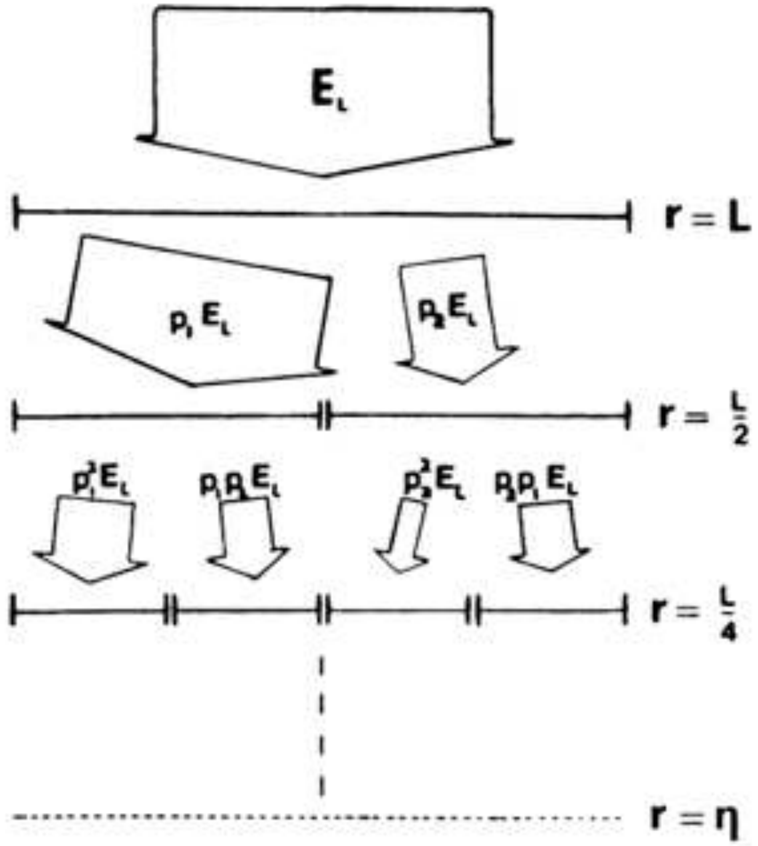


FIG. 3. One-dimensional version of a cascade model of eddies, each breaking down into two new ones. The flux of kinetic energy to smaller scales is divided into nonequal fractions p_1 and p_2. This cascade terminates when the eddies are of the size of the Kolmogorov scale, η.

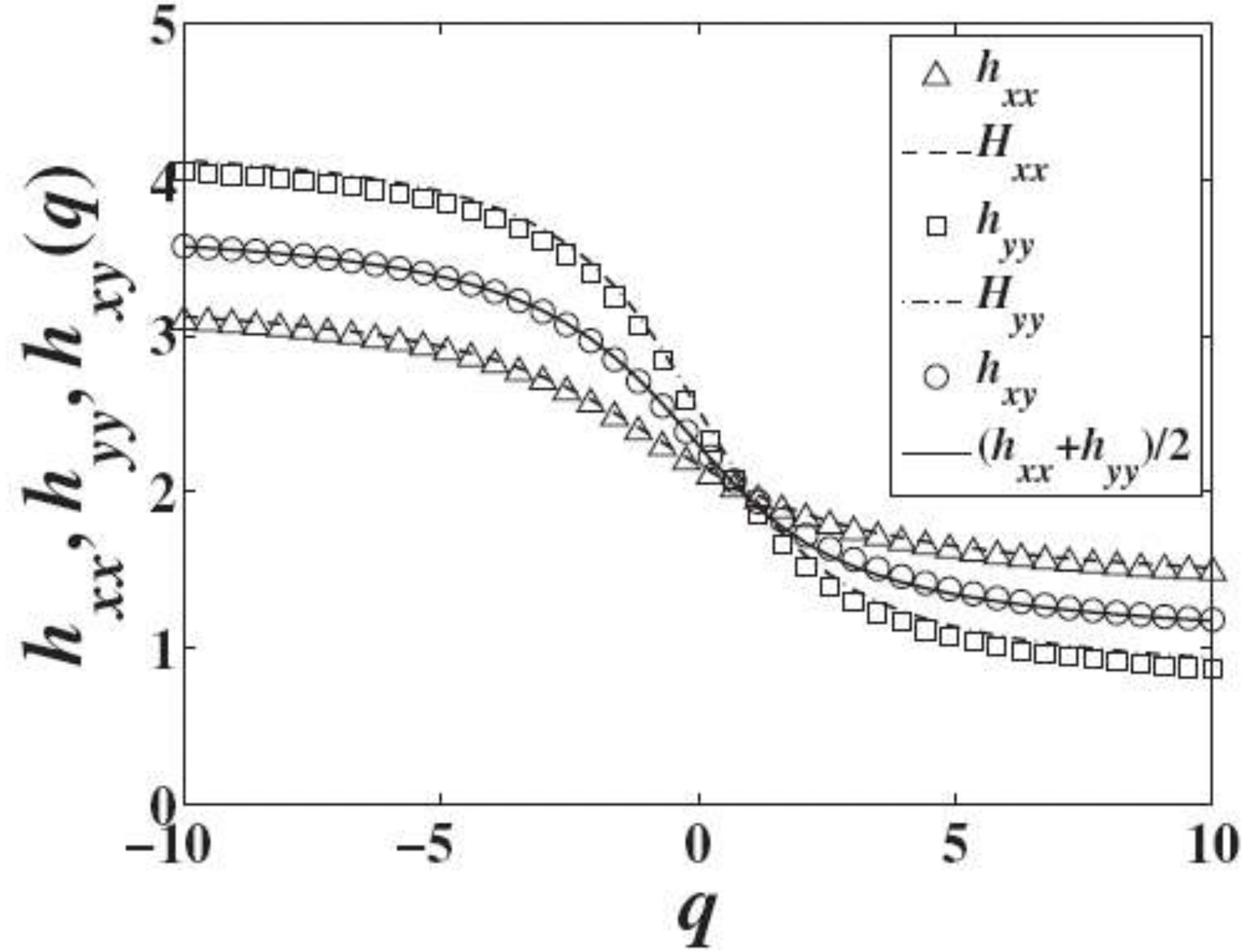


FIG. 4. Multifractal detrended cross-correlation analysis of two cross-correlated synthetic binomial measures from the p model. The size of each multifractal is 4096×4096 and the cross-correlation coefficient is 0.48. The numerical exponents $h_{xx}(q)$ and $h_{yy}(q)$ obtained from the multifractal detrended fluctuation analysis of X and Y are located approximately on the analytical curves $H_{xx}(q)$ and $H_{yy}(q)$. This example illustrates the relation $h_{xy}(q) = [h_{xx}(q)+h_{yy}(q)]/2$.

B. B. Mandelbrot, J. Fluid Mech. **62**, (1 .331 1974
C. Meneveau and K. R. Sreenivasan, (2 .Phys. Rev. Lett. **59**, 1424 1987
E. A. Novikov, Phys. Fluids A **2**, 814(3 1990
C. Meneveau and K. R. Sreenivasan, (4 .J. Fluid Mech. **224**, 429 1991

$$H_{zz}(q) = [2 - \log_2(p_{11}^q + p_{12}^q + p_{21}^q + p_{22}^q)]/q,$$

Cross-correlation exponent for stationary series

$$x(i,j) \qquad y(i,j) \qquad i = 1,\ldots,M \qquad j = 1,\ldots,N$$

$$\mu_x = \frac{1}{N}\sum_{i=1}^{N} x(i) \qquad \sigma_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left[x(i)-\mu_x\right]^2$$

$$\mu_y = \frac{1}{N}\sum_{i=1}^{N} y(i) \qquad \sigma_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left[y(i)-\mu_y\right]^2$$

$$C_x(\tau) = \frac{\left\langle (x(i+\tau)-\mu_x)(x(i)-\mu_x)\right\rangle}{\sigma_x^2} : \tau^{-\gamma_x}$$

$$C_{xy}(\tau) = \frac{\left\langle (x(i+\tau)-\mu_x)(y(i)-\mu_y)\right\rangle}{\sigma_x\sigma_y} : \tau^{-\gamma_{xy}}$$

$$F_v(s) = \frac{1}{s}\sum_{i=1}^{s}[Y_v(i)-\bar{Y}][X_v(i)-\bar{X}]$$

$$F_2^2(s) = \frac{1}{M_s}\sum_{v=1}^{M_s}\left[F_v(s)\right] \equiv \left\langle [Y_v(s)-\bar{Y}][X_v(s)-\bar{X}]\right\rangle$$

$$= sC_{xy}(0) + \sum_{i=1}^{s-1}[s-i][C_{xy}(i)+C_{xy}(-i)]\,;\; s^{1-\gamma_{xy}} + s^{2-\gamma_{xy}}$$

$$F_2^2(s) : s^{2\lambda} : s^{2-\gamma_{xy}} \rightarrow 2-\gamma_{xy} = 2\lambda \rightarrow \gamma_{xy} = 2-2\lambda$$

Cross-Correlation in the presence of trends

$$x(i,j) \qquad y(i,j) \qquad i = 1,\ldots,M \qquad j = 1,\ldots,N$$

$$F(s,v) = \frac{1}{s}\sum_{i}^{s}\left\{Y[(v-1)s+i] - y_v(i)\right\} \times \left\{X[(v-1)s+i] - x_v(i)\right\}$$

$$F_2^2(s) = \left\{\frac{1}{M_s}\sum_{v=1}^{M_s}\left[F(s,v)\right]\right\} : \; s^{2\lambda}$$

$$F_2^2(s) : \; s^{2\lambda} \; : \; s^{2-\gamma_{xy}} \rightarrow 2-\gamma_{xy} = 2\lambda \rightarrow \gamma_{xy} = 2-2\lambda$$

$$if \quad x \equiv y \rightarrow \gamma_{xx} = 2-2H$$

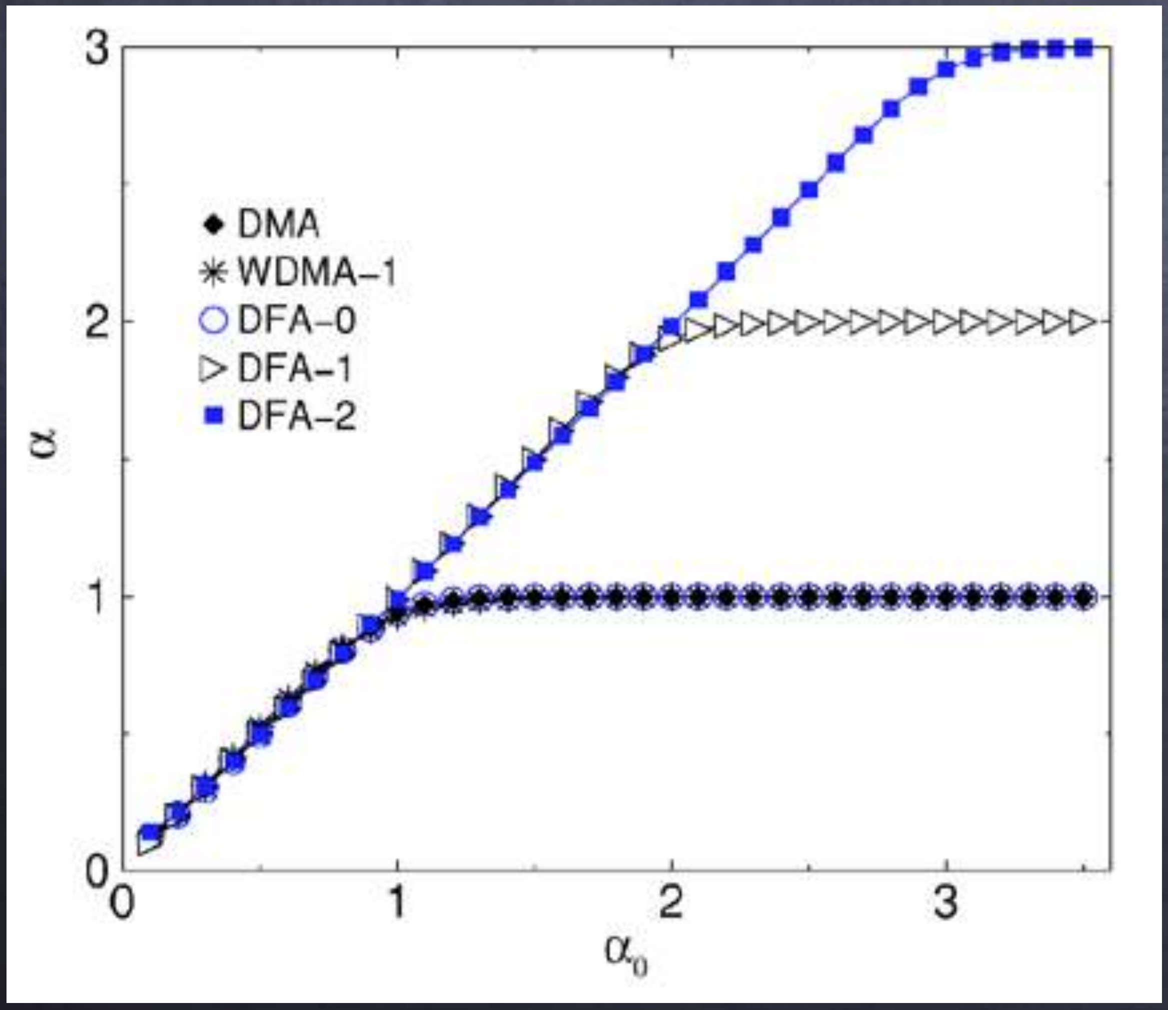
Strategy for using methods

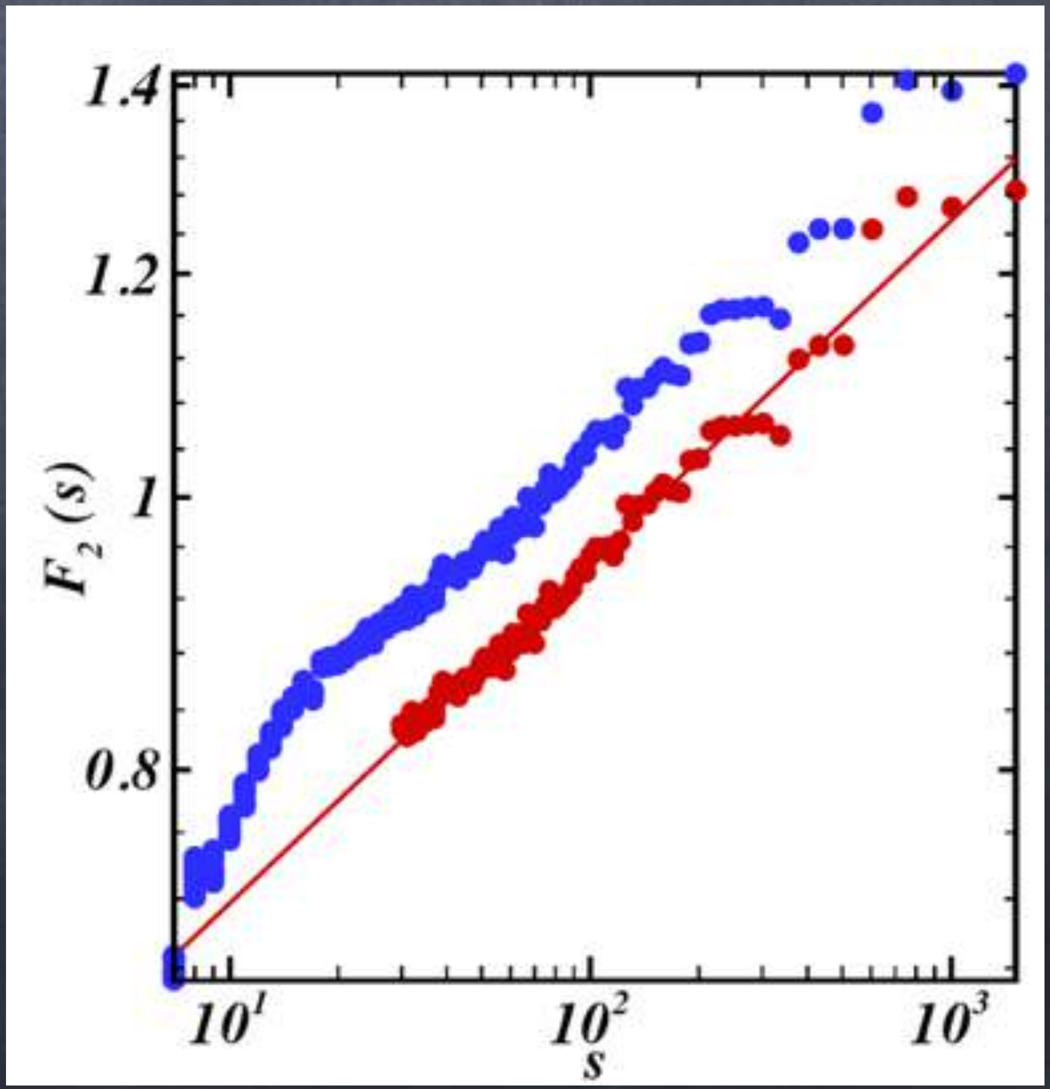
- For stationary anti-correlated signal i.e. $H<0.5$,
  SWV
- For stationary correlated signal, $H>0.5$,
  R/S
- For signal with superimposed trends,
  WTMM, MF-DFA, MF-TWDFA, DMA

More about DFA

1) The longer the time series, the better the agreement with the theory in all methods but DFA behaves more reliable than others

2) DFA cannot give correct results when h(q=2) ~ 0, In this case it is recommended to construct double profile and use DFA method

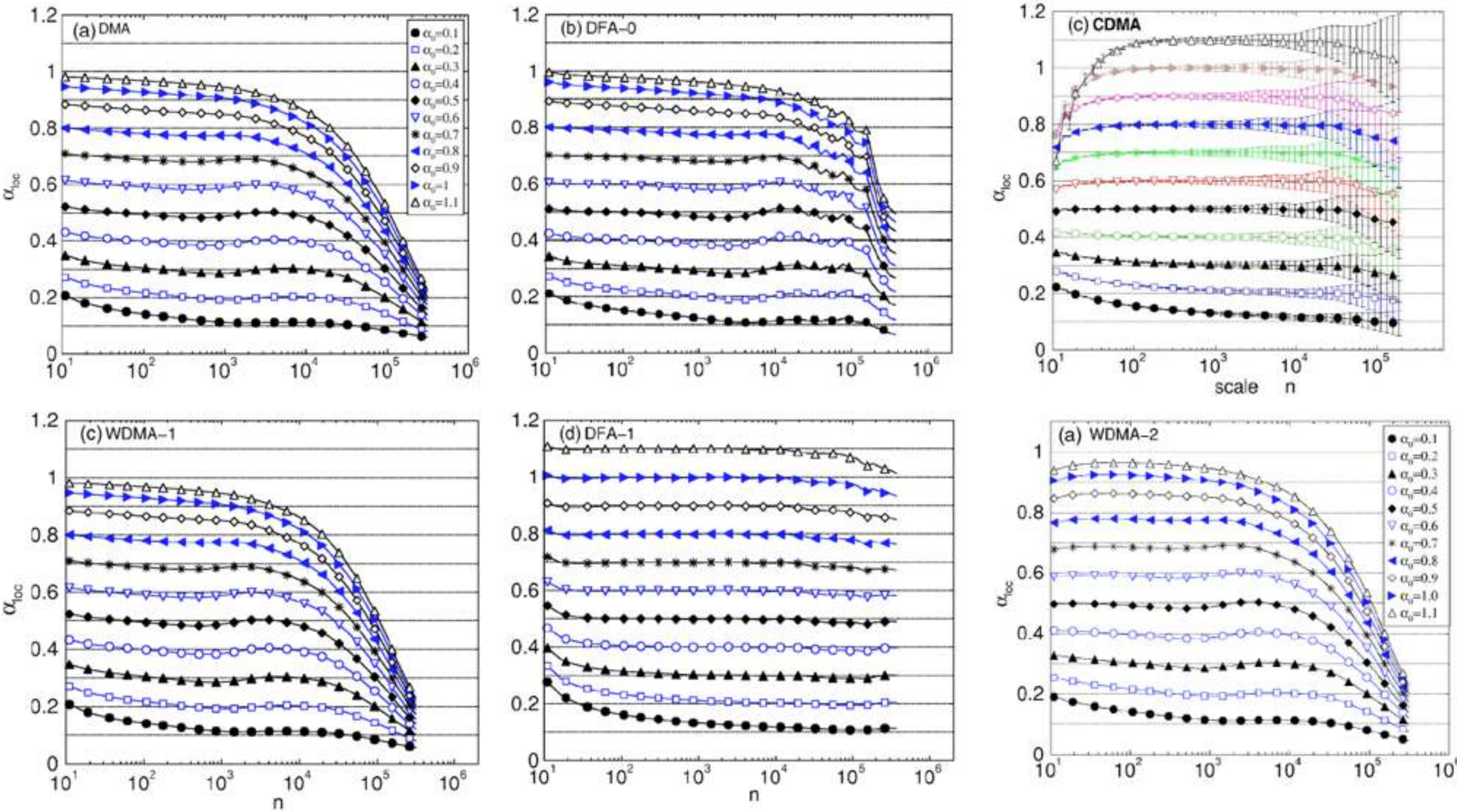




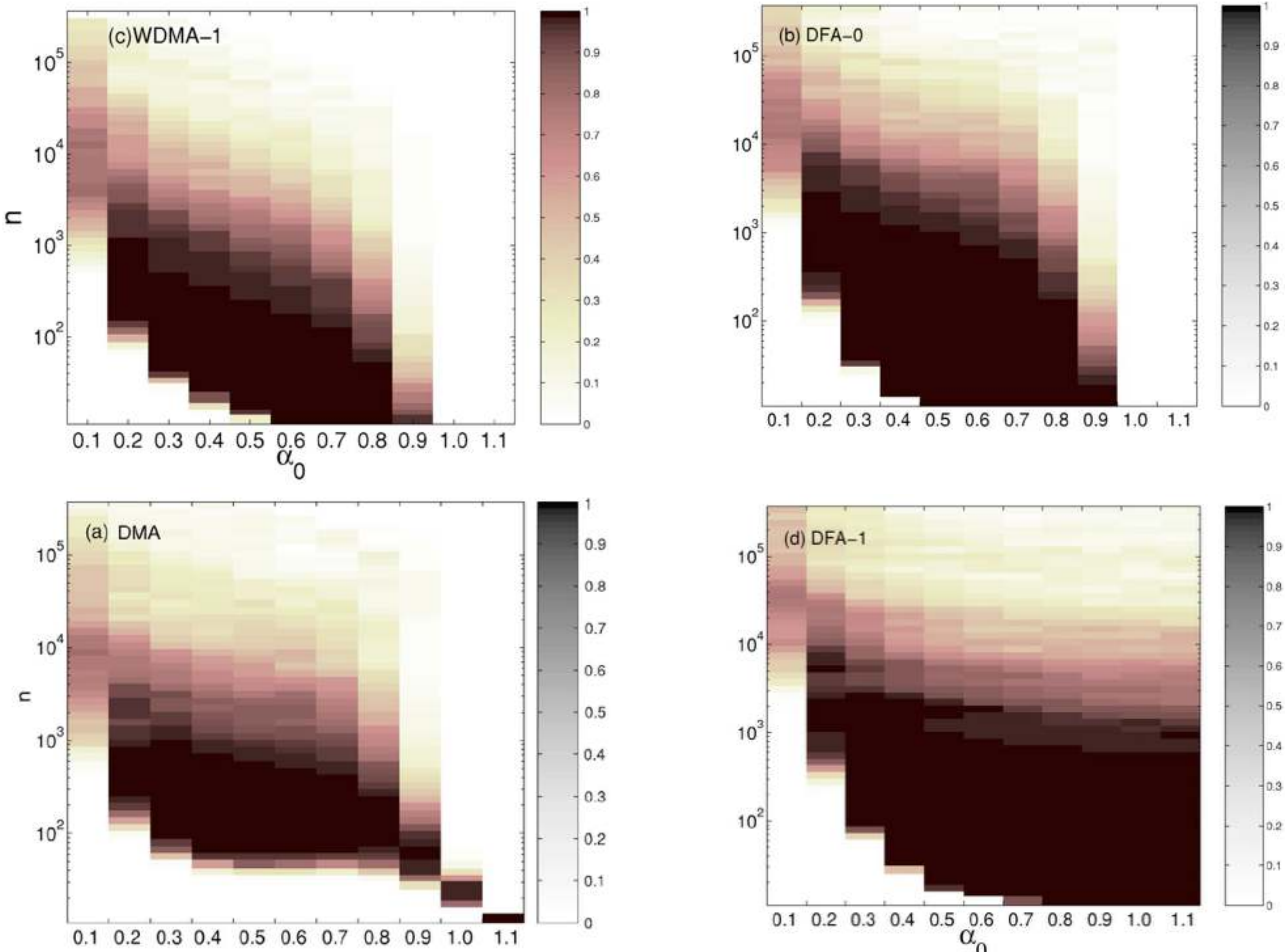
DMA (BDMA & CDMA) and MF-TWDFA

: Refer to

arXiv:cond-mat/0507395 (1
(PRE 71, 051101 (2005 (2
(PRE 73, 016117 (2006 (3
(JSTAT P06021 (2010 (4



Limei Xu et. al., PRE 71, 051101 (2005)



Limei Xu et. al., PRE 71, 051101 (2005)

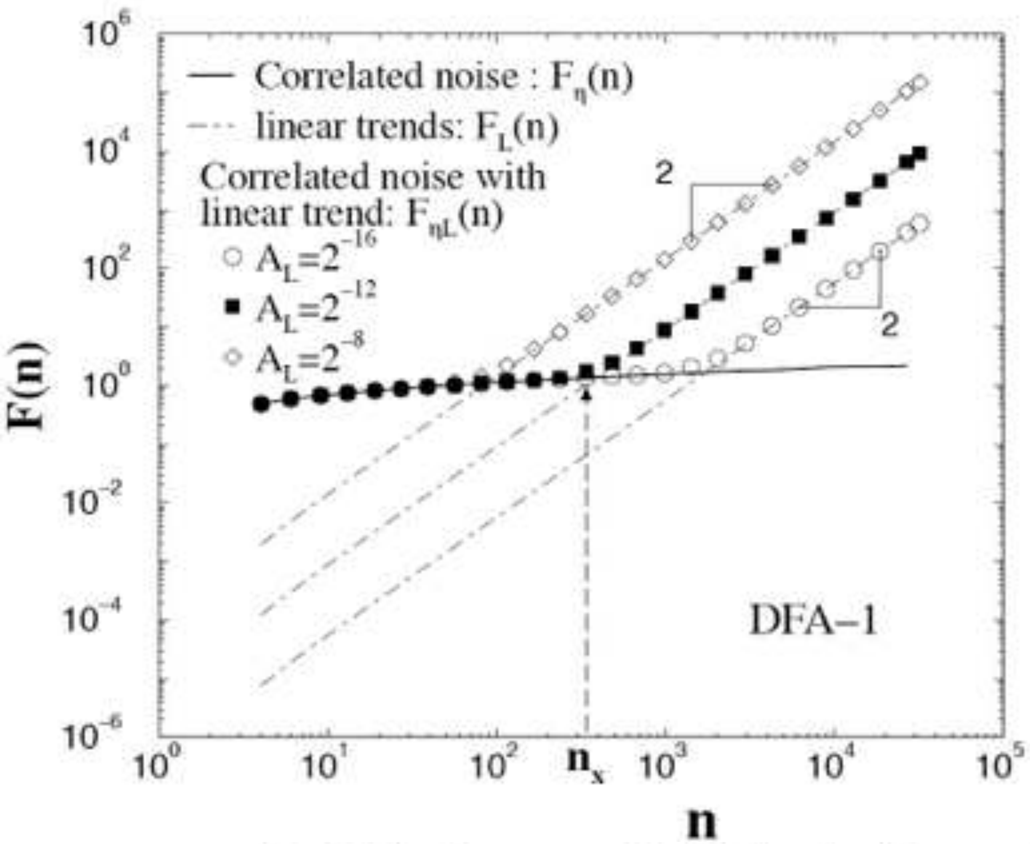
Crossover and effect of trends

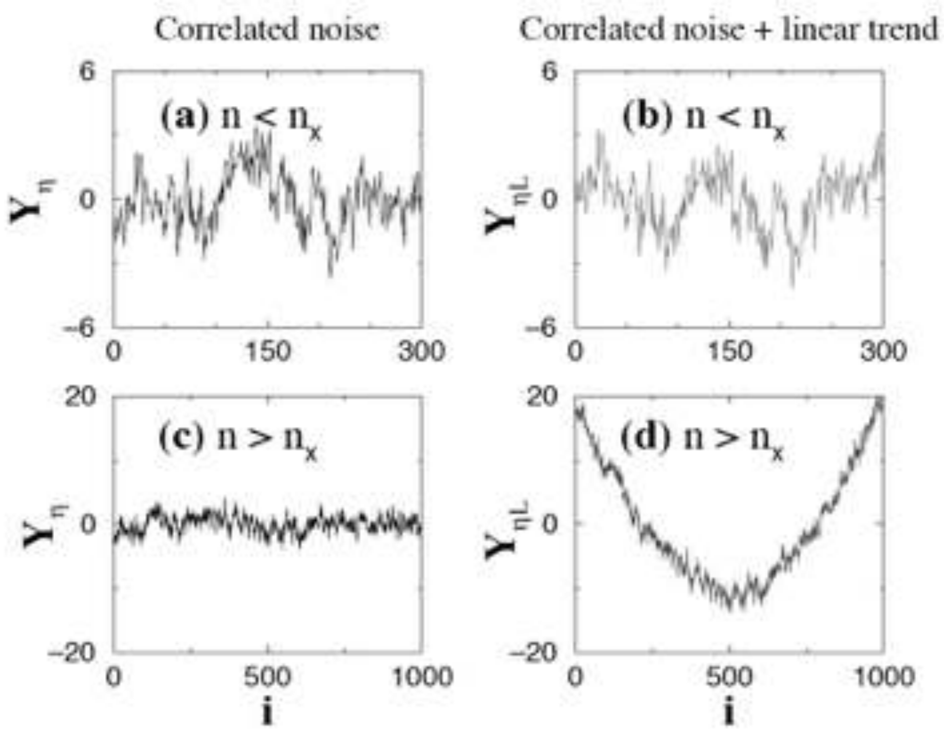
- Polynomial trends: MF-DFAm
- Sinusoidal trends: F-DFA, SVD, chaotic SVD and Empirical mode decomposition(EMD)

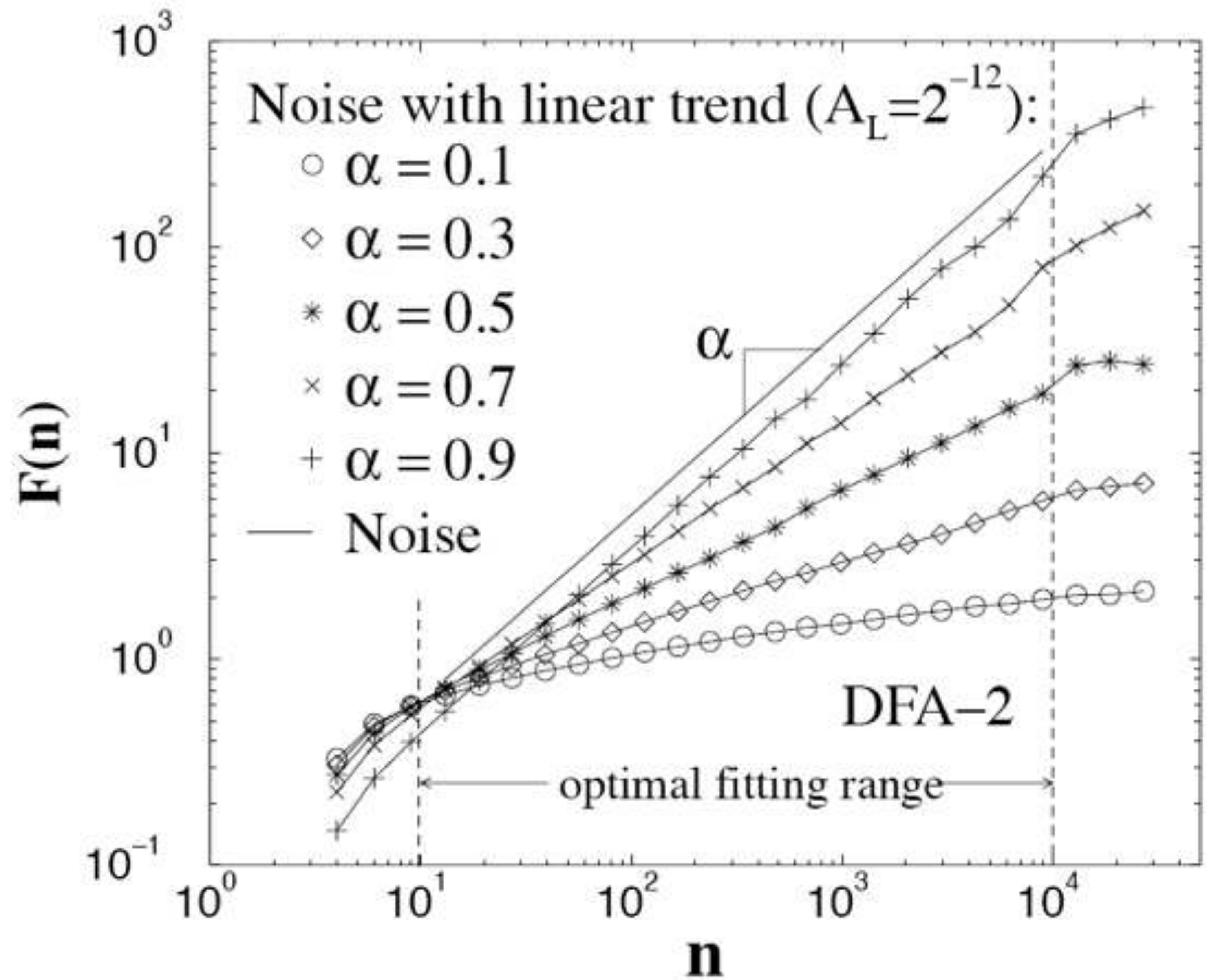
Z. Wu et al., PNAS, 104, 38 (2007)

Polynomial Trends

It has been demonstrated that by MF-DFAm, polynomial of order m-1 to be diminished

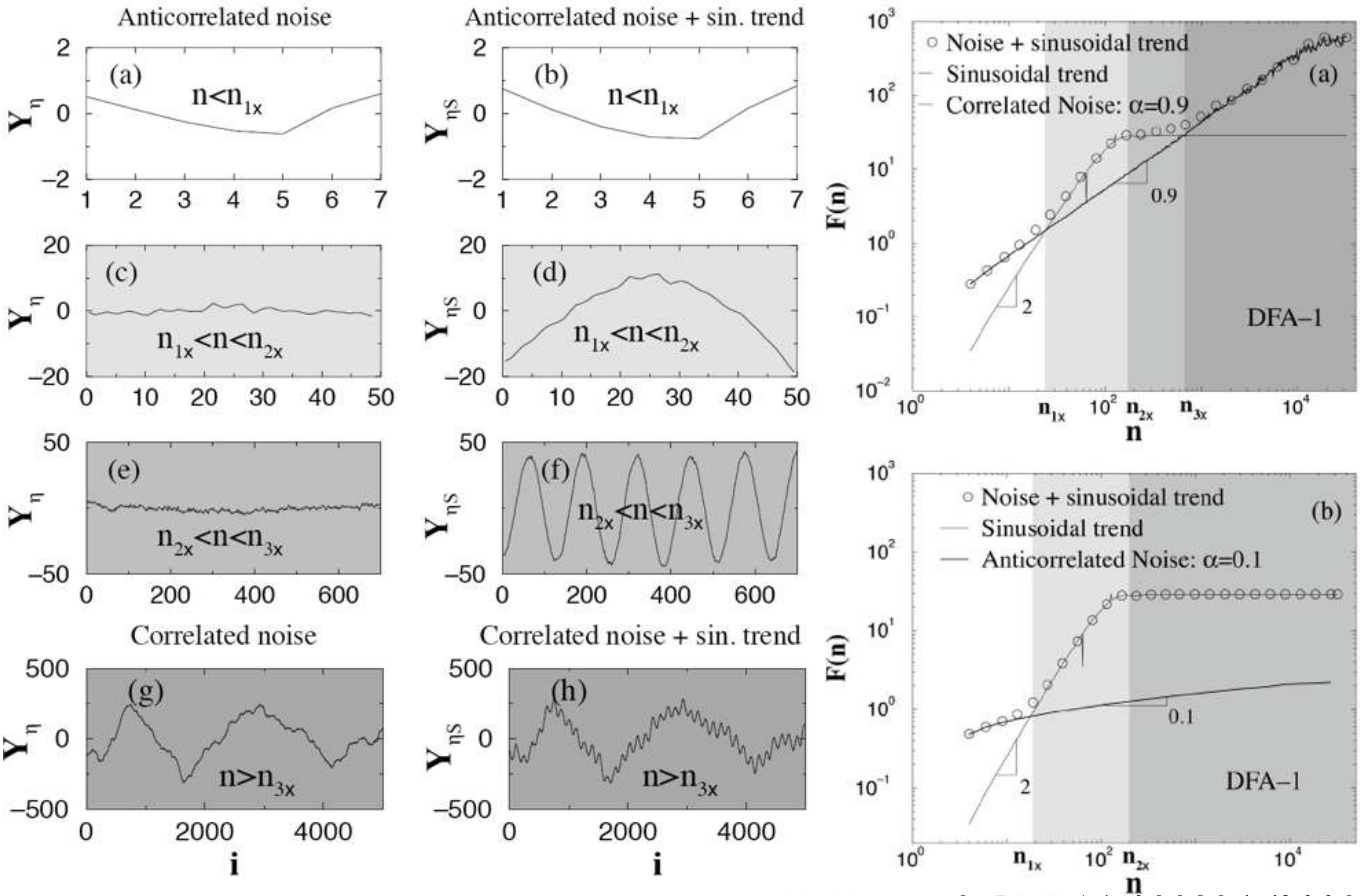






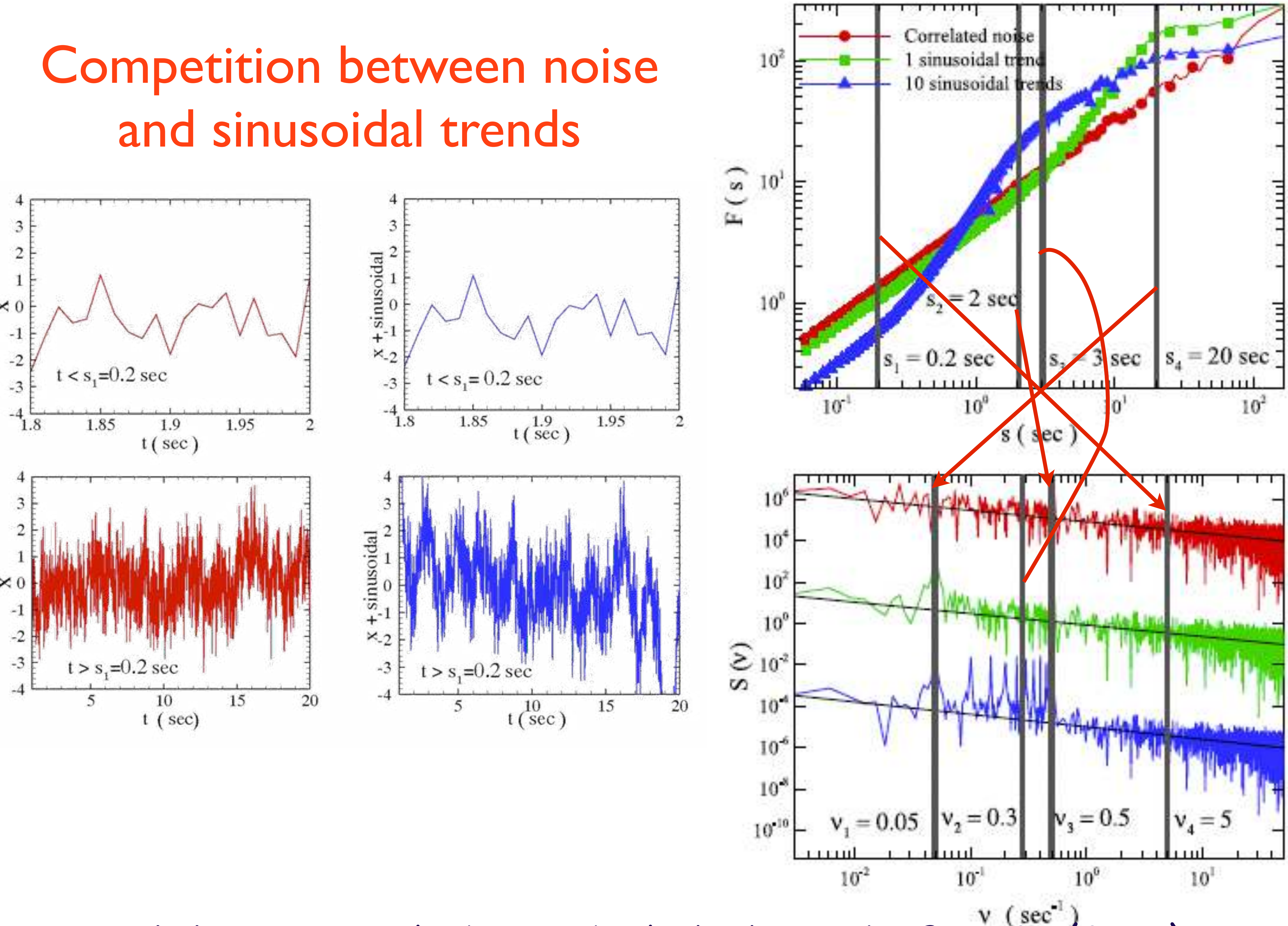
K. Hu et. al., PRE 64, 011114 (2001)

Sinusoidal Trends



K. Hu et. al., PRE 64, 011114 (2001)

Competition between noise and sinusoidal trends

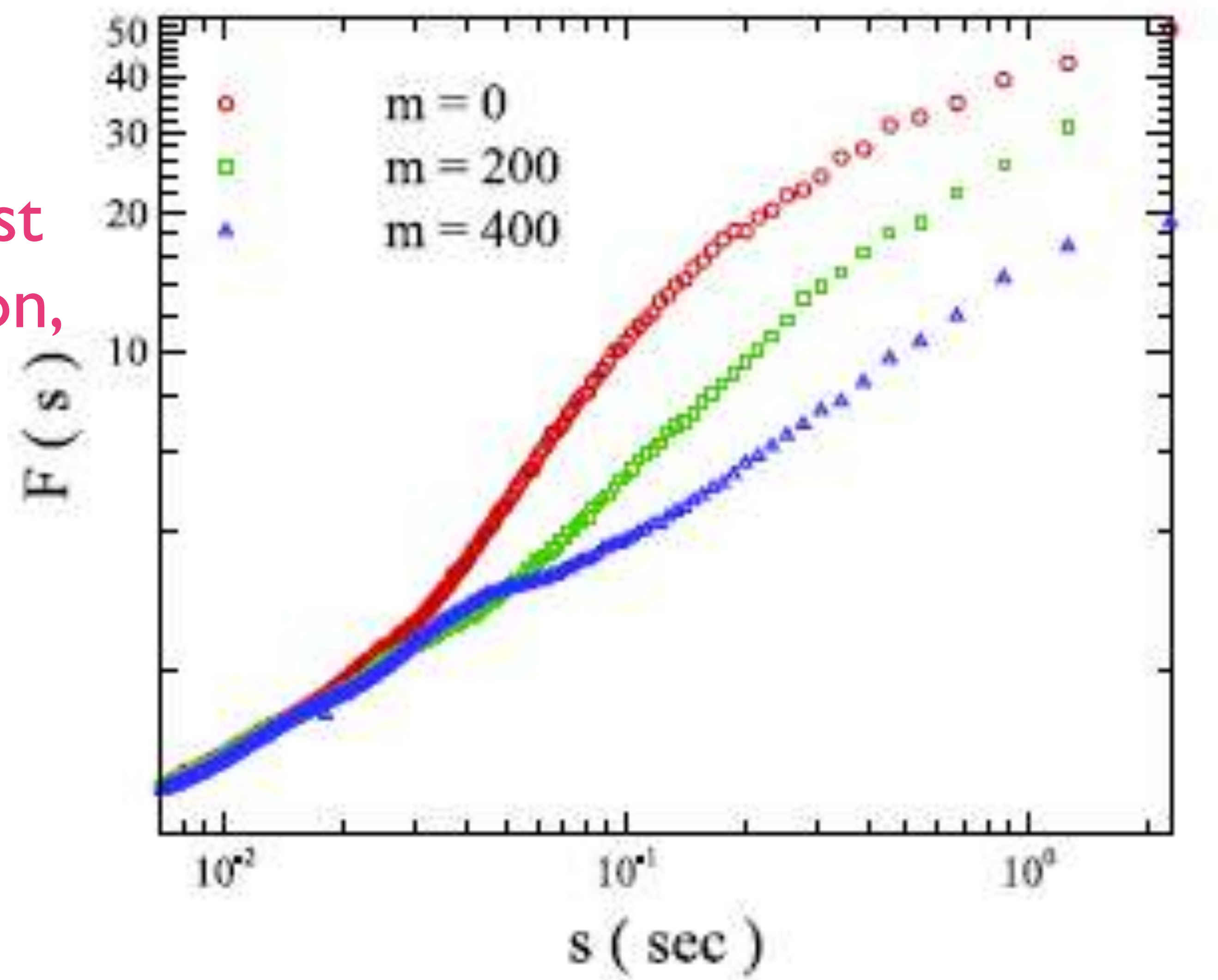


S. Kimiagar, M. Sadegh Movahed et. al., JSTAT P03020 (2009)

Fourier-Detrended

- Indeed, this method bases on High-pass filter

We transform the data set to the Fourier space and then truncate the first few coefficients of the Fourier expansion, finally by inverse transformation, the clean data will be retrieved



Physica A 357, 447-454 (2005); Physica A 354, 182-198 (2005); Chaos, Solitons and fractals 26, 777-784 (2005), Jstat P03020 (2009)

Singular Value Decomposition (SVD)

$$\{x_i\}; i = 1, ..., N \qquad d \le N - (d-1)\tau + 1$$

$$\mathbf{\Gamma} \equiv \begin{pmatrix} x_1 & x_{1+\tau} & \cdots & x_{1+N-(d-1)\tau-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_i & x_{i+\tau} & \cdots & x_{i+N-(d-1)\tau-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_d & x_{d+\tau} & \cdots & x_{d+N-(d-1)\tau-1} \end{pmatrix}$$

p will be given by power spectrum

$$\mathbf{\Gamma} = \mathbf{USV}^\dagger$$

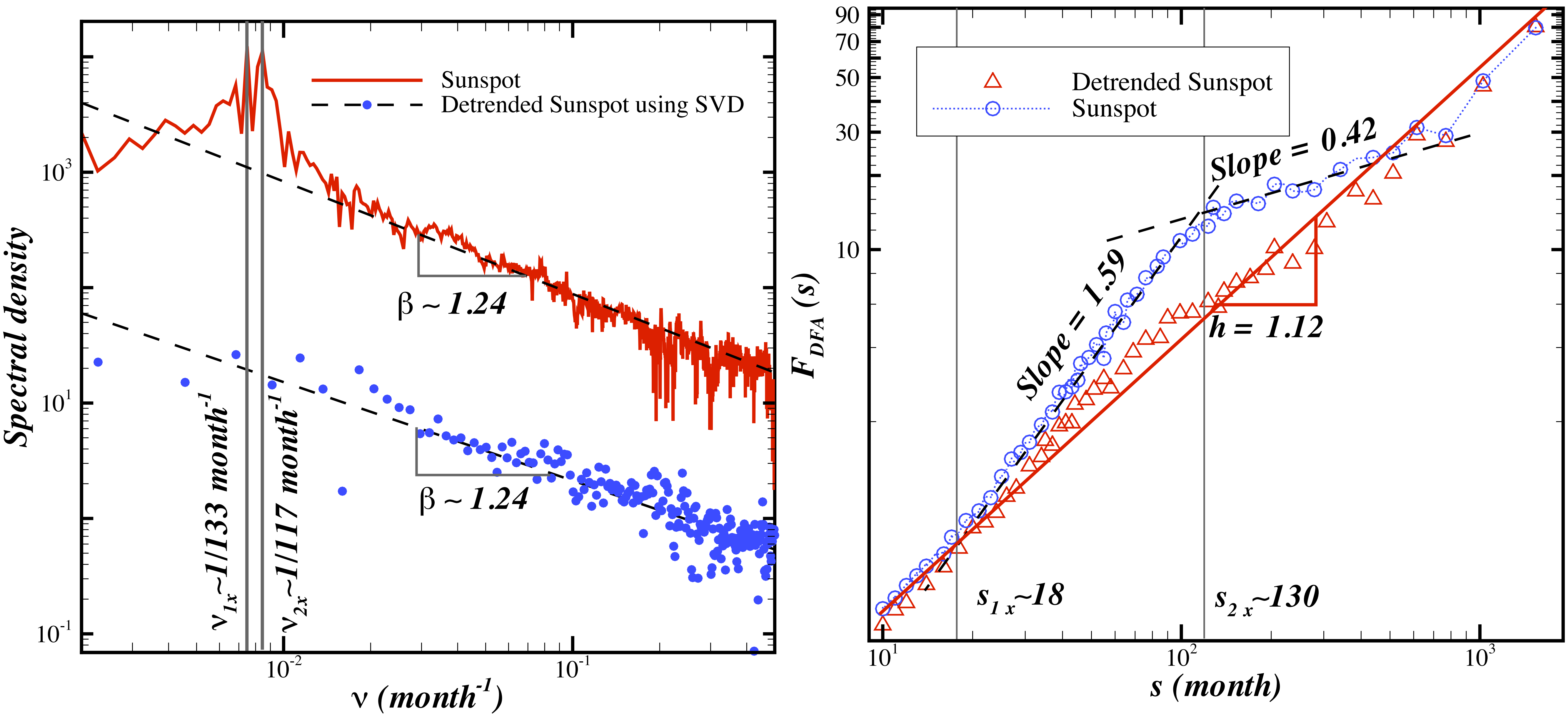
$$\mathbf{\Gamma}^\dagger\mathbf{\Gamma}\mathbf{v_i} = \lambda_i^2\mathbf{v_i}$$

$$\mathbf{\Gamma}\mathbf{\Gamma}^\dagger\mathbf{u_i} = \lambda_i^2\mathbf{u_i}$$

$$x^*_{i+j-1} = \Gamma^*_{ij}$$

The p dominant eigenvalue and associating eigendecomposed vector represent the superimposed trend and the remaining (d-p) demonstrates intrinsic fluctuations

S. hajian and M. Sadegh Movahed, arXiv:0908.0132

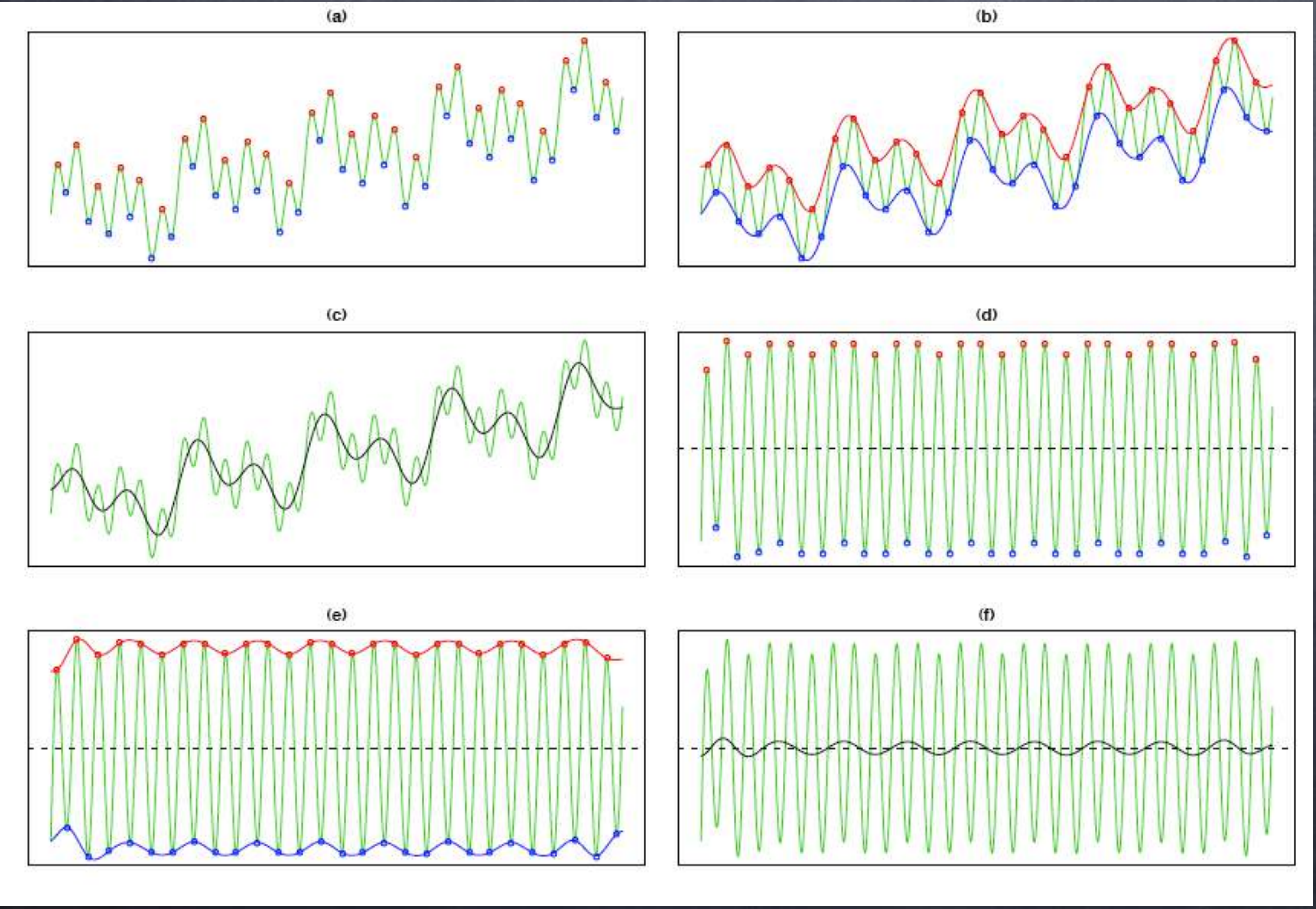


S. hajian and M. Sadegh Movahed, arXiv:0908.0132

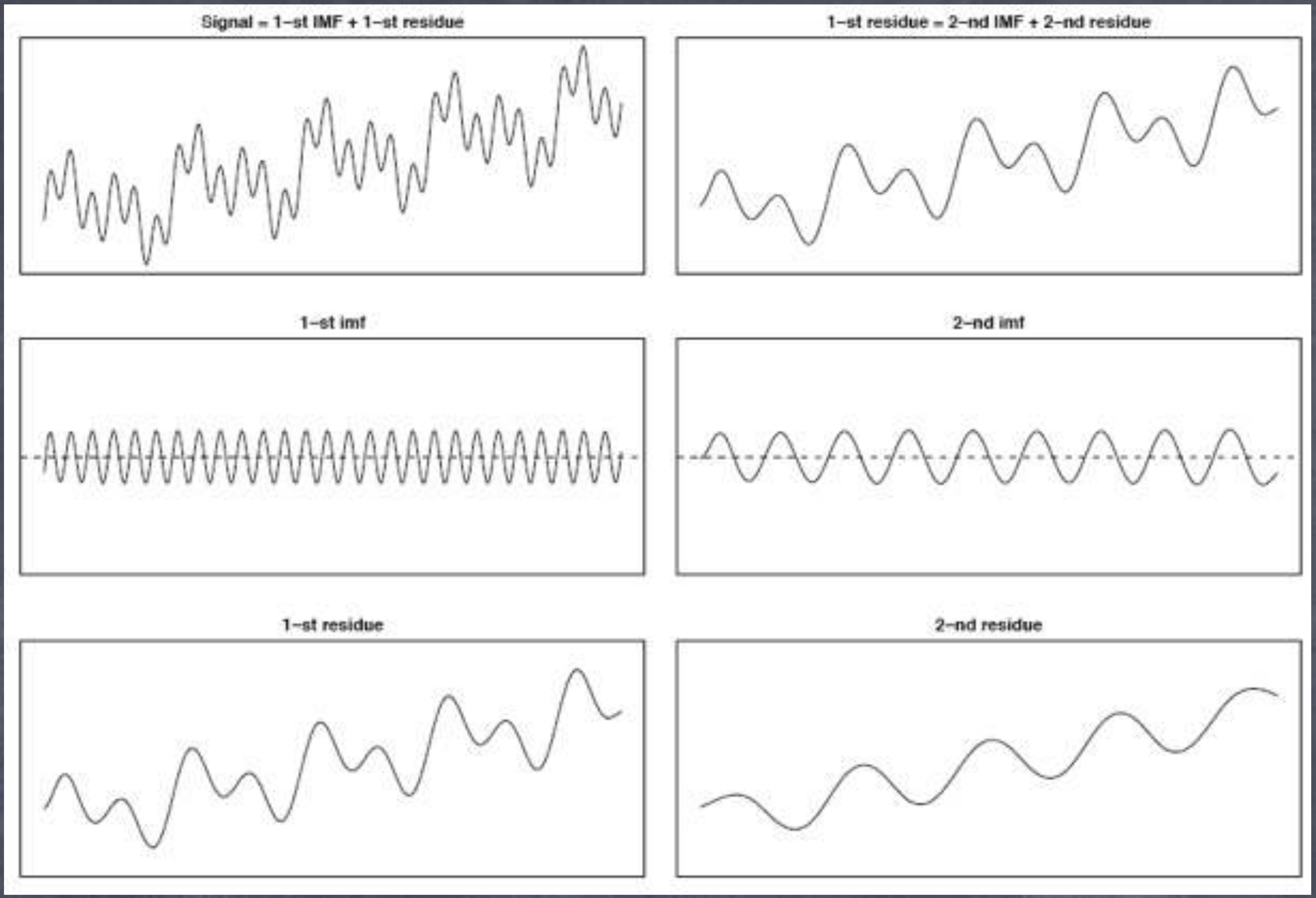
Empirical Mode decomposition (EMD)

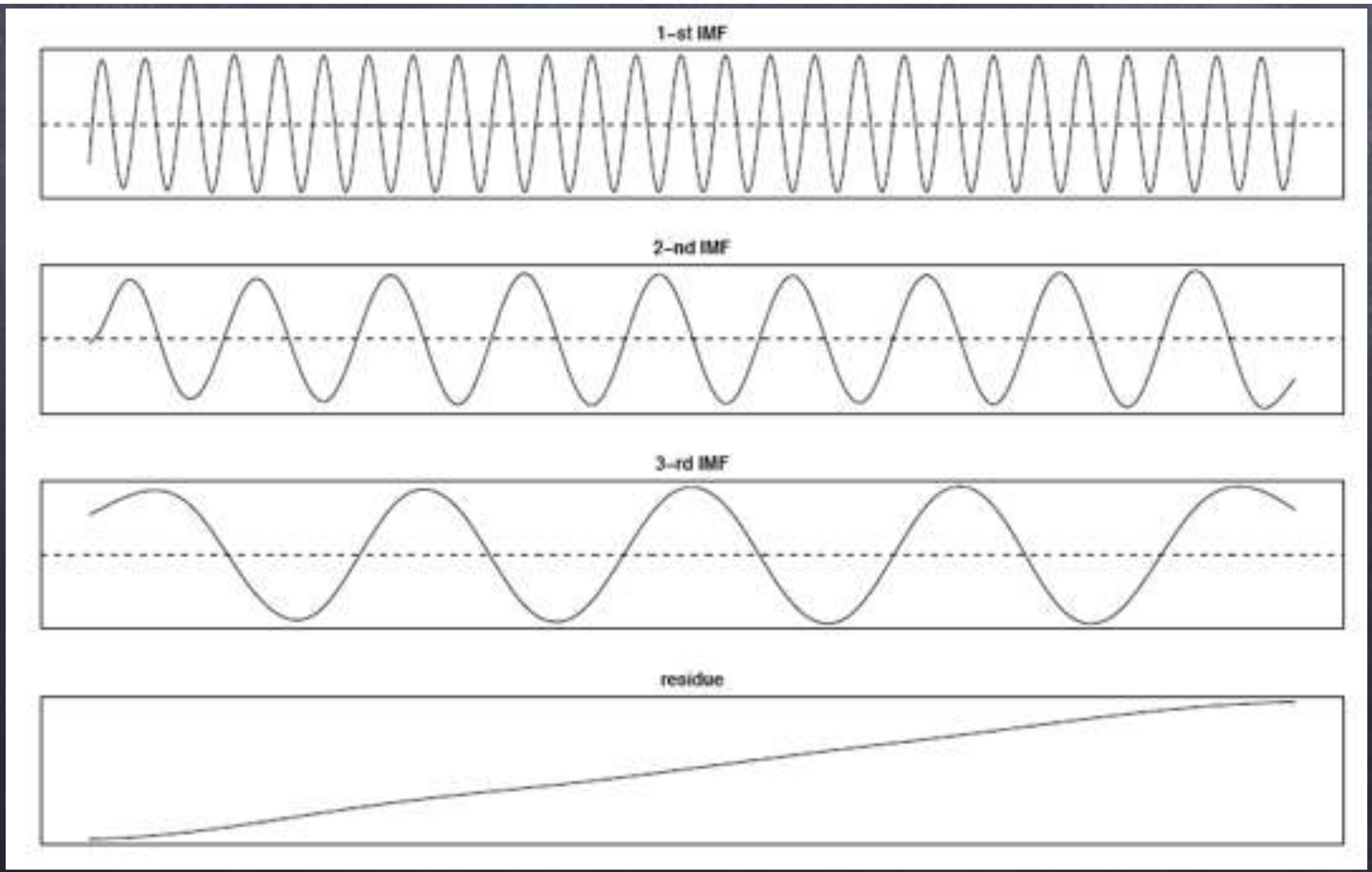
- This method is known as non-parametric method
- There is a good review by Norden E. Huang
  Proceedings: Mathematical, Physical and Engineering Sciences,
  Vol. 454, No. 1971 (Mar. 8, 1998)
- In this case, the intrinsic mode functions (IMFs) satisfy two conditions:
  1) The number of extrema and zero-crossing differs only by one
  2) The local average is zero

1) Identify the local extrema and find their average (Generating upper envelop and lower envelope)
2) Subtracting the envelop mean from signal
3) Check the IMF conditions

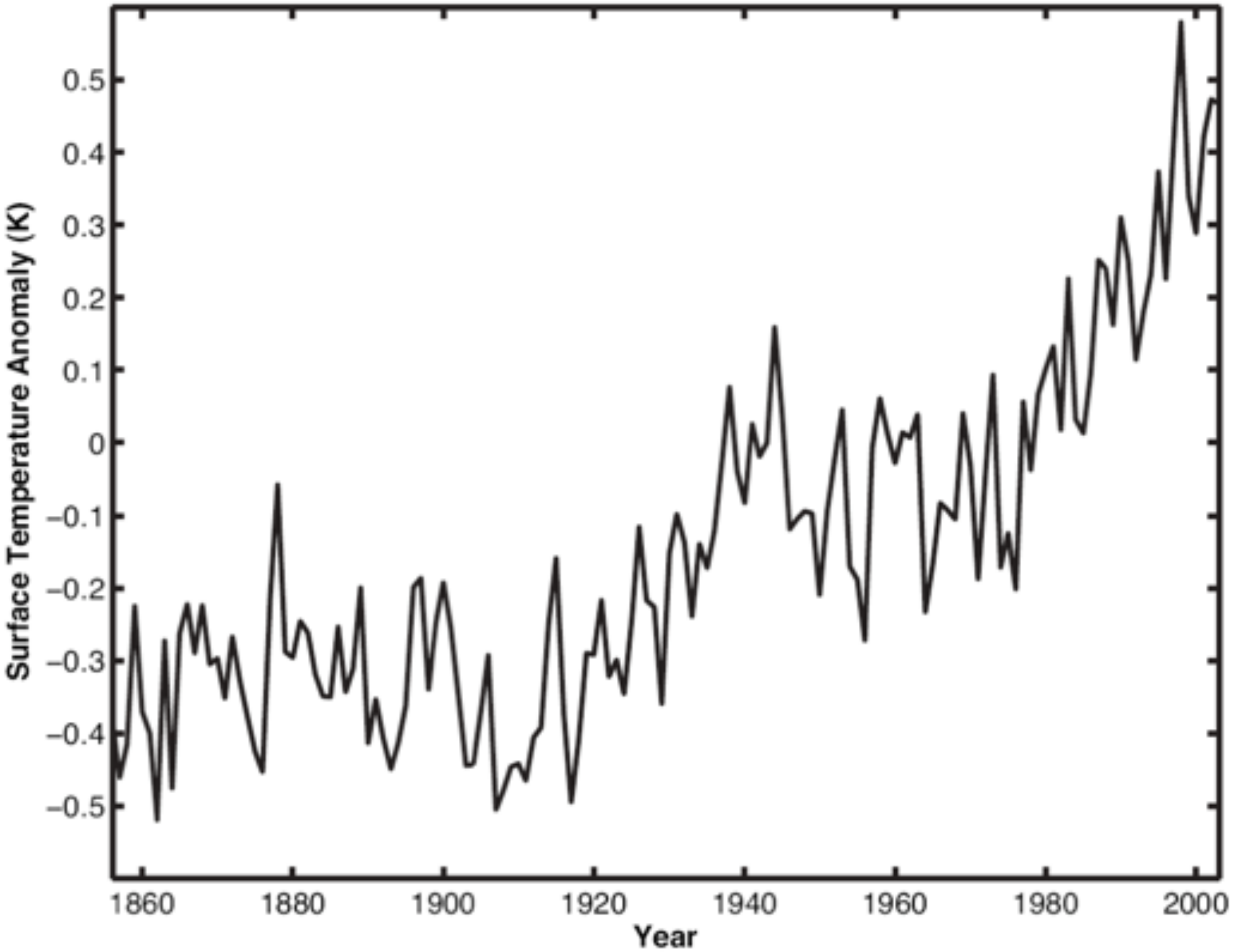


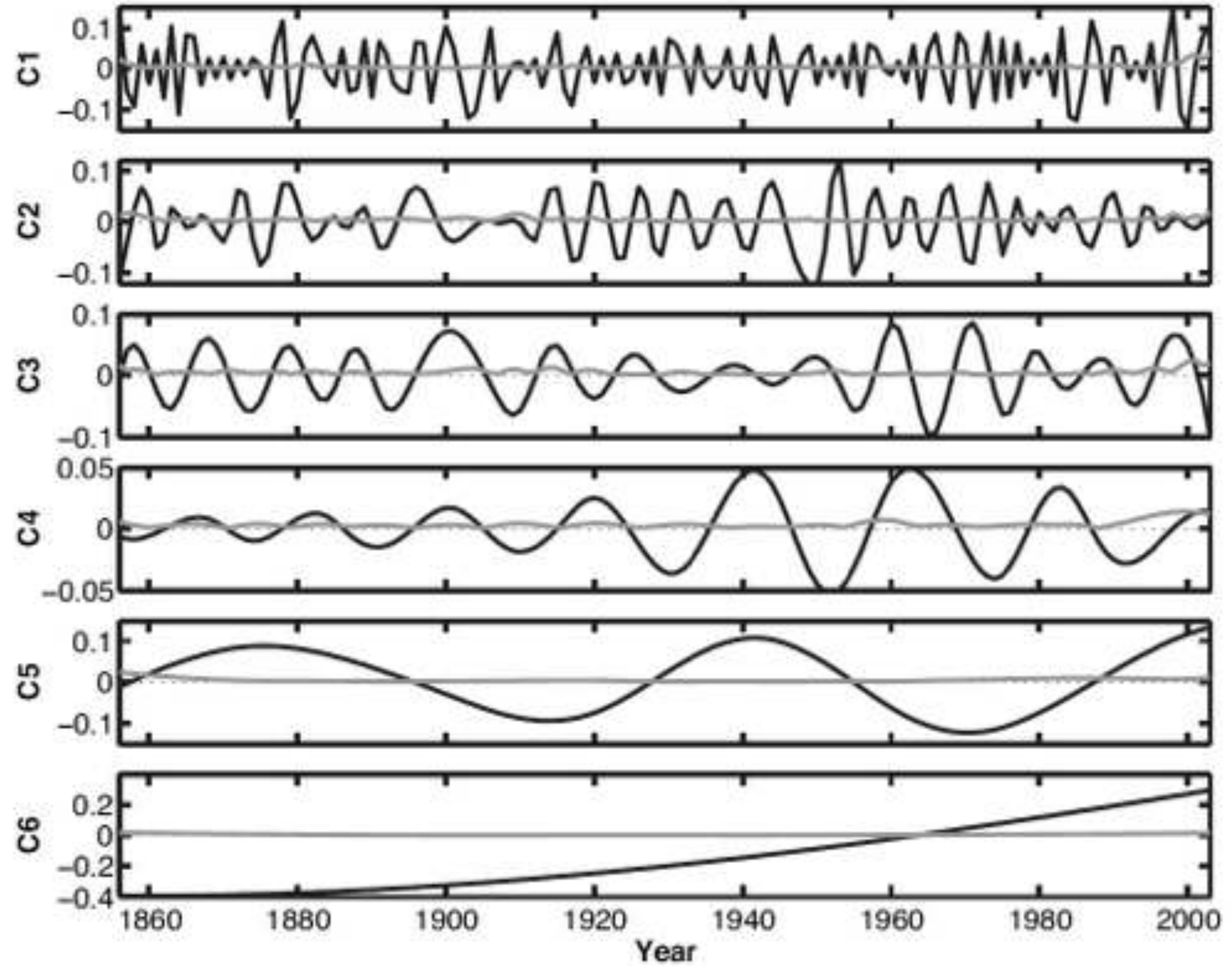
D. Kim et. all.,R Journal, Vol.1, 1 may 2009



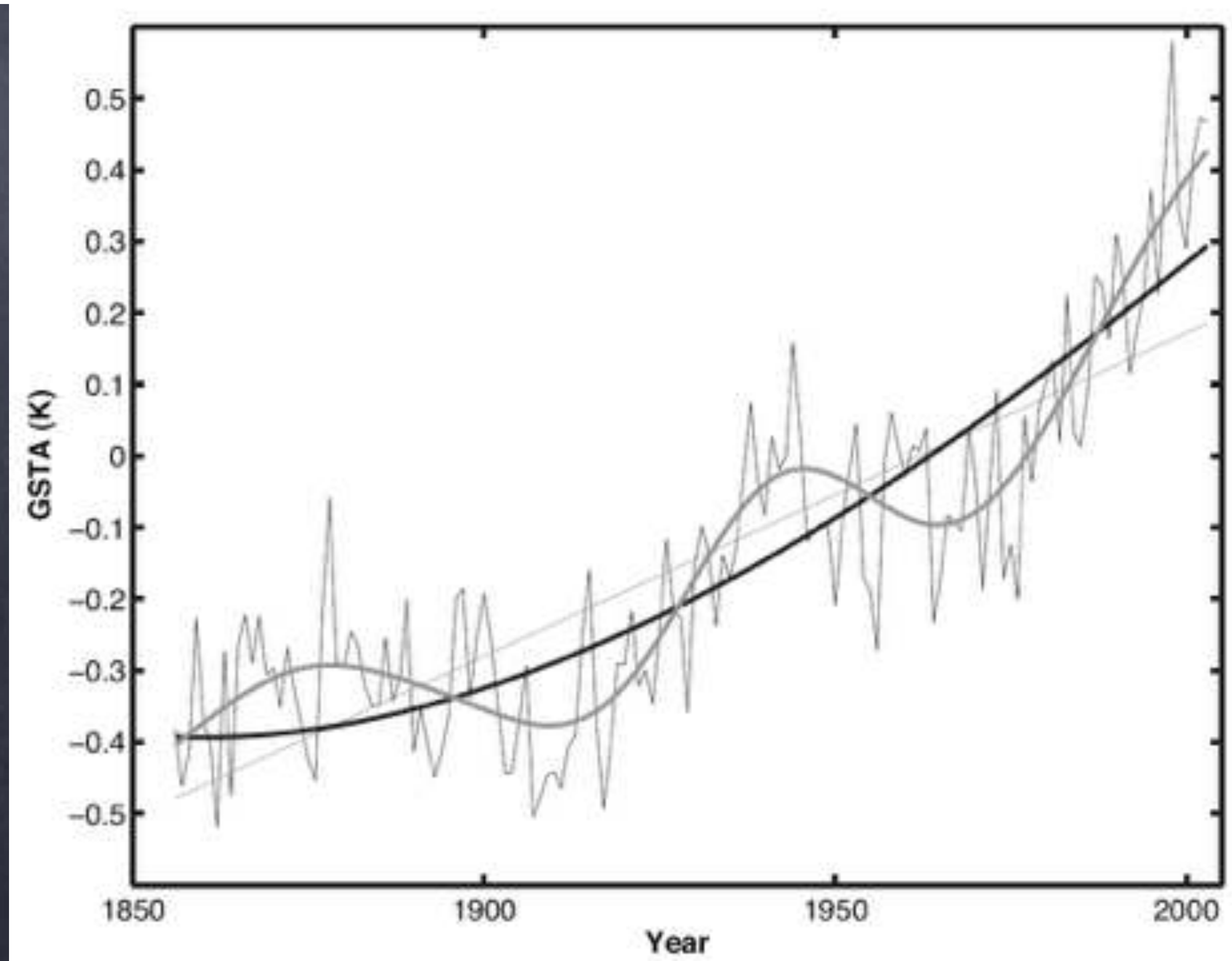


D. Kim et. all.,R Journal, Vol.1, 1 may 2009





Z. Hu, et. al., PNAS, 104, 38 (2007)



gray corresponds to vari
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various stoppir

Advantages and disadvantages

- The size of underlying data won't be invariant by using F-DFA, while the size will be preserved in SVD and EMD

User manual for MF-DFA code written by Sadegh Movahed

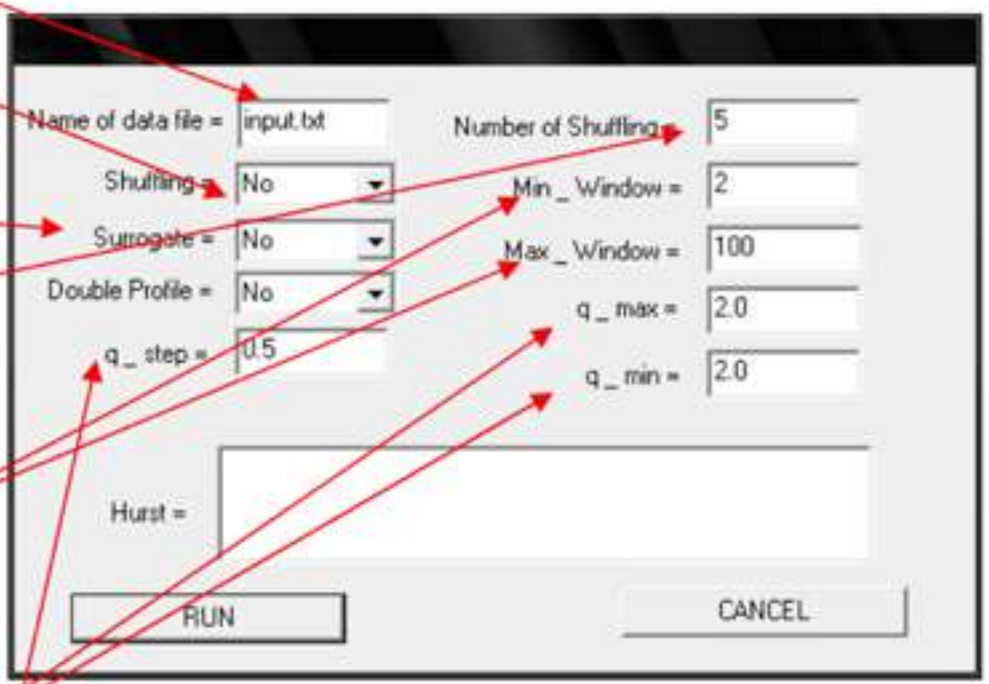
1: You should write the name of your data file in it

2: To shuffled data set you should select YES here.

3: If you want to surrogate your data, select YES for this option

4: This value shows the number of shuffling data set.

5: Here you should determine the maximum and minimum no. of windows, i.e. if you select “10” for maximum and “2” for minimum, your data set is divided to 2 up to 10 non-overlapping windows.



6: If you want to calculate just H=h(q=2) you should determine q=2., namely, q_max=q_min=2. To find the generalized Hurst exponent i.e. h(q) versus q(moment exponent), must q_min and q_max to be different. Just in this case you can find the singularity spectrum for data set.

7: Here the step of moment exponent is determined.

8: In some case, we have to use double profile for data. It is done by the proper option in my program.

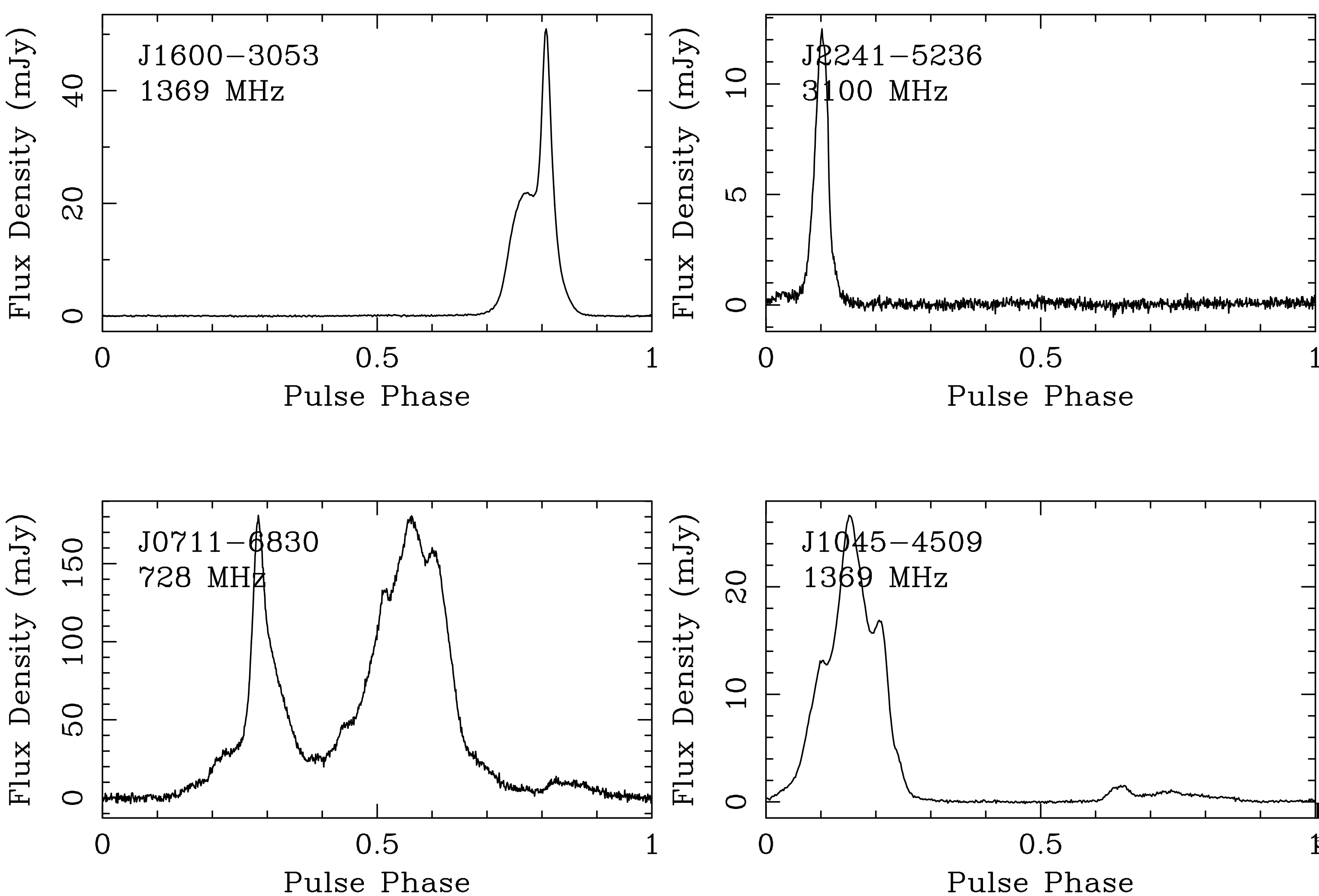
The name of output files are as follows:

1) hurst.txt gives generalized Hurst exponent versus q
2) log_f_s.txt gives the ln (F(s)) versus ln(s)
3) f_s.txt gives the fluctuation function versus “s”
4) tau.txt gives classical multifractal scaling exponent
5) D.txt gives generalized multifractal dimension
6) singularity.txt gives singularity spectrum
7) PDF.txt gives probability density function

Part 6
PTRs and GWB

Pulsar

- It was observed for the first time on 1967 by Bell and Hewish
- Received Nobel prize on 1974.
- Fast rotating object: Period~ msec -sec
- Pulse periods can be measured with accuracies approaching 1 part in 10^16
- A best model for pulsar is Neutron star



Dai, S. Hobbs, G. Manchester, R. N. Kerr, M. Shannon, R. M. van Straten, W., ... & Coles, W. A. *A study of multifrequency polarization pulse profiles of millisecond pulsars*. Monthly Notices of the Royal Astronomical Society, 449(3), 3223-3262. 2015.

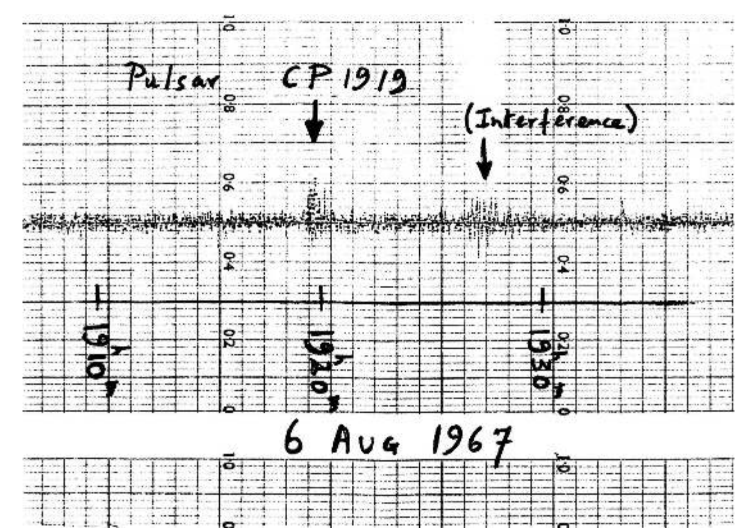


FIGURE 13 Discovery of the first pulsar, PSR 1919+21 ("CP" stands for Cambridge Pulsar). (Figure from Lyne and Graham-Smith, *Pulsar Astronomy*, ©Cambridge University Press, New York, 1990. Reprinted with the permission of Cambridge University Press.)

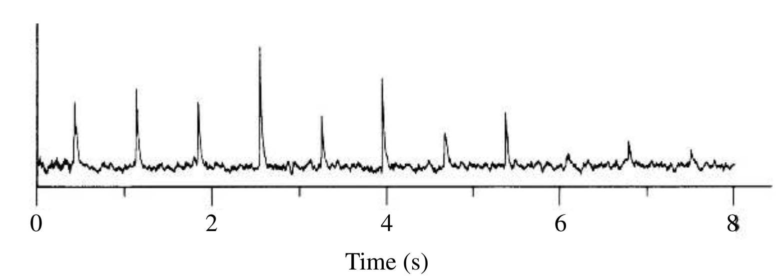
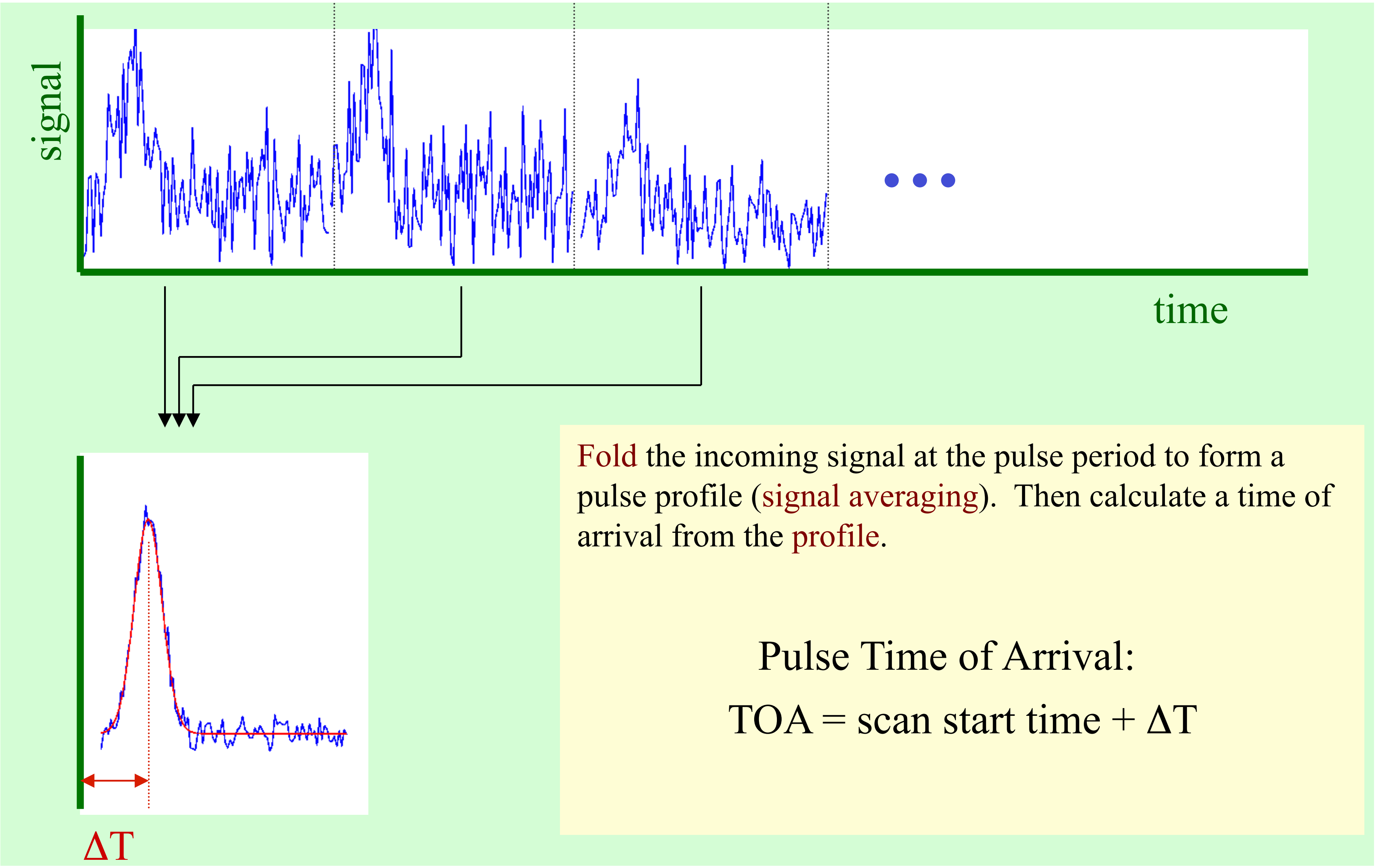


FIGURE 19 Pulses from PSR 0329+54 with a period of 0.714 s. (Figure adapted from Manchester and Taylor, *Pulsars*, W. H. Freeman and Co., New York, 1977.)

Pulsar's Time of Arrival

Pulsar timing is the regular monitoring of the rotation of pulsar by tracking (nearly exactly) the times of arrival of the radio pulses



Fold the incoming signal at the pulse period to form a pulse profile (signal averaging). Then calculate a time of arrival from the profile.

Pulse Time of Arrival:

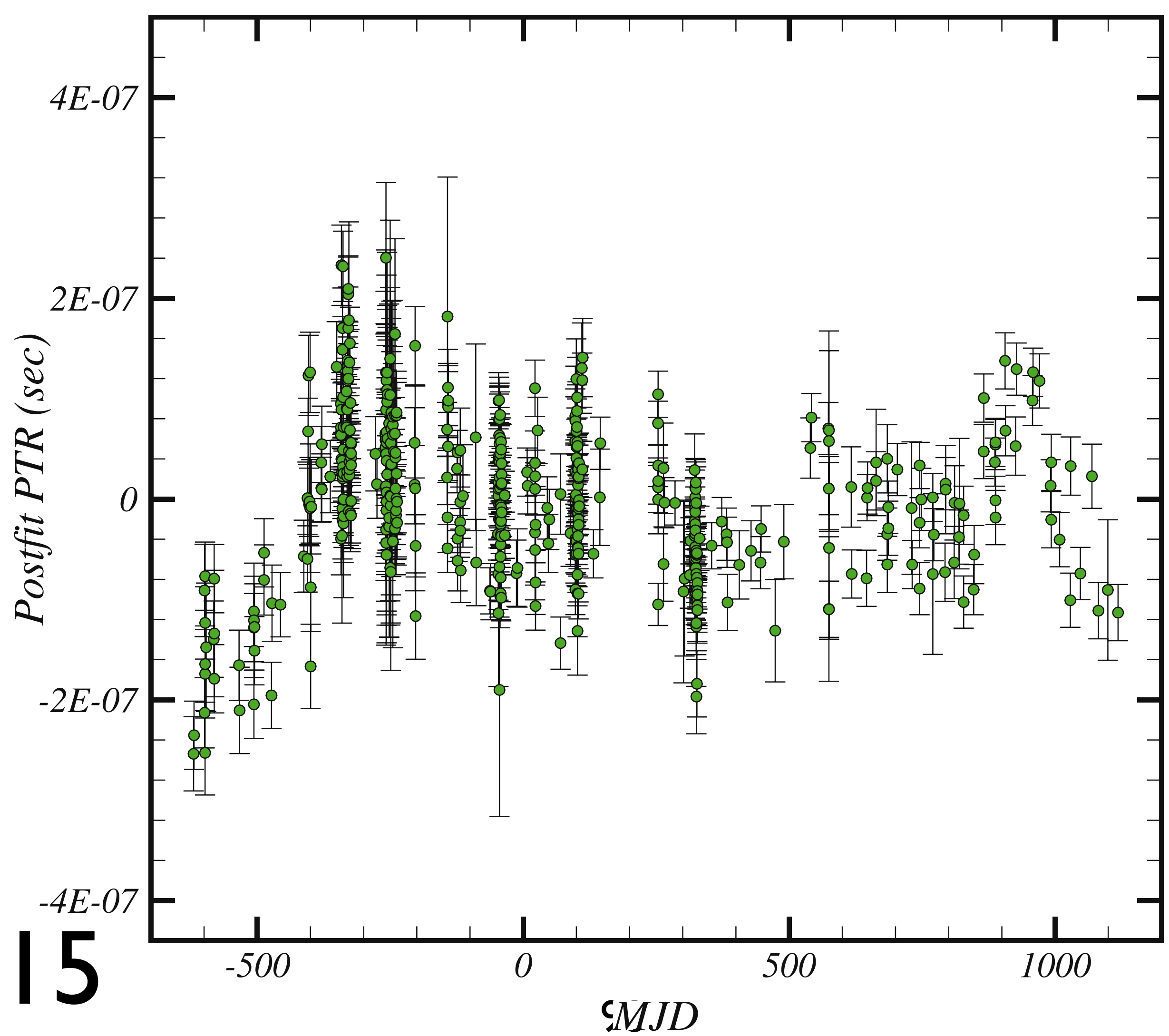
TOA = scan start time + ΔT

David Nice

Pulsar's Timing residual

Residual is difference between measured pulse's time of arrival and expected time of arrival:
Residual=Observed ToA-Computed ToA or vise versa

$$\Delta t = \Delta_c + \Delta_A + \Delta_{E_\odot} + \Delta_{R_\odot} + \Delta_{S_\odot} - D/f^{۲} + \Delta_{VP} + \Delta_B$$



PSR J0437-4715

Pulsar Timing Array

1) Parkes Pulsar Timing Array (PPTA),
   64-meter Radio Telescope from 1961
2) Square Kilometer Array (SKA)
3) International Pulsar Timing Array (IPTA)
4) European Pulsar Timing Array
   (Including 5 radio telescopes)
5) Indian Pulsar Timing Array (2016)

San Basilio, Sardinia, Italy

Nançay, Nançay, France

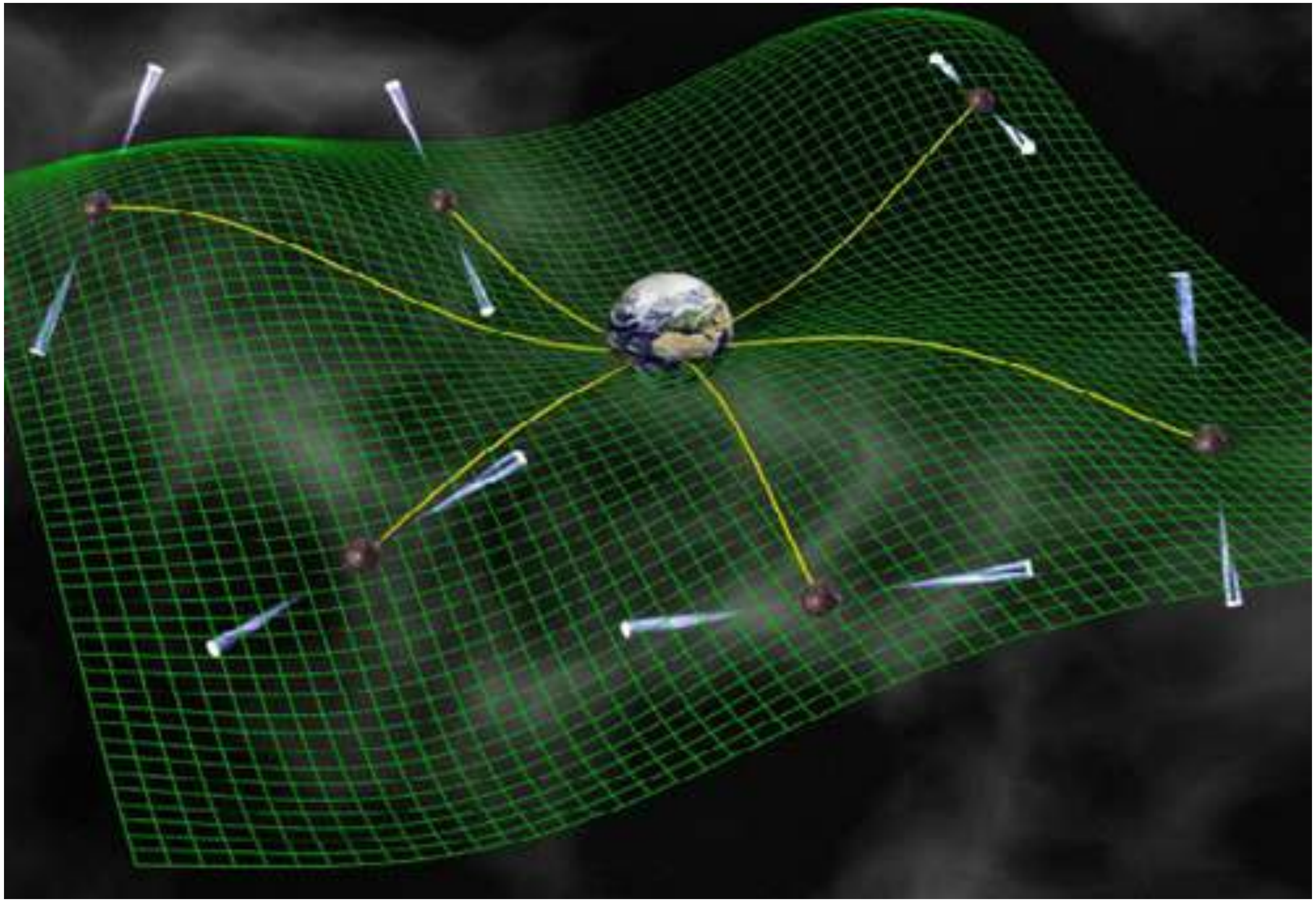
Cheshire East, United Kingdom

Australia

Hooghalen, Netherlands

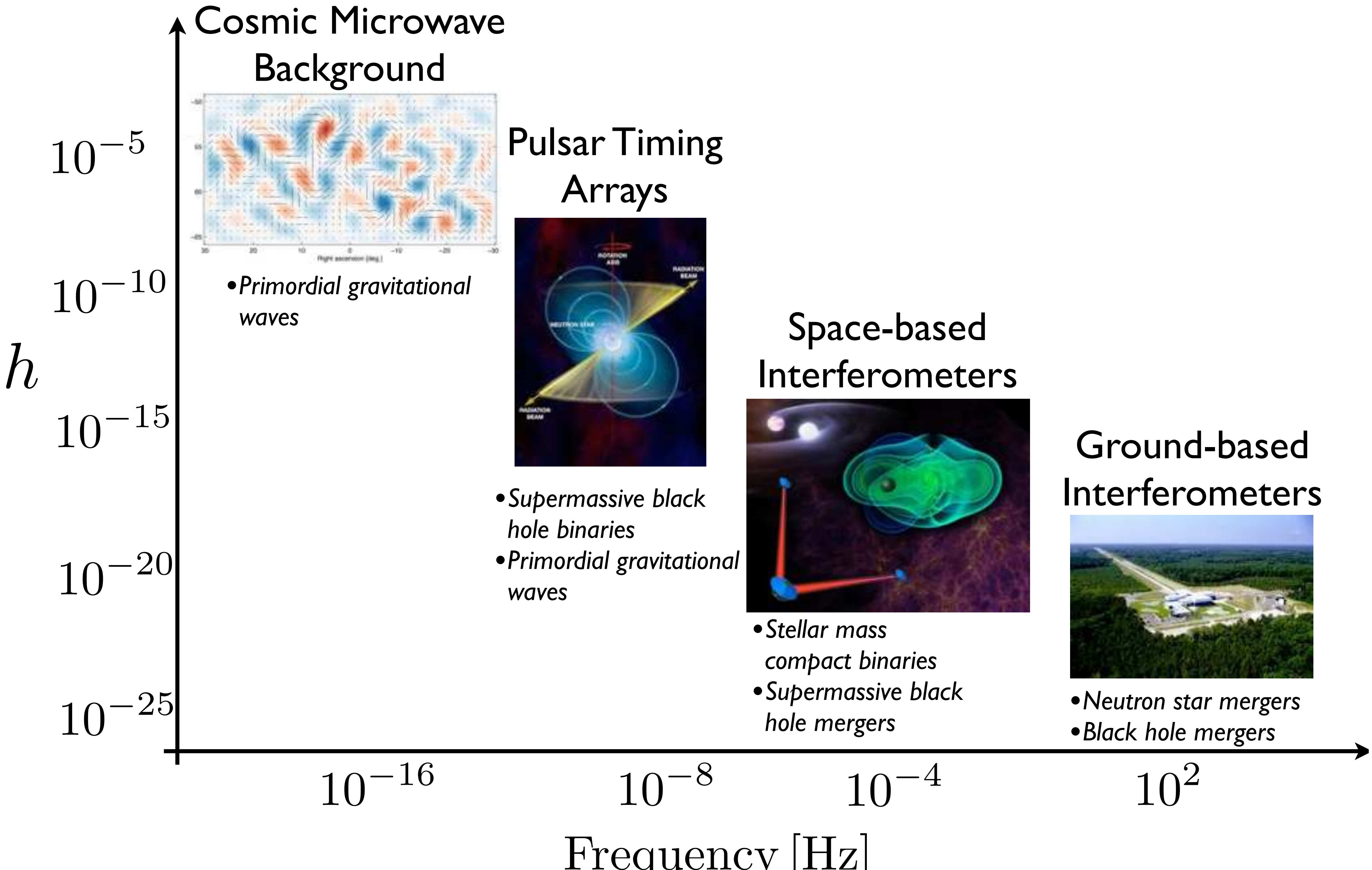
North Rhine-Westphalia, Germany

Applications of Pulsar



f ~ 1/(weeks-years) (10^{-6} – 10^{-9} Hz)

The big picture of gravitational-wave astronomy



Ierapetra, Crete

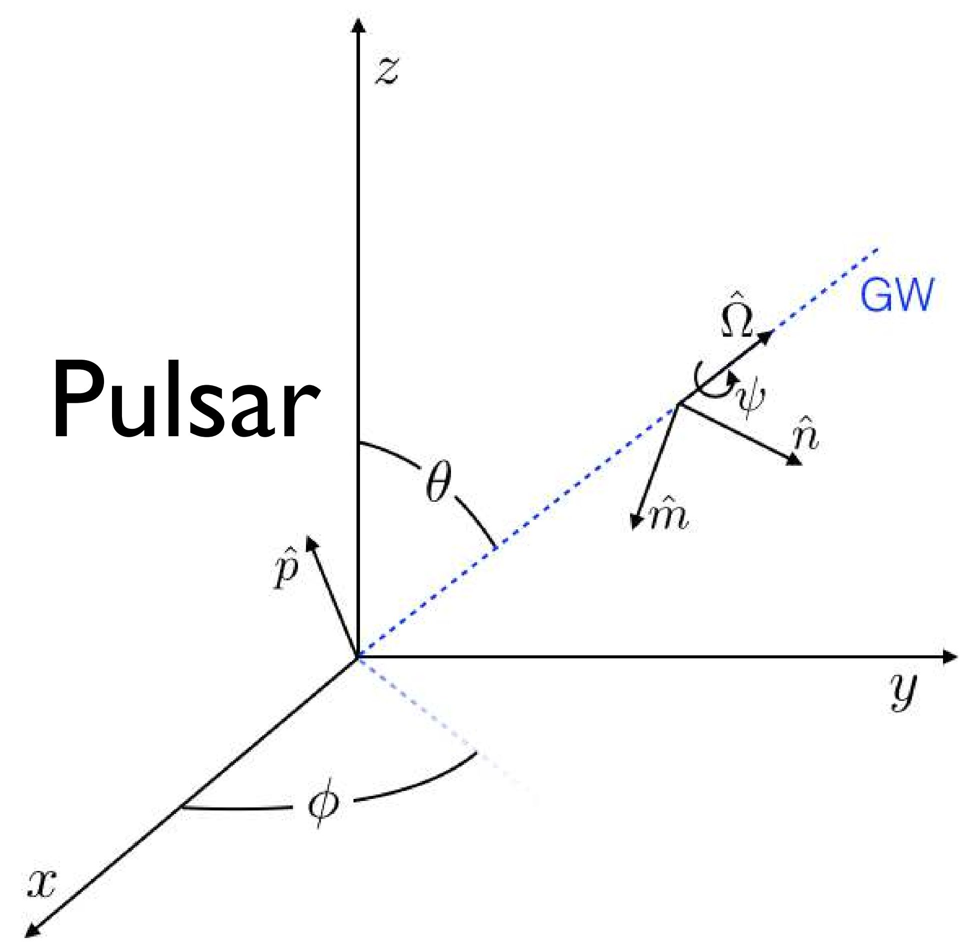
PTR and GWB model

$$h_{\mu\nu}^{TT}(t,z) = h_{\mu\nu}^{TT}(t-\frac{z}{c}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

Characteristic strain spectrum is:

$$\mathcal{H}_c(f) = \mathcal{A}_{\rm yr}\left(\frac{f}{f_{1\ \rm yr}}\right)^{\zeta},$$

Pulsar

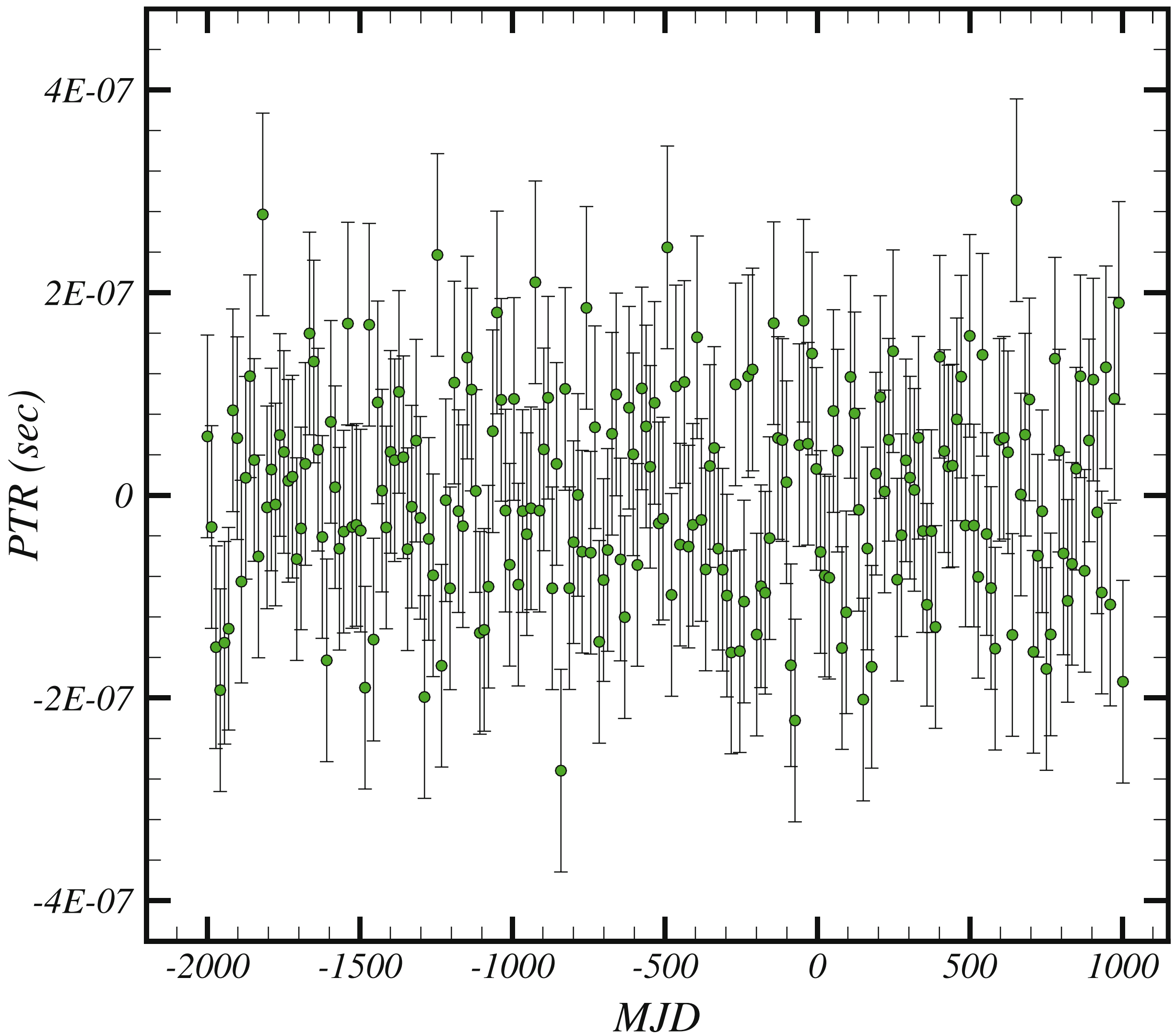


arXiv:1410.8256

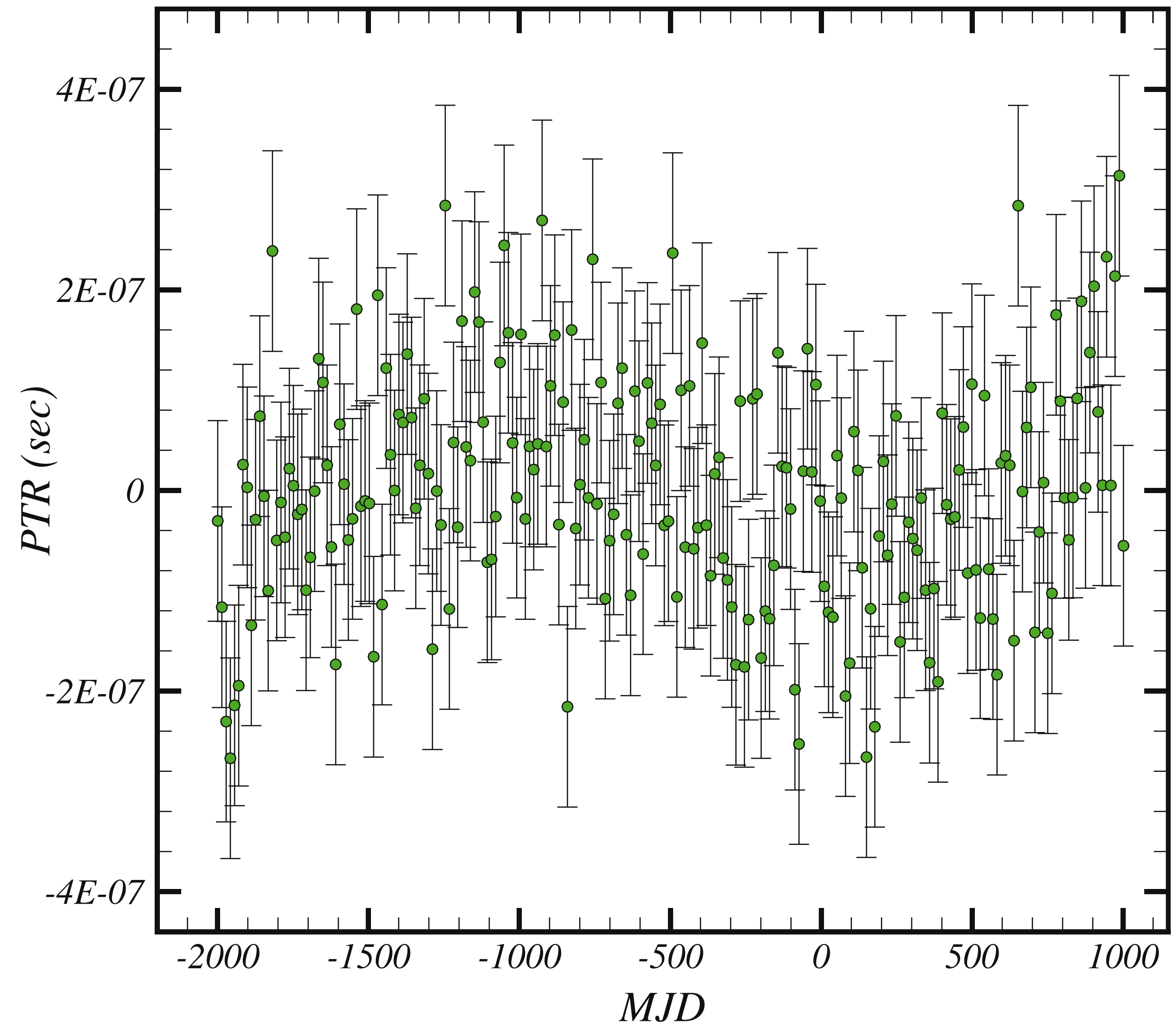
Synthetic Datasets

$$\mathcal{A}_{\rm yr} = 10^{-15}$$
$$\zeta = -2/3$$

Pure PTR

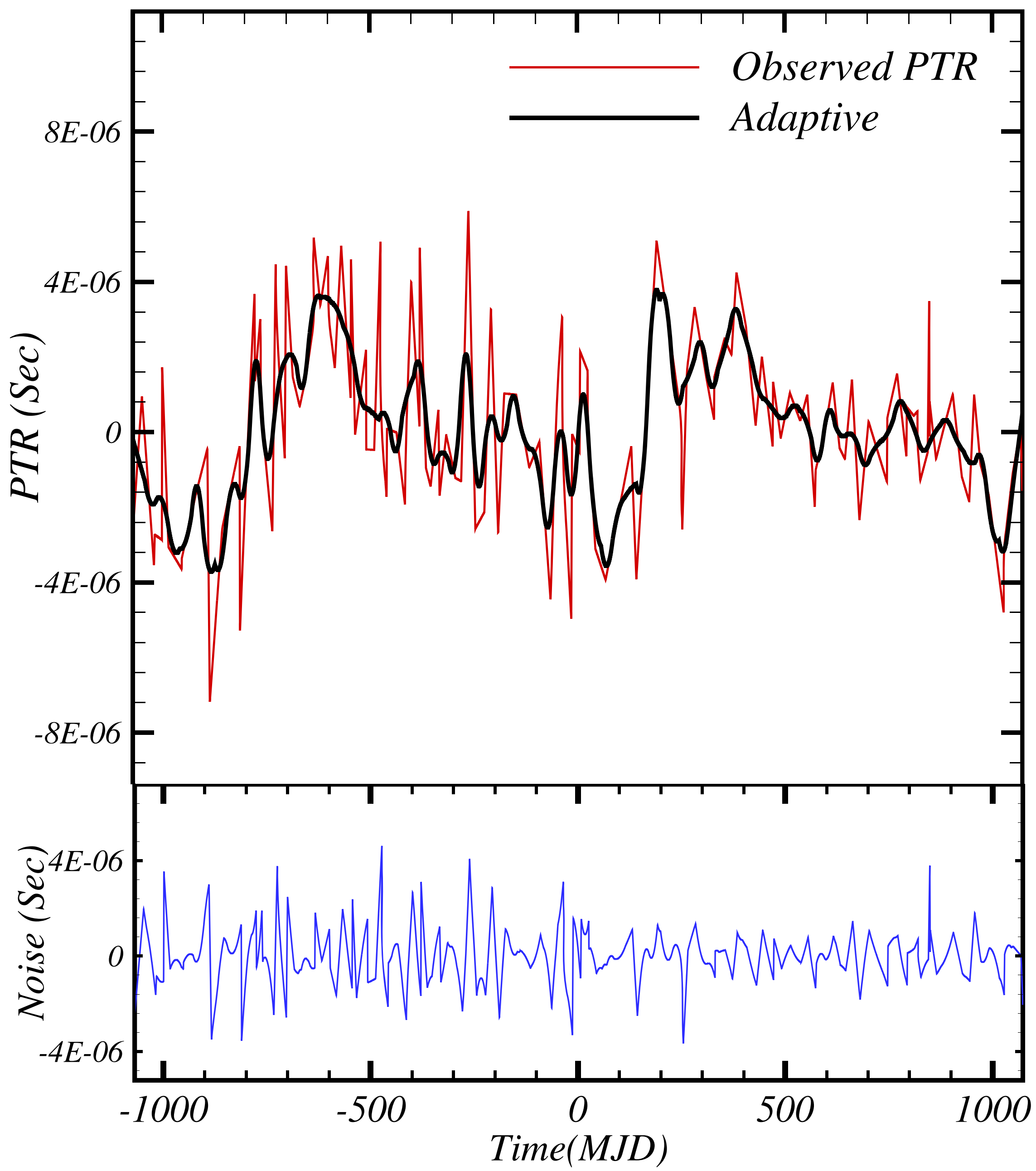


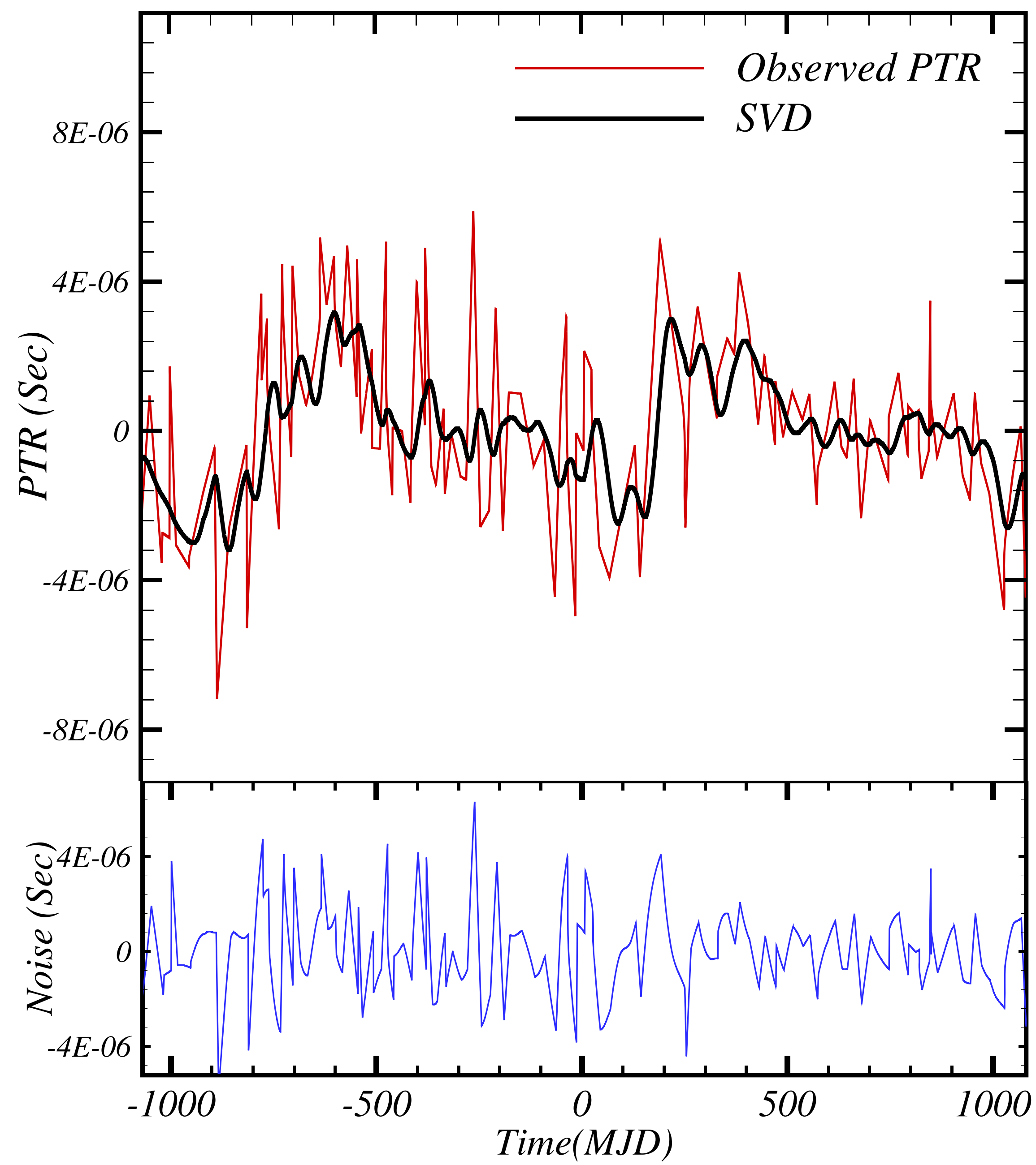
PTR+GW



TEMPO2 software package

Trend and Noise models





Generalized form of Hellings & Downs (1983)

$$\mathcal{C}_\times(\tau,\Theta_{ab}) = \langle PTR_a(t,\hat{n}_a) PTR_b(t+\tau,\hat{n}_b)\rangle_t \qquad \Theta_{ab} = \arccos|\hat{n}_a.\hat{n}_b|$$

$$PTR_a(t) = PTR_a^{\mathrm{pure}}(t) + \mathcal{B}_a R_{\mathrm{GWB}}(t)$$

$$PTR_b(t) = PTR_b^{\mathrm{pure}}(t) + \mathcal{B}_b R_{\mathrm{GWB}}(t)$$

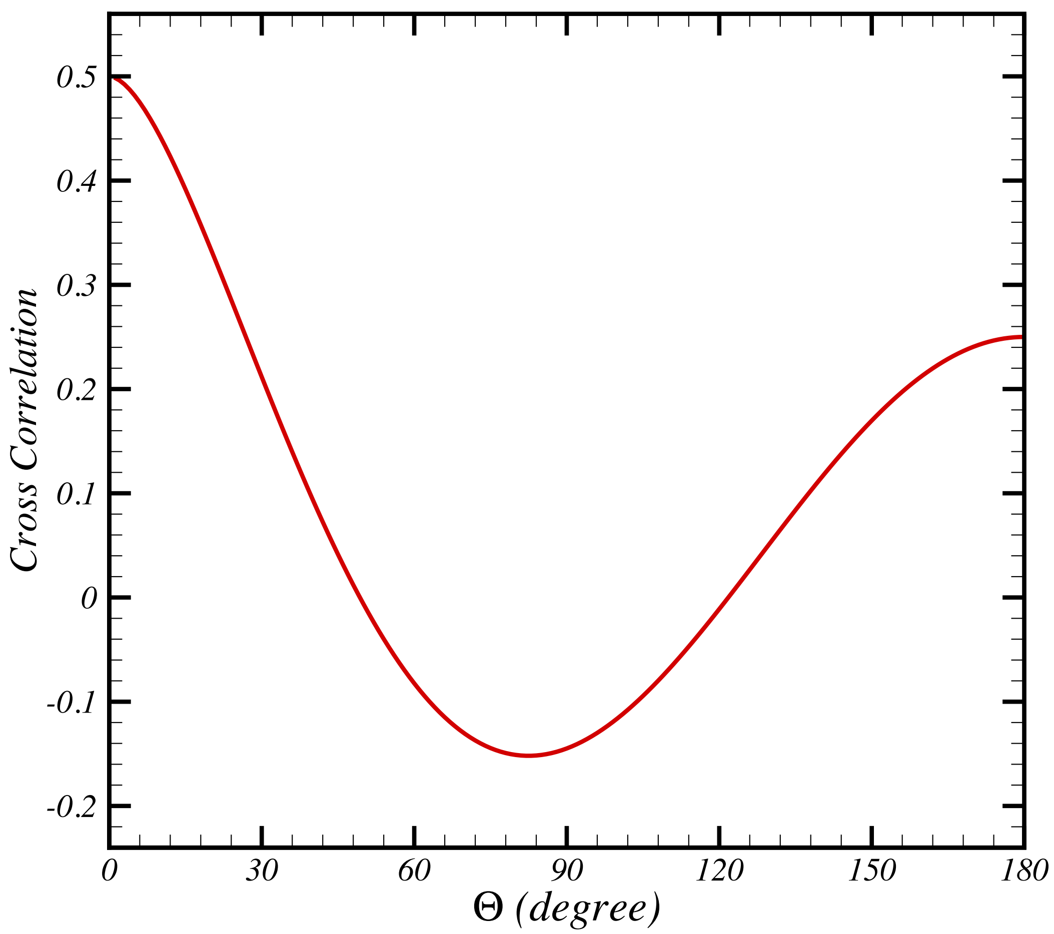
$$\mathcal{B}_i \equiv -\frac{1}{2}\cos(2\phi_i)(1-\cos(\theta_i))$$

$$\overline{\mathcal{C}}_\times(\tau,\Theta) = \langle\, \mathcal{C}_\times(\tau,\Theta_{ab})\rangle_{\mathrm{pairs}} \sim \overline{\Gamma}(\Theta) \times \tau^{\gamma_\times}$$

$$\overline{\Gamma}(\Theta) = \frac{3}{2}\psi\ln(\psi) - \frac{\psi}{4} + \frac{1}{2},$$

$$\psi \equiv [1-\cos(\Theta)]/2$$

$$\gamma_\times = 2 - 2H_\times = 2 - 2h_\times(q=2)$$



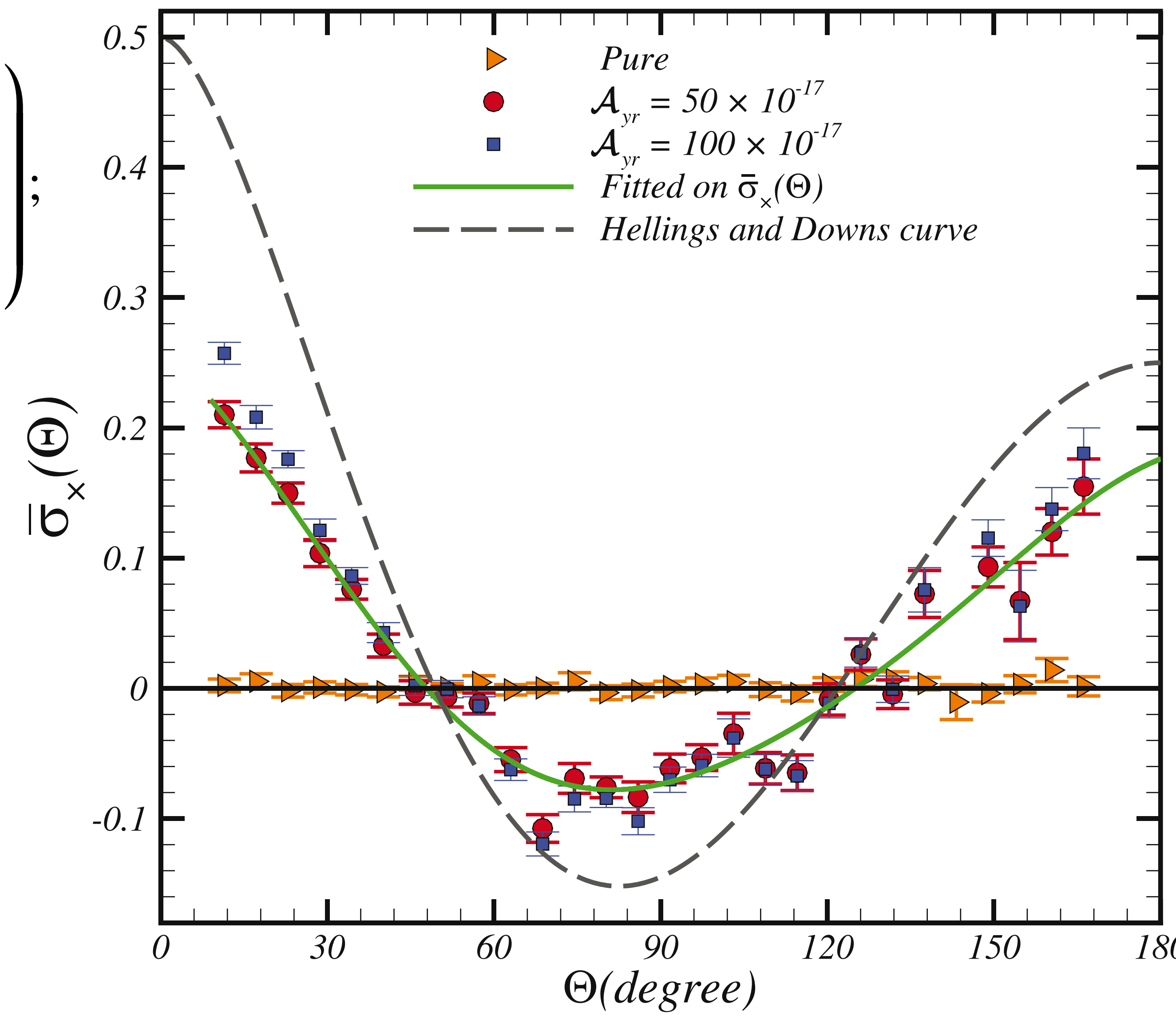
arXiv:1410.8256

New measure

$$\sigma_\times(\Theta_{ab}) \equiv \sum_s \left(\frac{\sum_{\nu=1}^{2N_s} \mathcal{E}_\times(s,\nu)}{\sqrt{\left[\sum_{\nu=1}^{2N_s} \mathcal{E}_a(s,\nu)\right]\left[\sum_{\nu=1}^{2N_s} \mathcal{E}_b(s,\nu)\right]}} \right);$$

$$\Theta_{ab} = \arccos|\hat{n}_a.\hat{n}_b|$$

$$\bar{\sigma}_\times(\Theta) = \frac{1}{4\pi}\int d\Omega \sigma_\times(\Theta_{ab}).$$



4 Strategies for GWB detection

1) $$\Delta h_1(\mathcal{A}_{yr},\zeta) \equiv \sum_{q=q_{min}}^{q=q_{max}} |\mathcal{G}_{h(q)}(\mathcal{A}_{yr},\zeta) - \mathcal{G}_{h(q)}(\mathcal{A}_{yr}=0)|$$

$$\left(\frac{\mathcal{A}_{yr}}{10^{-17}}\right) = a\Delta h_1^2 + b\Delta h_1 + c$$

2) $$\Delta h_2(\mathcal{A}_{yr},\zeta) \equiv \sum_{q=q_{min}}^{q_{max}} |h(q;\mathcal{A}_{yr},\zeta) - h_{\rm shuf}(q;\mathcal{A}_{yr},\zeta)|$$

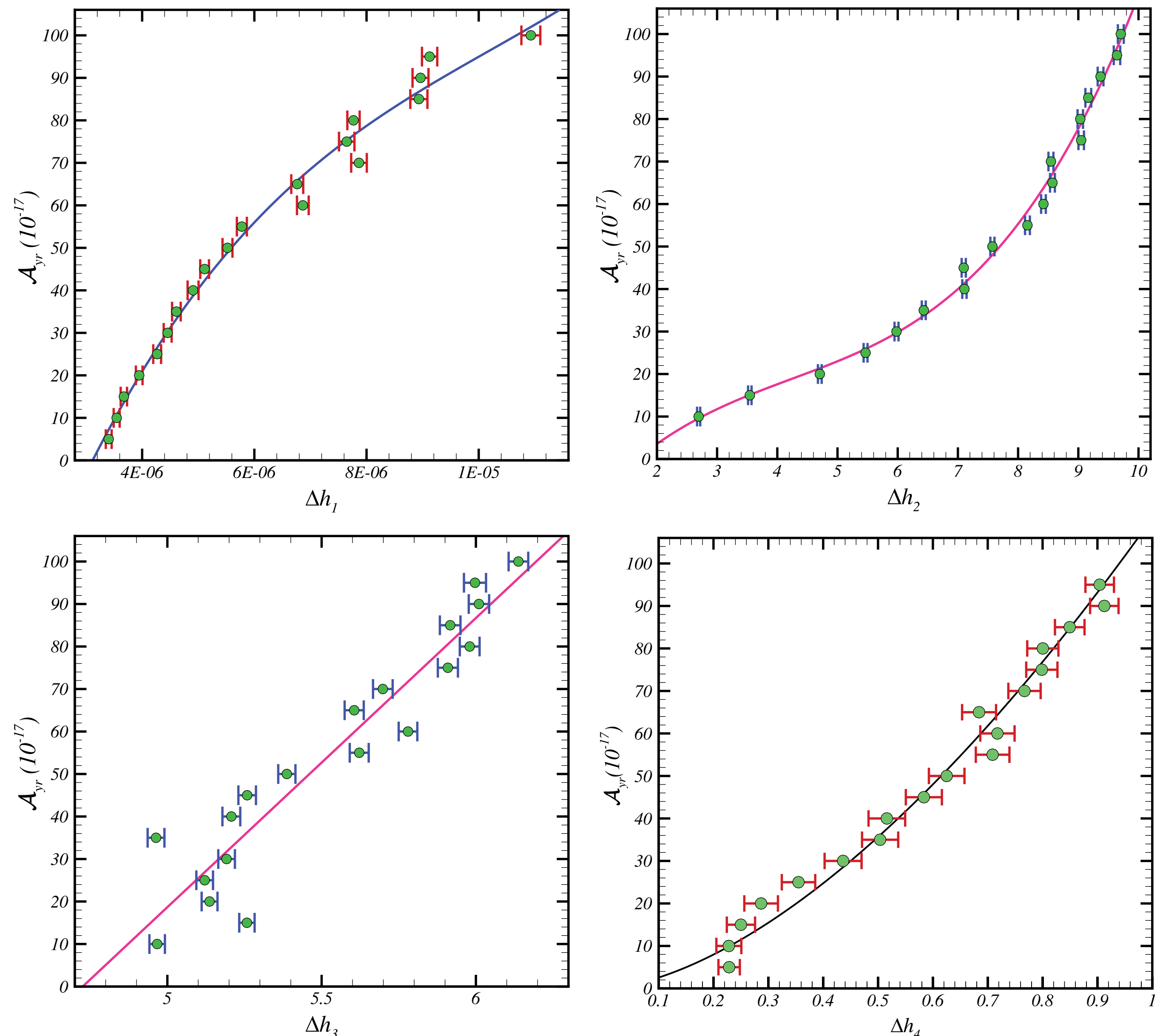
$$\left(\frac{\mathcal{A}_{yr}}{10^{-17}}\right) = a\Delta h_2^3 + b\Delta h_2^2 + c\Delta h_2$$

3) $$\Delta h_3(\mathcal{A}_{yr},\zeta) \equiv \sum_{q=q_{min}}^{q_{max}} |h(q;\mathcal{A}_{yr},\zeta) - h_{\rm sur}(q,\mathcal{A}_{yr},\zeta)|$$

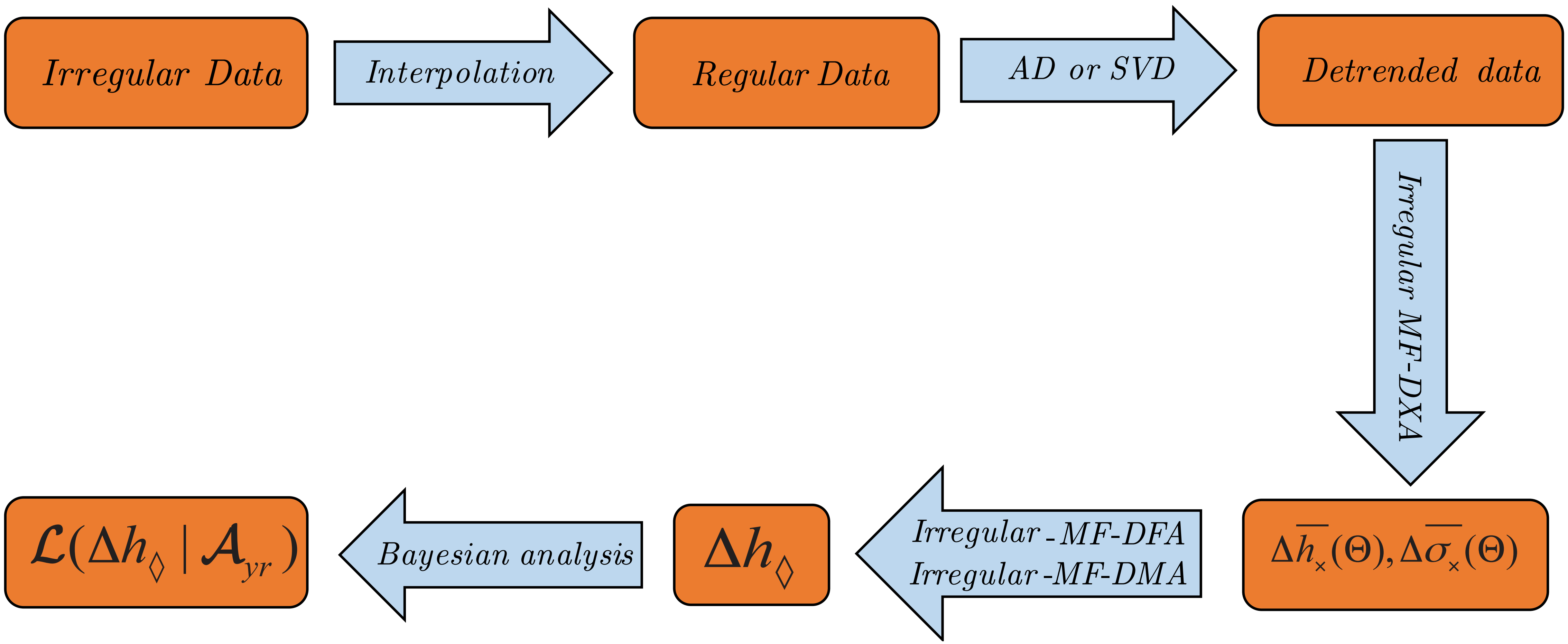
$$\left(\frac{\mathcal{A}_{yr}}{10^{-17}}\right) = a\Delta h_3 + b$$

4) $$\Delta h_4(\mathcal{A}_{yr},\zeta) \equiv |\Delta\alpha(\mathcal{A}_{yr},\zeta) - \Delta\alpha(\mathcal{A}_{yr}=0)|$$

$$\left(\frac{\mathcal{A}_{yr}}{10^{-17}}\right) = a\Delta h_4^b + c$$



Pipeline for GWB detection



Eghdami, I., et al., APJ, 864:162 (18 pp), 2018

Observed PTR, Parkes Pulsar Timing Array (PPTA)

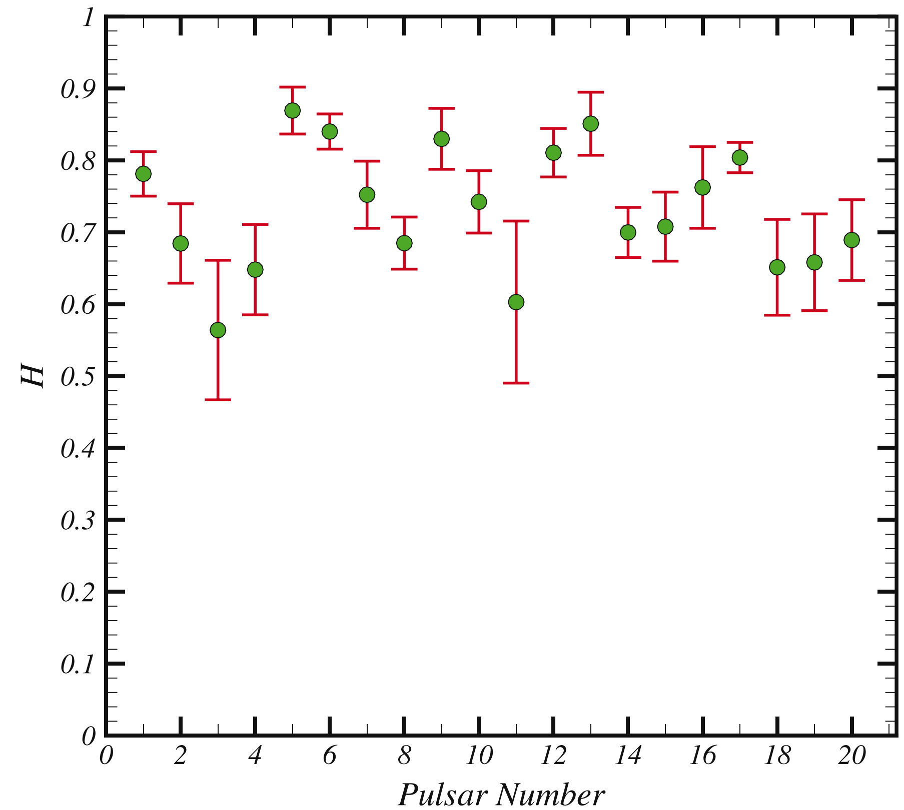
Table 1

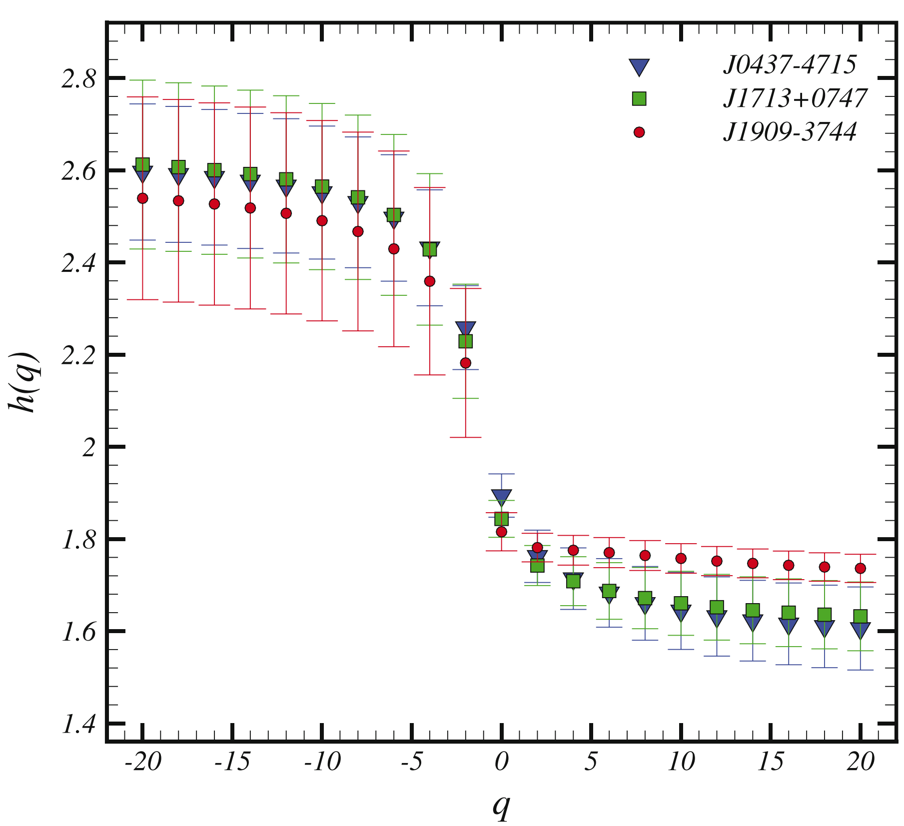
Hurst Exponent, H, Width of Singularity Spectrum, $\Delta\alpha$, Scaling Exponent of Temporal Autocorrelation, γ, rms, Total Time Span (TTS) of Post-fit Timing Residuals, and Upper Limit on Dimensionless Amplitude of GWB of 20 MSPs Observed in PPTA Project

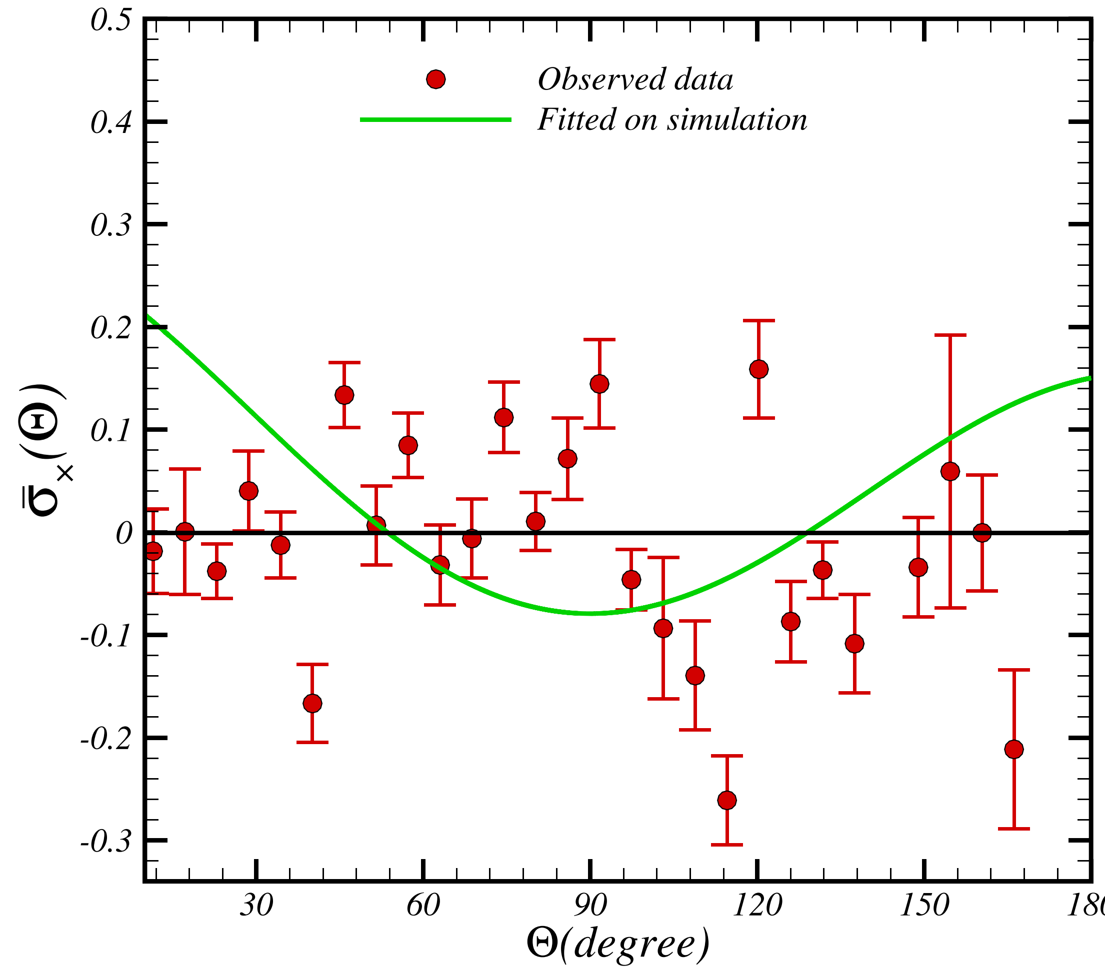
PSR Number	PSR Name	H	$\Delta\alpha$	γ	rms (μs)	TTS (yr)	$\mathcal{A}_{\rm yr}^{\rm up}(95\%)$
1	J0437–4715	0.78 ± 0.03	0.89 ± 0.06	-1.56 ± 0.06	0.08	4.76	5.0×10^{-15}
2	J0613–0200	0.68 ± 0.06	1.22 ± 0.04	-1.37 ± 0.11	1.07	5.99	7.0×10^{-15}
3	J0711–6830	0.56 ± 0.10	1.40 ± 0.08	-1.13 ± 0.19	0.89	5.99	6.0×10^{-15}
4	J1022+1001	0.65 ± 0.06	1.04 ± 0.04	-1.30 ± 0.13	1.72	5.88	8.5×10^{-15}
5	J1024–0719	0.87 ± 0.03	1.60 ± 0.03	-1.74 ± 0.07	1.13	5.99	…
6	J1045–4509	0.84 ± 0.02	1.29 ± 0.04	-1.68 ± 0.05	2.77	5.94	…
7	J1600–3053	0.75 ± 0.05	1.34 ± 0.04	-1.50 ± 0.09	0.68	5.93	…
8	J1603–7202	0.68 ± 0.04	1.29 ± 0.05	-1.37 ± 0.07	2.14	5.99	2.5×10^{-15}
9	J1643–1224	0.83 ± 0.04	0.89 ± 0.02	-1.66 ± 0.08	1.64	5.87	…
10	J1713+0747	0.74 ± 0.04	1.20 ± 0.05	-1.48 ± 0.09	0.31	5.71	2.0×10^{-15}
11	J1730–2304	0.60 ± 0.11	1.79 ± 0.04	-1.21 ± 0.23	1.47	5.93	…
12	J1732–5049	0.81 ± 0.03	1.56 ± 0.03	-1.62 ± 0.07	2.22	5.08	2.0×10^{-15}
13	J1744–1134	0.85 ± 0.04	1.52 ± 0.03	-1.70 ± 0.09	0.32	5.87	…
14	J1824–2452A	0.70 ± 0.03	1.26 ± 0.05	-1.40 ± 0.07	2.44	5.75	10.0×10^{-15}
15	J1857+0943	0.71 ± 0.05	1.45 ± 0.02	-1.42 ± 0.10	0.84	5.93	…
16	J1909–3744	0.76 ± 0.06	1.32 ± 0.06	-1.52 ± 0.11	0.13	5.75	6.0×10^{-15}
17	J1939+2134	0.80 ± 0.02	1.25 ± 0.02	-1.61 ± 0.04	0.68	5.88	…
18	J2124–3358	0.65 ± 0.07	1.23 ± 0.04	-1.30 ± 0.13	1.90	5.99	6.0×10^{-15}
19	J2129–5721	0.66 ± 0.07	1.54 ± 0.04	-1.32 ± 0.13	0.80	5.86	7.0×10^{-15}
20	J2145–0750	0.69 ± 0.06	1.29 ± 0.05	-1.38 ± 0.11	0.78	5.99	…

Eghdami, I., et al., APJ, 864:162 (18 pp), 2018

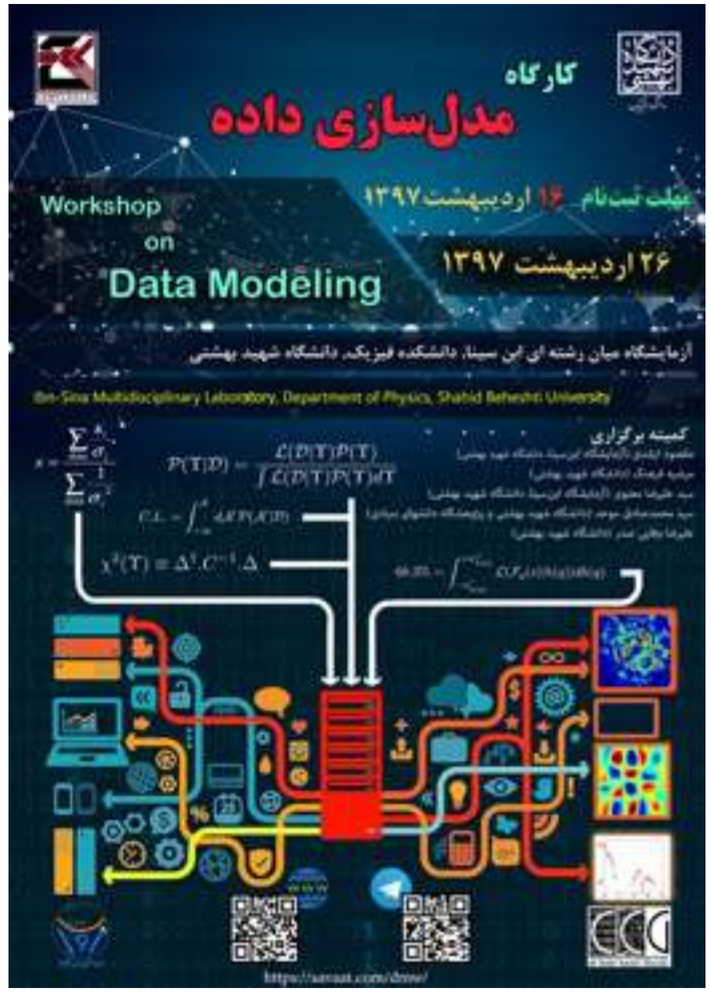
Observed PTR, Parkes Pulsar Timing Array (PPTA)







Eghdami, I., et al., APJ, 864:162 (18 pp), 2018



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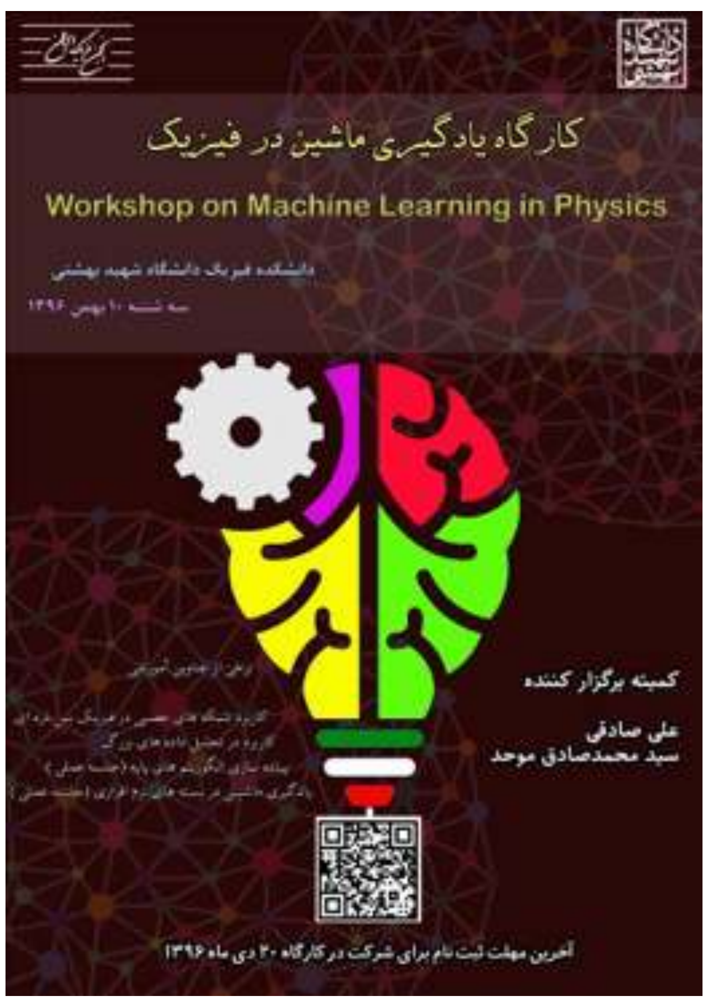

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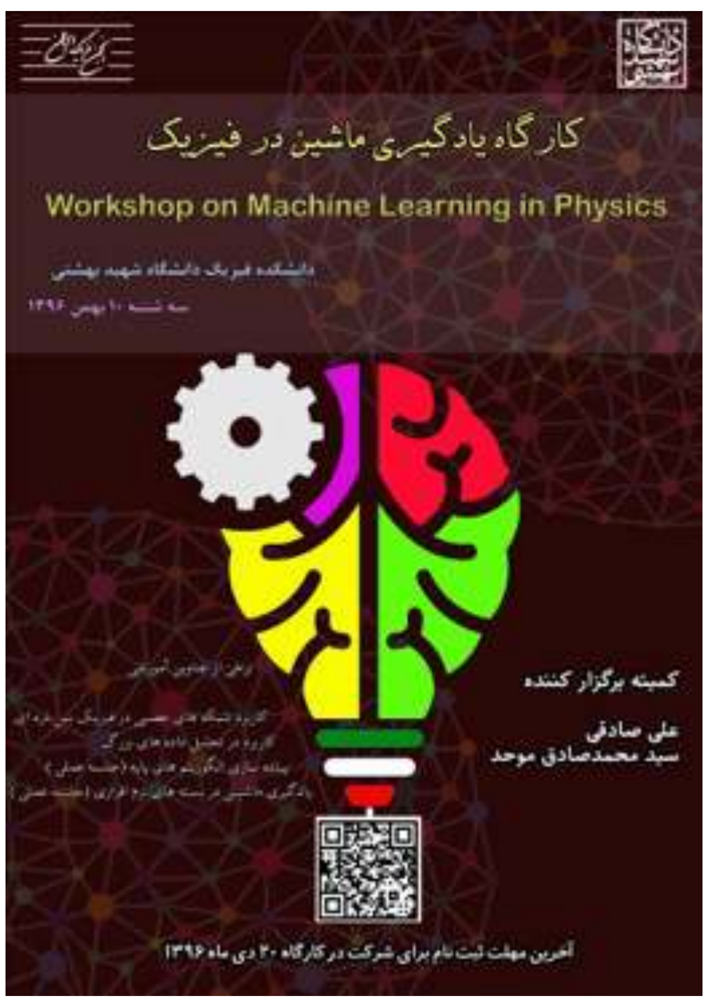

Computational Cosmology Group
Of Shahid Beheshti University

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