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Of Shahid Beheshti University

Geometrical and Topological properties of Stochastic fields: Perturbation Theory Approach

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Outline

1) Stochastic fields in Various fields of researches

2) Perturbative expansion of Statistics

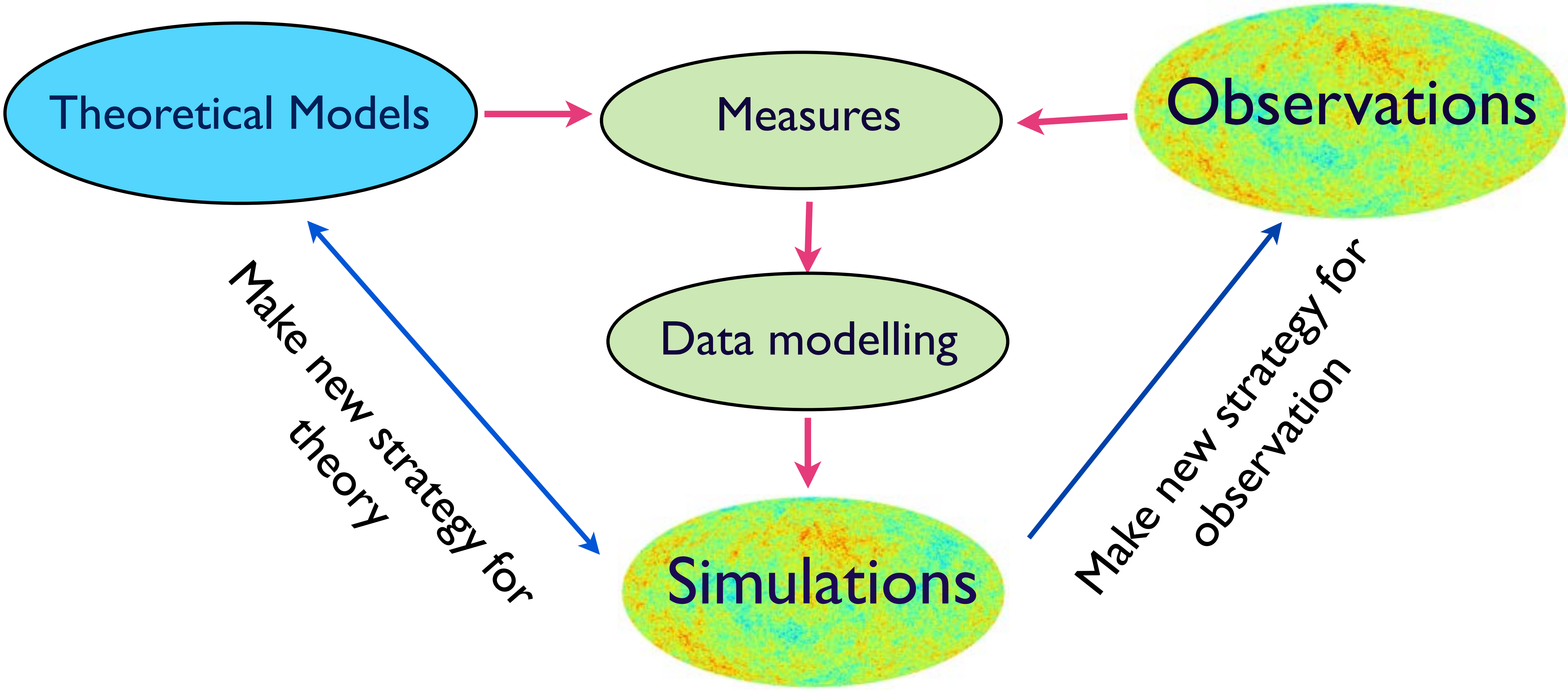
3) Excursion and Critical sets in a 2D stochastic field

4) Clustering of Peaks and Up-crossing

5) Results and researches in progress

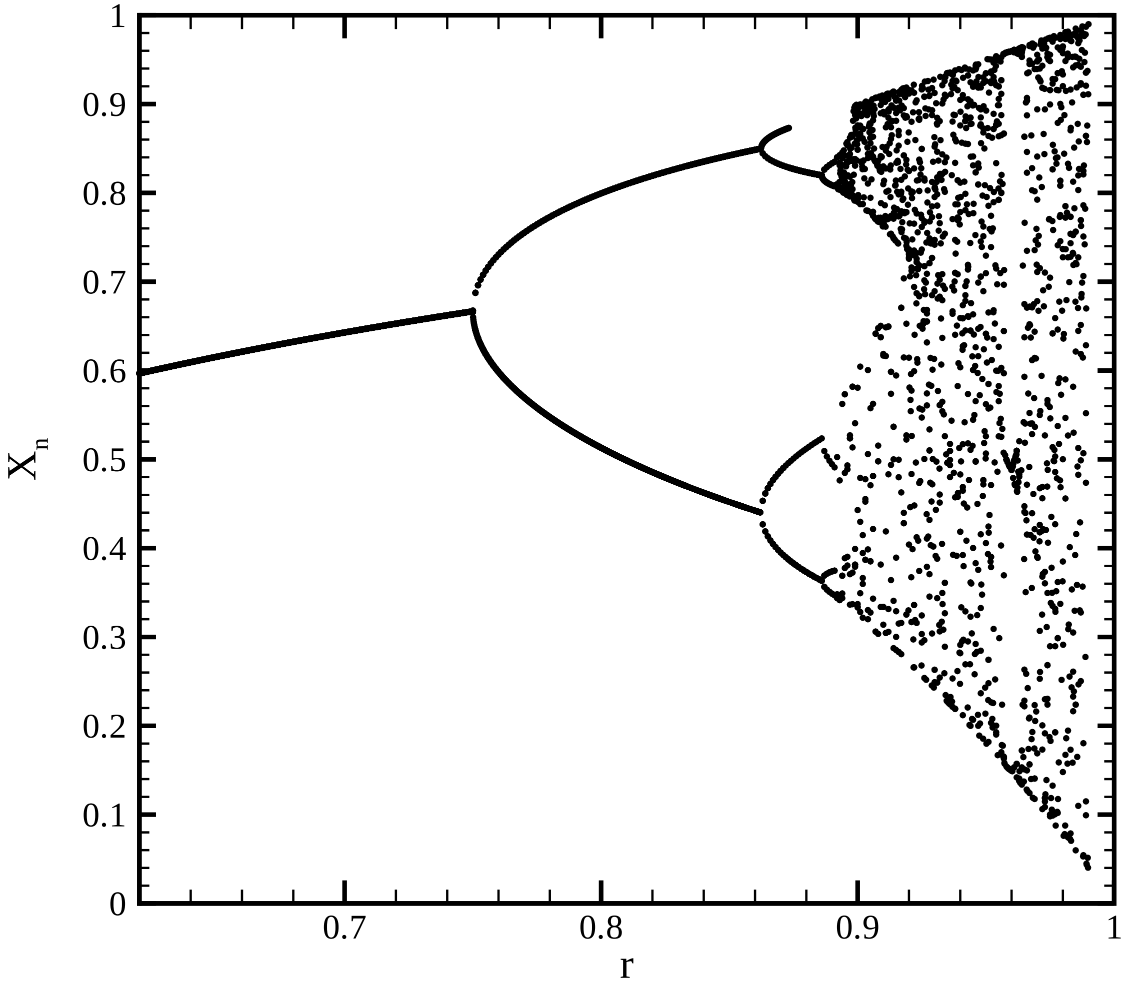
My road map

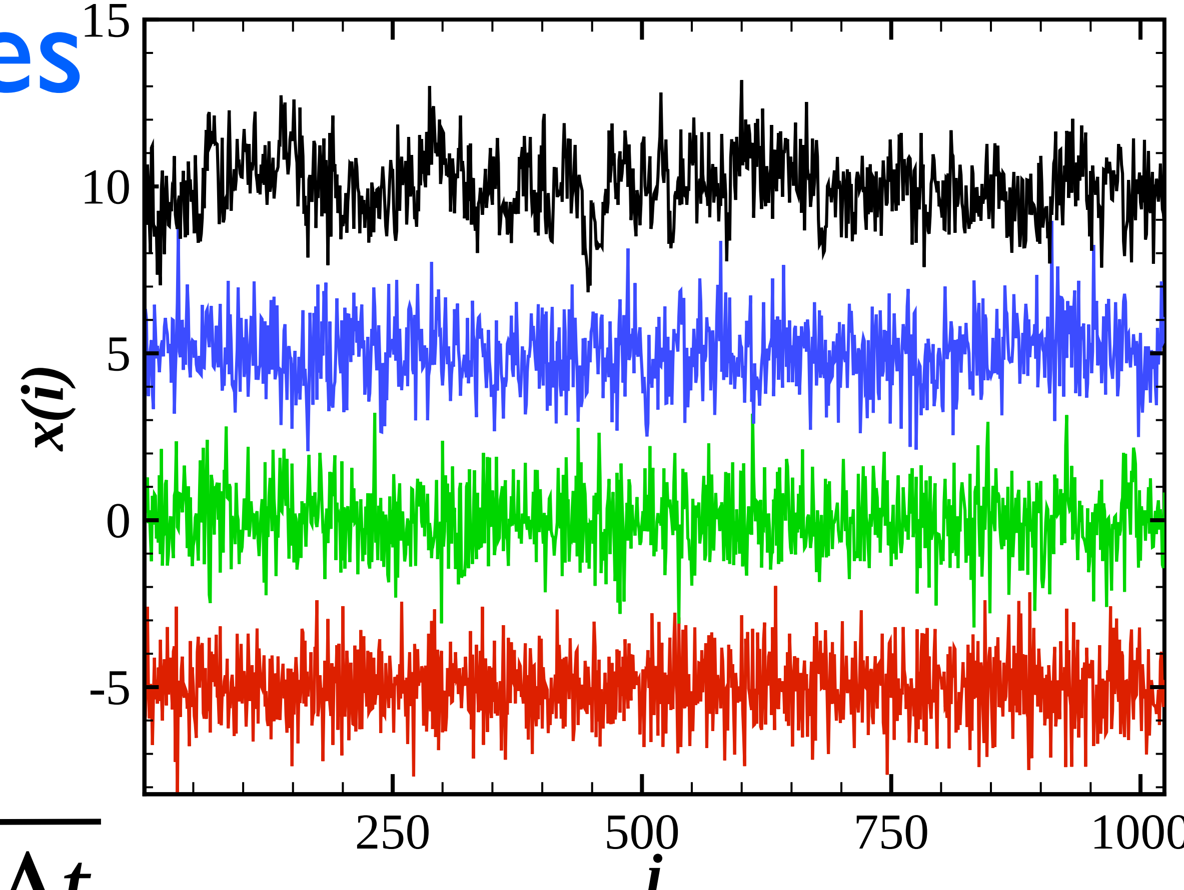
Model - Observation & Experiments - Statistical analysis



Classification of processes

- Deterministic processes
- Chaotic processes
- Stochastic Processes
  1) Purely Random processes
  2) Dependent processes
  3) Markov processes



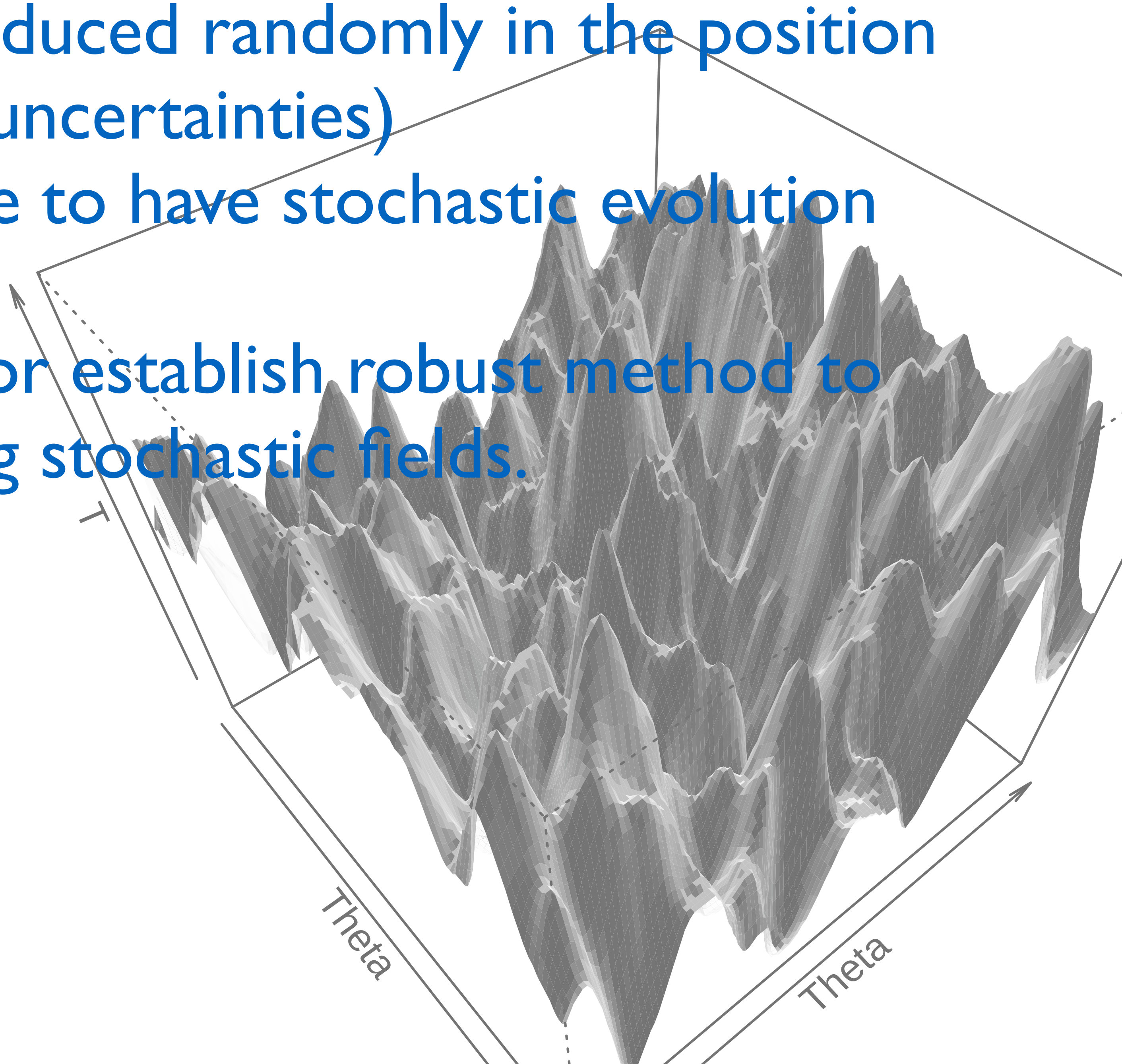


$$x_{n+1} = 4rx_n(1 - x_n)$$

$$x(t + \Delta t) = x(t) - \gamma\, x(t)\Delta t + \eta(t)\sqrt{\Delta t}$$

Why stochastic field?

1) Random fields are ubiquitous in physics.
2) In the nature, there are many reasons to produce initial conditions in the random frameworks,
   e.g. Initial fluctuations are produced randomly in the position coordinate (due to quantum uncertainties)
2) In addition, it could be possible to have stochastic evolution (temporal and/or spatial)
3) Consequently, we should use or establish robust method to find reliable results for underlying stochastic fields.

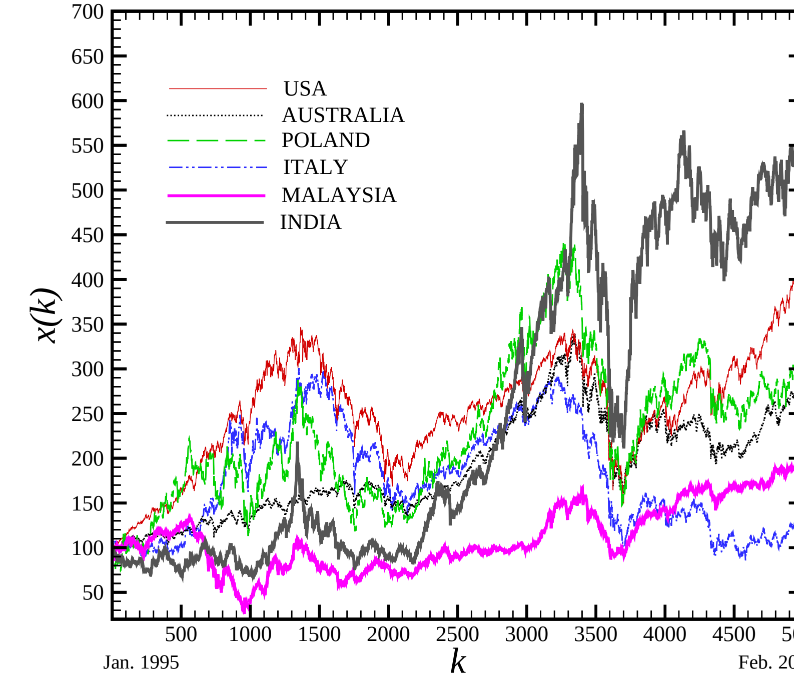


Paulo Ferreira, Andreia Dionsio, S.M.S Movahed (in progress)

Some examples

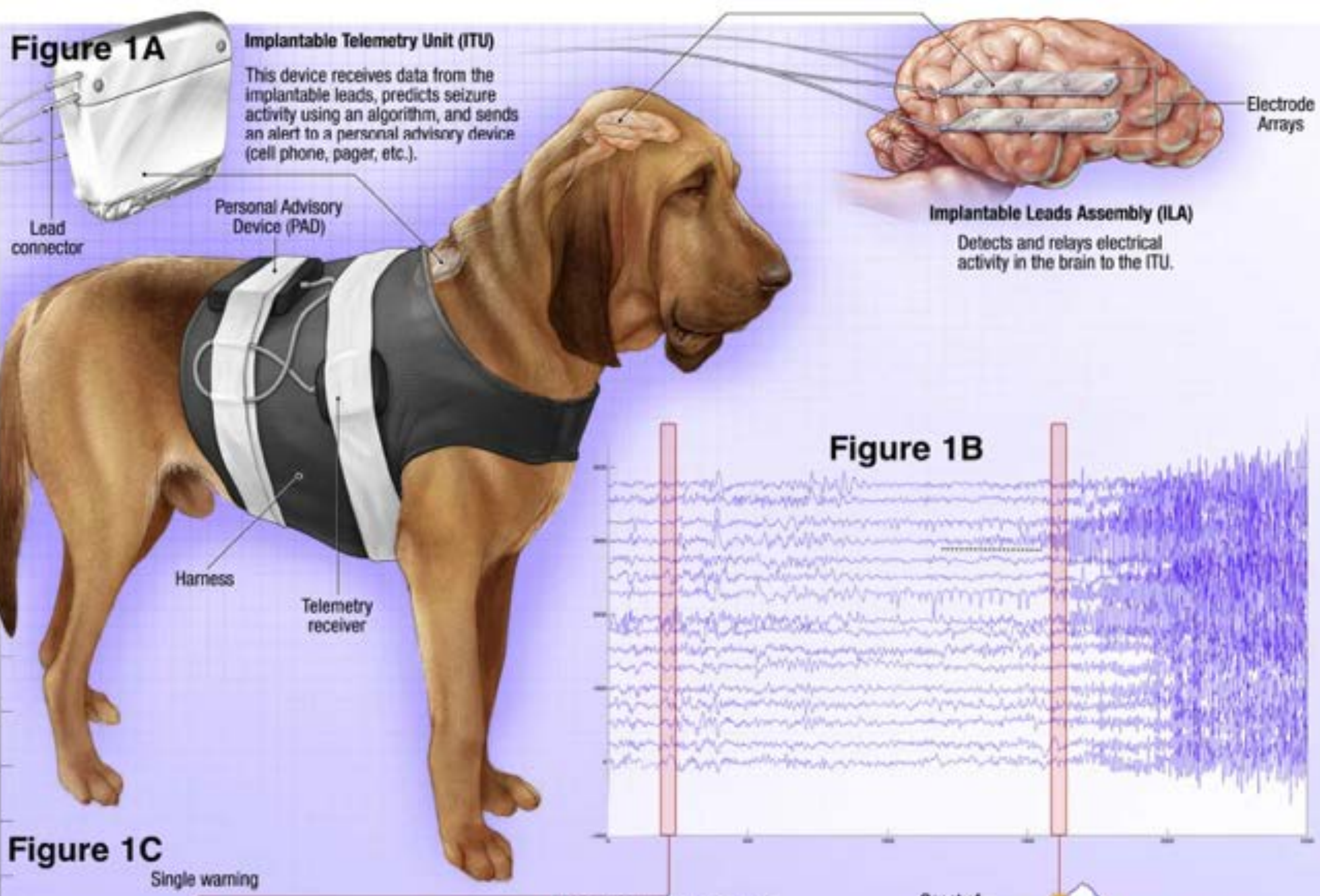
- 1+1D fields

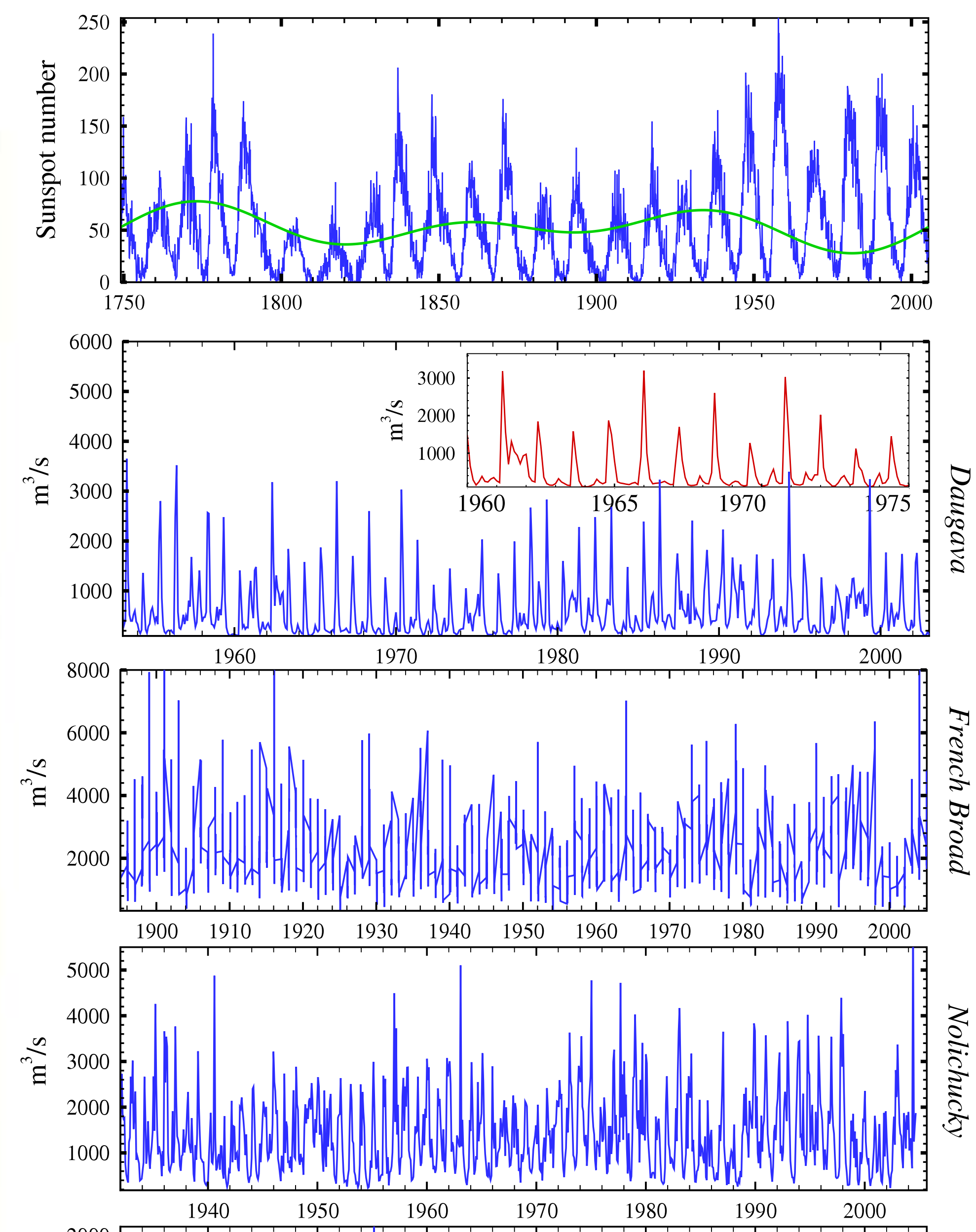
  (Earthquake, Heartbeat, Epilepsy, Stock index, Climate indexes, …)



S. Hajian, M.S. Movahed / Physica A 389 (2010) 4942–4957

https://www.kaggle.com/c/seizure-prediction

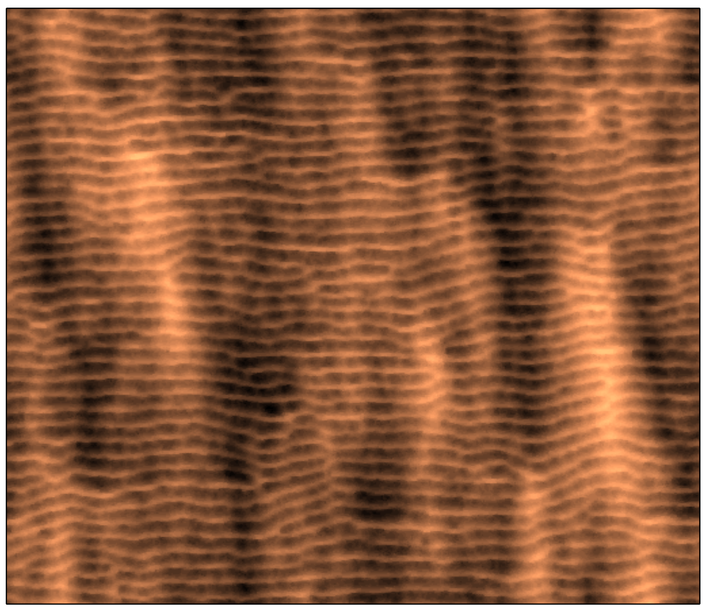




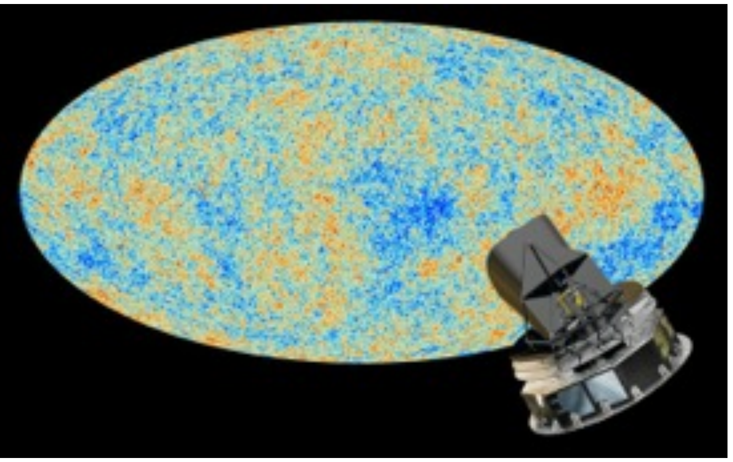
Some examples

- 2D (1+2D) fields
  (CMB, Rough surfaces, …)

Anisotropic surface

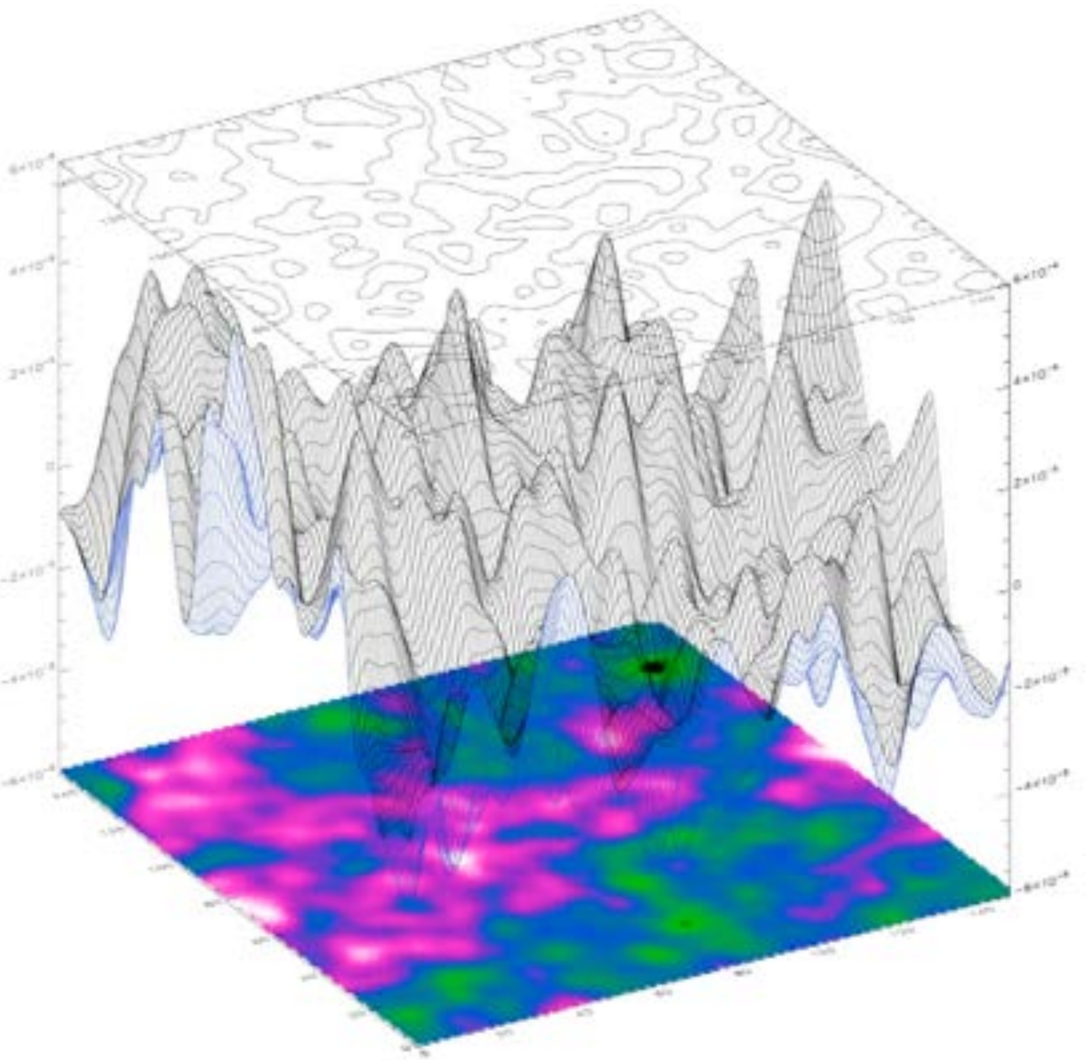


S.M. Vaez Allaei et al., (in progress)



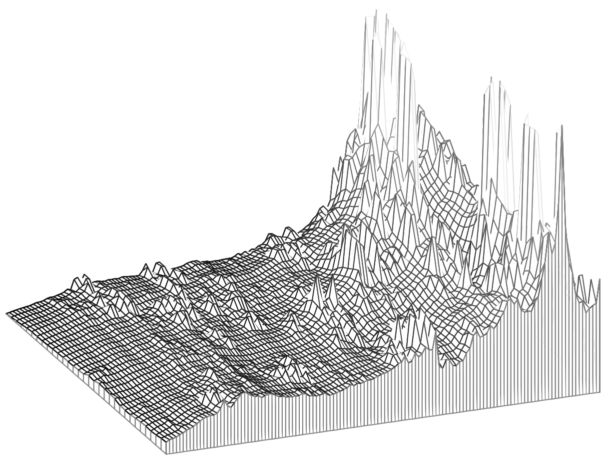
Planck Satellite results (2013)

Gaussian stochastic field
primordial quantum fluctuations

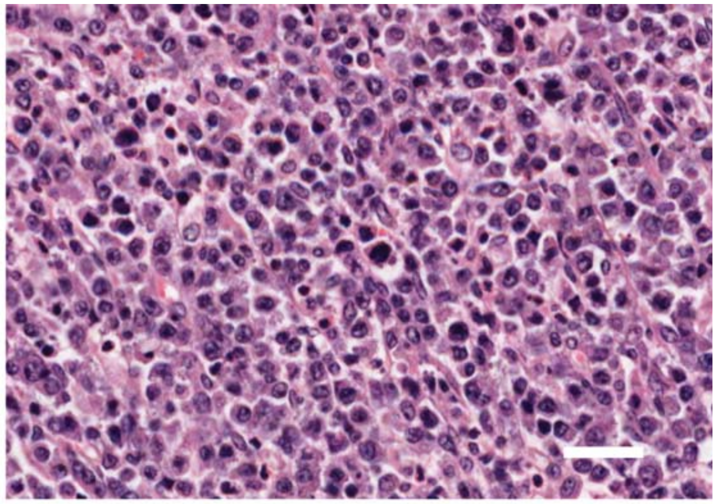


S.M.S. Movahed et. al., MNRAS 2013
S.M.S. Movahed et. al., JCAP 2011

Multifractal singular and
smoothed surfaces



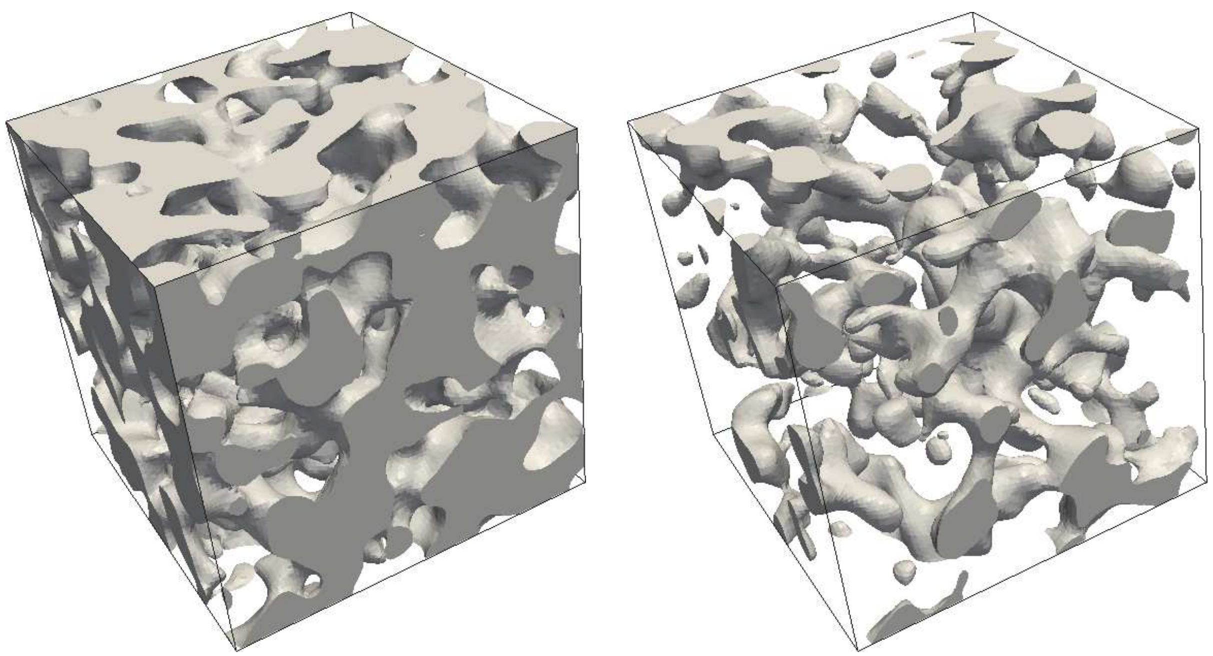
S. Hosseinabadi et al., PRE 2012



Magnetic Resonance in Medicine 71:402–410 (2014)

Some examples

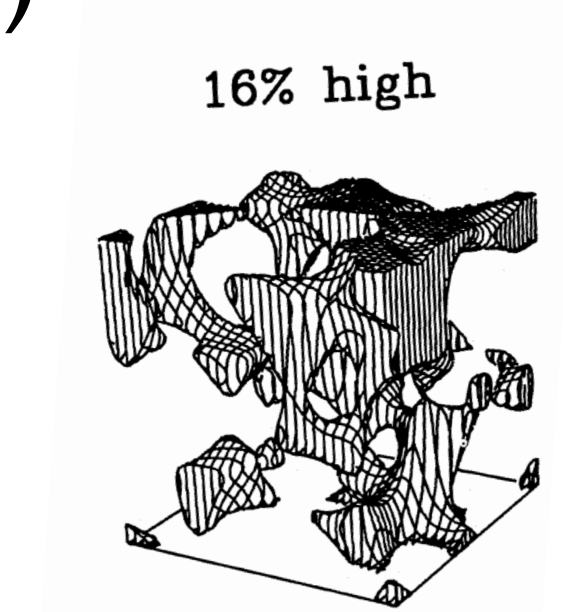
- 3D (1+3D) fields:
  (Large scale structure, …)



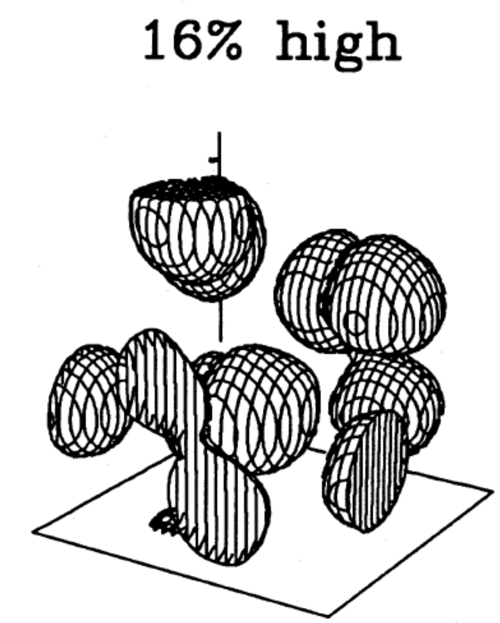
Emmanuel Roubin Thesis, 2013

THE ASTROPHYSICAL JOURNAL, 465:499–514, 1996 July 10

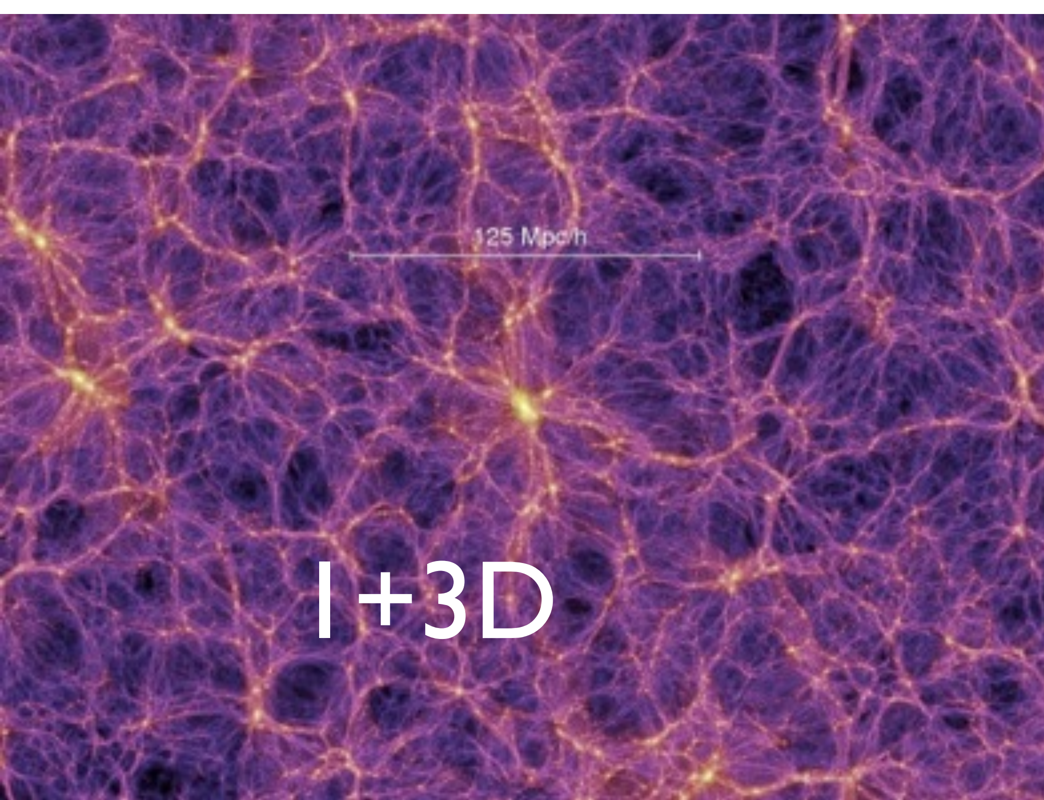
J. RICHARD GOTT III, RENYUE CEN,[1] AND JEREMIAH P. OSTRIKER

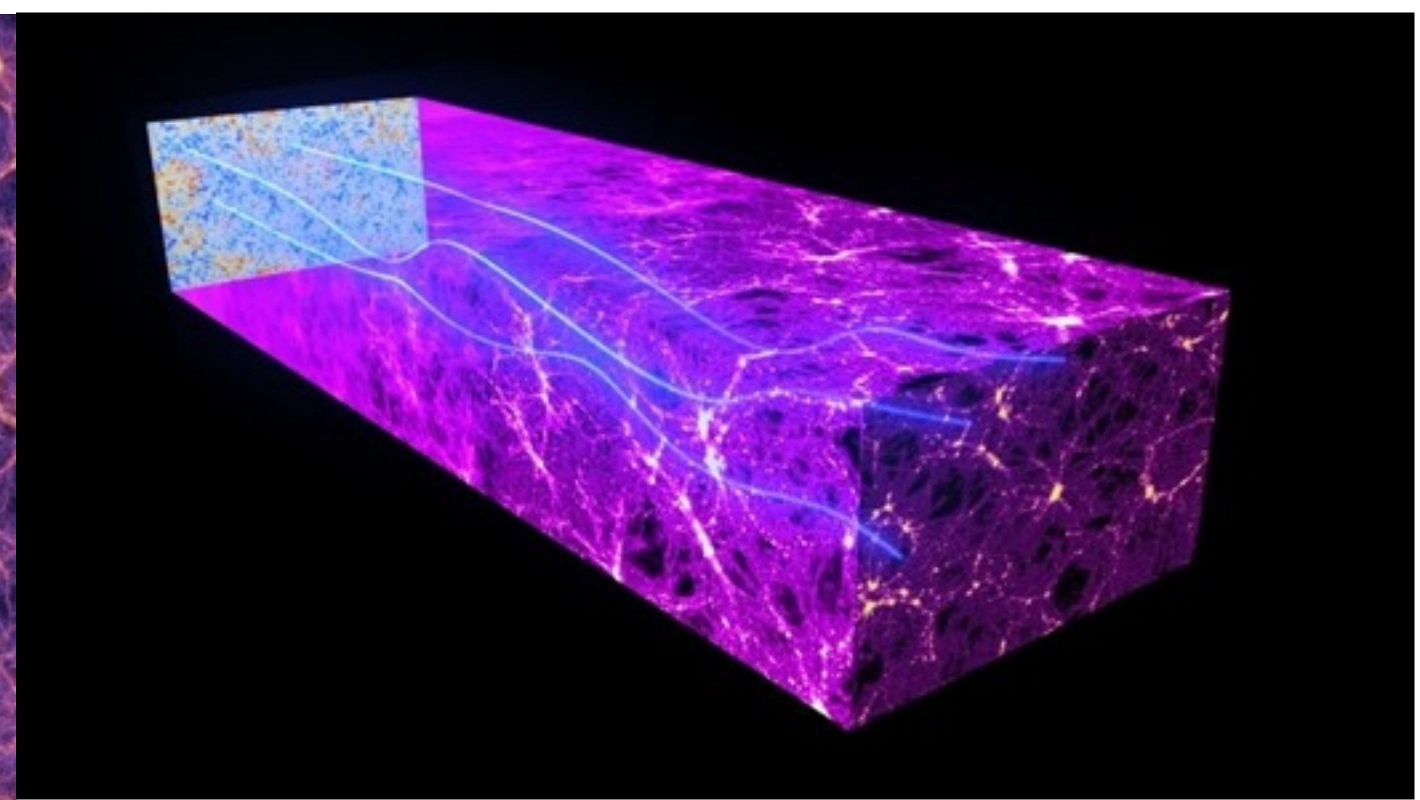


Swiss cheese: isolated voids surrounded on all sides by walls



Meatball: isolated clusters in low density connected background





http://www.esa.int

Cloud structure

http://www.brockmann-consult.de/

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Feature of Stochastic fields by mathematics

Density contrast field 1+4D

$$\delta(t;\bar{X}) \equiv \frac{\rho(t;\bar{X}) - \left\langle \rho(t;\bar{X}) \right\rangle}{\left\langle \rho(t;\bar{X}) \right\rangle}$$

#Velocity contrast field 1+4D

$$\delta\vec{V}(t;\bar{X}) \equiv \frac{\vec{V}(t;\bar{X}) - \left\langle \vec{V}(t;\bar{X}) \right\rangle}{\left\langle \vec{V}(t;\bar{X}) \right\rangle}$$

Gravitational field 1+4D

$$\delta\Phi(t;\bar{X}) \equiv \frac{\Phi(t;\bar{X}) - \left\langle \Phi(t;\bar{X}) \right\rangle}{\left\langle \Phi(t;\bar{X}) \right\rangle}$$

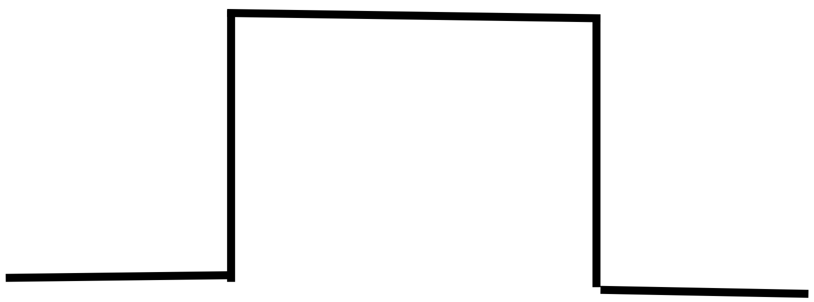
Height or Temperature fields 1+3D and 1+2D

$$\delta T(t;\bar{X}) \equiv \frac{T(t;\bar{X}) - \left\langle T(t;\bar{X}) \right\rangle}{\left\langle T(t;\bar{X}) \right\rangle}$$

Preparing real field: Smoothed stochastic field

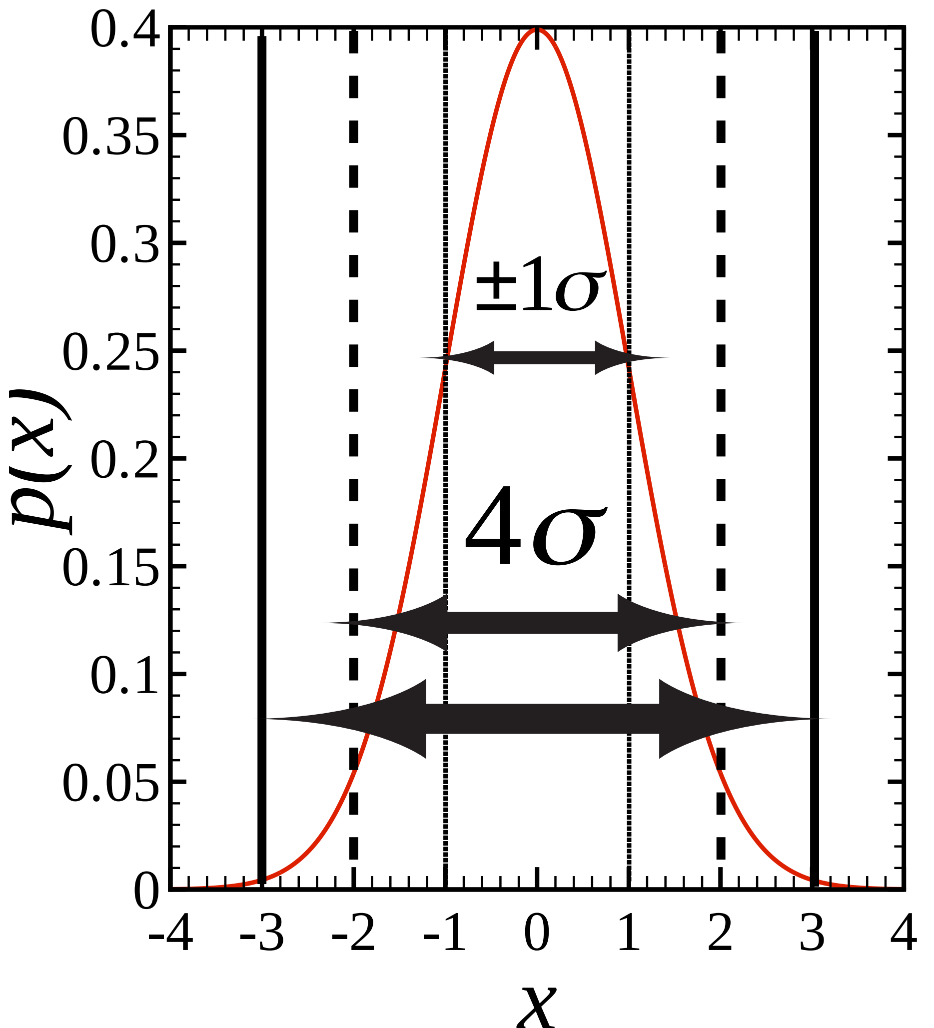
To cuts the high-frequency fluctuations (Low-pass filter)

$$f_{smoothed}(\vec{r}) = \int d^d\vec{r}\,' W_R\left(\left|\vec{r} - \vec{r}\,'\right|\right) f(\vec{r}\,')$$



$$W_R(r) \sim \Theta(R - r)$$

$$W_R(r) \sim \exp\left(-\frac{r^2}{2R^2}\right)$$



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Probability density function of features in an arbitrary smoothed stochastic field

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Perturbative expansion of Statistics I

$$f \to f' \equiv f - \langle f \rangle \quad \to \langle f' \rangle = 0 \qquad \sigma_0^2 = \langle f^2 \rangle = \frac{1}{(2\pi)^{d/2}} \int d^d k P(k) \quad \alpha \equiv \frac{f}{\sigma_0}$$

$$A_{\mu\nu\eta...} = \big(\alpha(r_\mu), \alpha(r_\mu)_{;1}, \alpha(r_\mu)_{;2}, \alpha(r_\mu)_{;3}, \alpha(r_\mu)_{;11}, \alpha(r_\mu)_{;22}, \alpha(r_\mu)_{;33}, \alpha(r_\mu)_{;12}, \alpha(r_\mu)_{;13}, \alpha(r_\mu)_{;23},$$
$$\big(\alpha(r_\nu), \alpha(r_\nu)_{;1}, \alpha(r_\nu)_{;2}, \alpha(r_\nu)_{;3}, \alpha(r_\nu)_{;11}, \alpha(r_\nu)_{;22}, \alpha(r_\nu)_{;33}, \alpha(r_\nu)_{;12}, \alpha(r_\nu)_{;13}, \alpha(r_\nu)_{;23}, ...\big)$$

$$Z_A(\lambda) \equiv \langle \exp(i\lambda \bullet A) \rangle_A = \int_{-\infty}^{+\infty} d^N A \, P(A) \exp(i\lambda \bullet A)$$

$$= 1 + \sum_{n=1} \frac{i^n}{n!} \left(\sum_{\mu_1=1}^{N} \sum_{\mu_2=1}^{N} \cdots \sum_{\mu_a=1}^{N} \sum_{\nu_1=1}^{N} \sum_{\nu_2=1}^{N} \cdots \sum_{\nu_b=1}^{N} M^{(a+b=n)}_{\mu_1\mu_2..\mu_a;\nu_1,\nu_2..\nu_b} \lambda_{\mu_1} \lambda_{\mu_2} \dots \lambda_{\mu_a} \lambda_{\nu_1} \lambda_{\nu_2} \dots \lambda_{\nu_b} \right)$$

Moment

$$\ln(Z_A(\lambda)) = \sum_{n=1} \frac{i^n}{n!} \left(\sum_{\mu_1=1}^{N} \sum_{\mu_2=1}^{N} \cdots \sum_{\mu_a=1}^{N} \sum_{\nu_1=1}^{N} \sum_{\nu_2=1}^{N} \cdots \sum_{\nu_b=1}^{N} K^{(a+b=n)}_{\mu_1\mu_2..\mu_a;\nu_1,\nu_2..\nu_b} \lambda_{\mu_1} \lambda_{\mu_2} \dots \lambda_{\mu_a} \lambda_{\nu_1} \lambda_{\nu_2} \dots \lambda_{\nu_b} \right)$$

Cumulant

Free energy

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Perturbative expansion of Statistics II

$$Z_A(\lambda) \equiv \left\langle \exp\left(i\lambda \bullet A\right)\right\rangle_A = \int_{-\infty}^{+\infty} d^N A \, P(A) \exp\left(i\lambda \bullet A\right)$$

$$Z_A(\lambda) = \exp\left(-\frac{1}{2}\lambda^T \bullet K^{(2)} \bullet \lambda\right)$$

$$\times \sum_{n=3} \frac{i^n}{n!}\left(\sum_{\mu_1=1}^{N}\sum_{\mu_2=1}^{N}\cdots\sum_{\mu_a=1}^{N}\sum_{\nu_1=1}^{N}\sum_{\nu_2=1}^{N}\cdots\sum_{\nu_b=1}^{N} K^{(a+b=n)}_{\mu_1\mu_2..\mu_n;\nu_1,\nu_2..\nu_n}\lambda_{\mu_1}\lambda_{\mu_2}\ldots\lambda_{\mu_a}\lambda_{\nu_1}\lambda_{\nu_2}\ldots\lambda_{\nu_b}\right)$$

$$K^{(2)} \equiv \left\langle A \otimes A \right\rangle$$

$$= \begin{pmatrix} K_{\mu_{11}} & K_{\mu_{12}} & \cdots & K_{\mu_{1n}} & K_{\mu_1\nu_1} & \cdots & K_{\mu_1\nu_n} \\ \vdots & & & & & & \cdots \\ K_{\nu_n\mu_1} & K_{\nu_n\mu_2} & \cdots & K_{\nu_n\mu_n} & K_{\nu_n\nu_1} & \cdots & K_{\nu_n\nu_n} \end{pmatrix}_{2n\times 2n = N\times N}$$

Perturbative expansion of Statistics III

$$Z_A(\lambda) \equiv \langle \exp(i\lambda \bullet A) \rangle_A = \int_{-\infty}^{+\infty} d^N A \, P(A) \exp(i\lambda \bullet A)$$

$$P(\vec{A}) = \frac{1}{(2\pi)^N} \int_{-\infty}^{+\infty} d^N \lambda \, Z_A(\lambda) \exp(-i\lambda \bullet A)$$

$$= \exp\left(\sum_{n=3}^{\infty} \frac{(-1)^n}{n!} \left(\sum_{\mu_1=1}^{N} \sum_{\mu_2=1}^{N} \cdots \sum_{\mu_a=1}^{N} \sum_{\nu_1=1}^{N} \sum_{\nu_2=1}^{N} \cdots \sum_{\nu_b=1}^{N} K^{(a+b=n)}_{\mu_1 \mu_2 .. \mu_a ; \nu_1 , \nu_2 .. \nu_b} \frac{\partial^n}{\partial A_{\mu_1} \ldots \partial A_{\mu_a} \partial A_{\nu_1} \ldots \partial A_{\nu_b}} \right) \right)$$

$$\times P_G(\vec{A})$$

$$P_G(\vec{A}) = \frac{\exp\left(-\frac{1}{2} \vec{A}^T \bullet \left(K^{(2)} \right)^{-1} \bullet \vec{A} \right)}{(2\pi)^{N/2} \sqrt{Det \left| K^{(2)} \right|}}$$

Covariance matrix or The inverse of Fisher information matrix

Perturbative expansion of Statistics IV

$$\langle F\rangle_A = \int_{-\infty}^{+\infty} d^N A\, P(A) F$$

$$\langle F\rangle_A = \left\langle \exp\left(\sum_{n=3}^{\infty} \frac{(-1)^n}{n!} \left(\sum_{\mu_1=1}^{N} \sum_{\mu_2=1}^{N} \cdots \sum_{\mu_a=1}^{N} \sum_{\nu_1=1}^{N} \sum_{\nu_2=1}^{N} \cdots \sum_{\nu_b=1}^{N} K^{(a+b=n)}_{\mu_1\mu_2..\mu_a;\nu_1,\nu_2..\nu_b} \frac{\partial^n}{\partial A_{\mu_1}\ldots\partial A_{\mu_a}\partial A_{\nu_1}\ldots\partial A_{\nu_b}} \right)\right) F \right\rangle_G$$

$$\langle F\rangle_A = \langle F\rangle_G + \frac{1}{3!} \sum_{\mu_1=1}^{N} \sum_{\mu_2=1}^{N} \sum_{\mu_3=1}^{N} K^{(3)}_{\mu_1\mu_2\mu_3} \left\langle F_{;\mu_1\mu_2\mu_3} \right\rangle_G + \ldots.$$

$$F \equiv \delta(\alpha - \beta) \quad \alpha \equiv \frac{f}{\sigma_0}$$

$$P(\alpha) = \int dA_{\mu_2} dA_{\mu_3} \ldots dA_{\mu_N} \delta(\alpha - \beta) P(\vec{A})$$

$$P(f) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{\alpha^2}{2}\right) + \frac{1}{3!} K^{(3)}_{111} \left\langle \frac{\partial^3 \delta(\alpha-\beta)}{\partial \beta^3} \right\rangle_G + \ldots$$

$$P(f) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{\alpha^2}{2}\right) \left[1 + \frac{1}{6} K^{(3)}_{111} H_3(\alpha) + \mathrm{O}(\sigma_0^3)\right]$$

Skewness

Hermit polynomial

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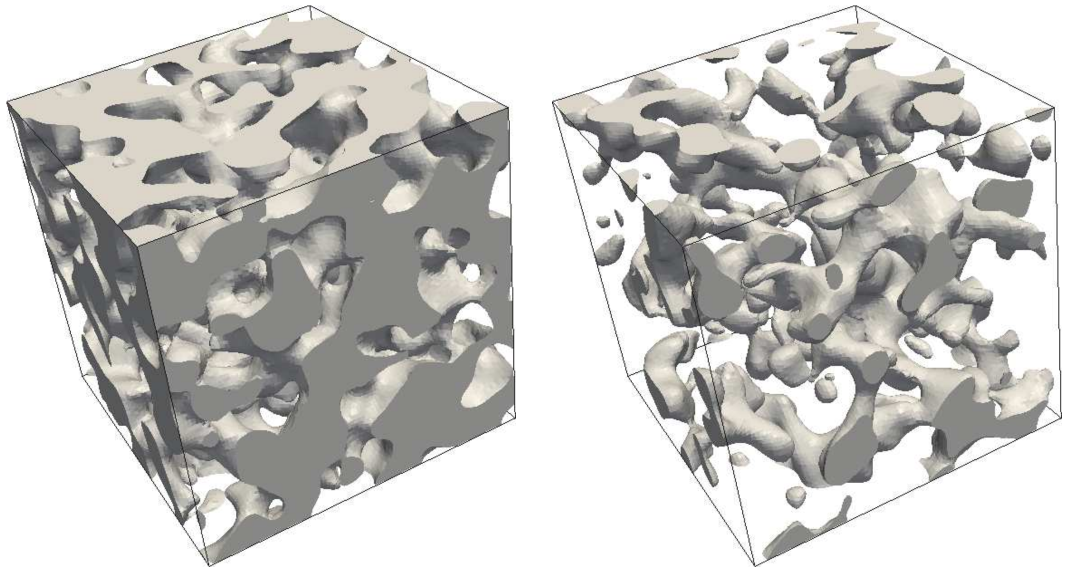
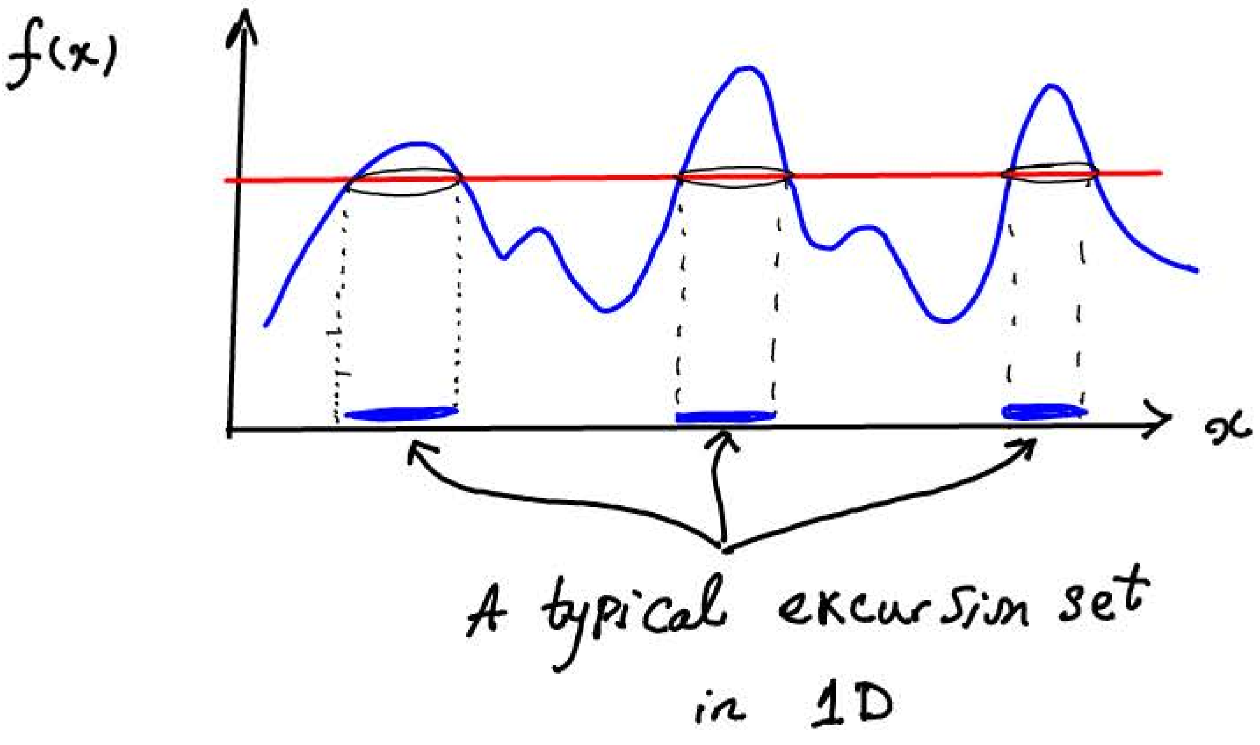
Excursion sets:

In principle, an excursion set is defined as an arbitrary feature considered in an arbitrary condition in the underlying stochastic field



Emmanuel Roubin Thesis, 2013



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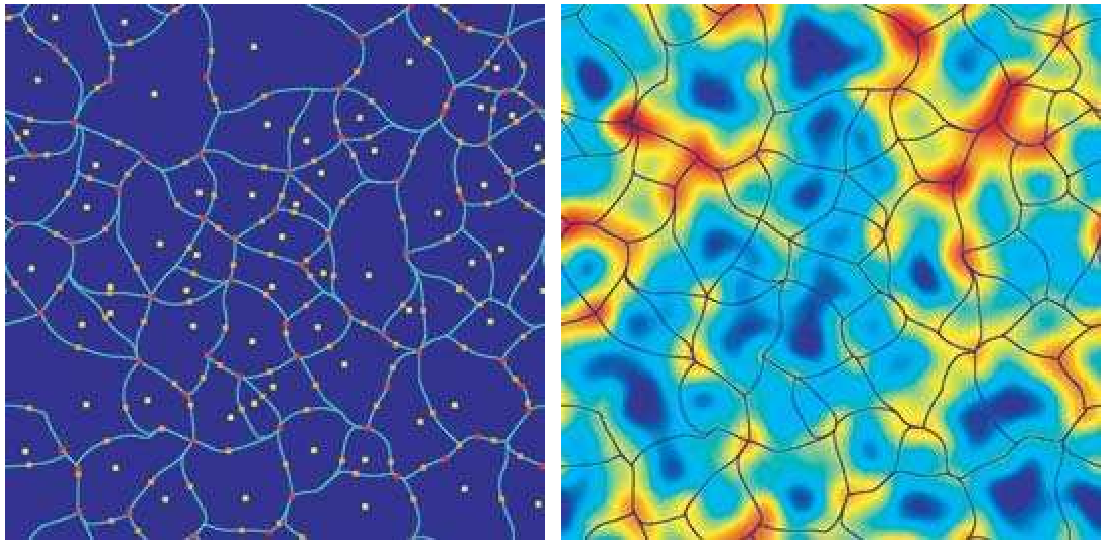
Critical sets

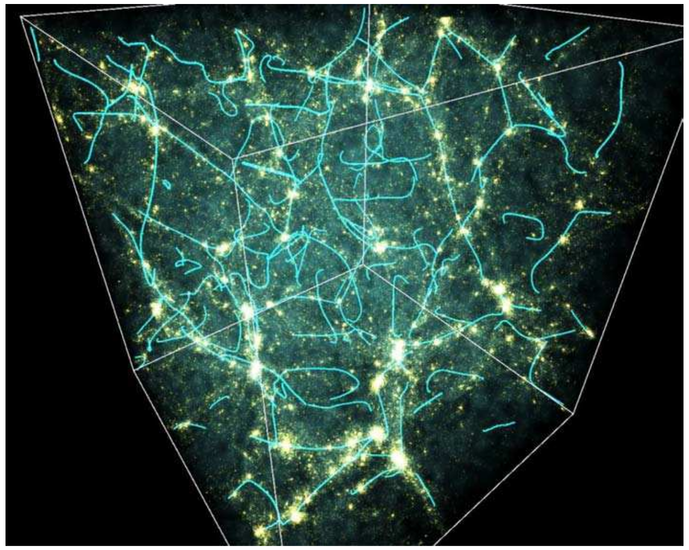
In principle, a critical set is defined as extrema point or path in the underlying stochastic field

Skeleton as a probe of filamentary 2D & 3D

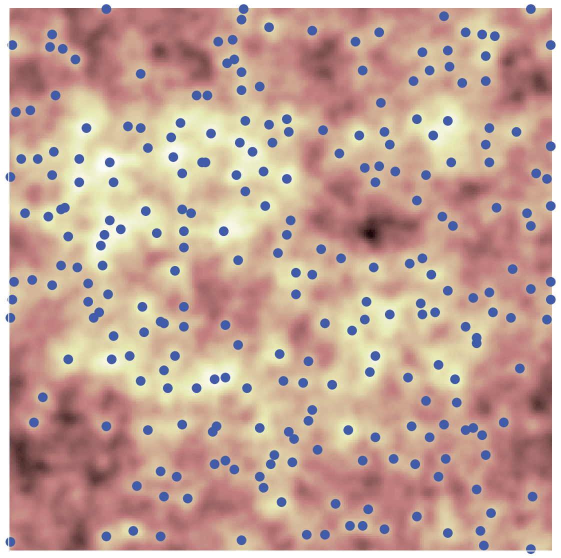
Peaks



Mon. Not. R. Astron. Soc. **366,** 1201–1216 (2006)



Mon. Not. R. Astron. Soc. **383,** 1655–1670 (2008)



S.M.S. Movahed et. al., (2013)

Skeleton is given by the set of points where the gradient is aligned with local curvature major axis and simultaneously, second component of local curvature is negative

Statistical measures for Excursion and Critical sets

- Topological measures

  - Genus # of handles - # of isolated regions ($C \equiv \int k dA = 4\pi(1-g)$)
  - Minkowski Functionals (T. Matsubara et.al. 2013)
  - Euler characteristics # of maxima + # of minima - # of saddle points
  - Gaussian curvature

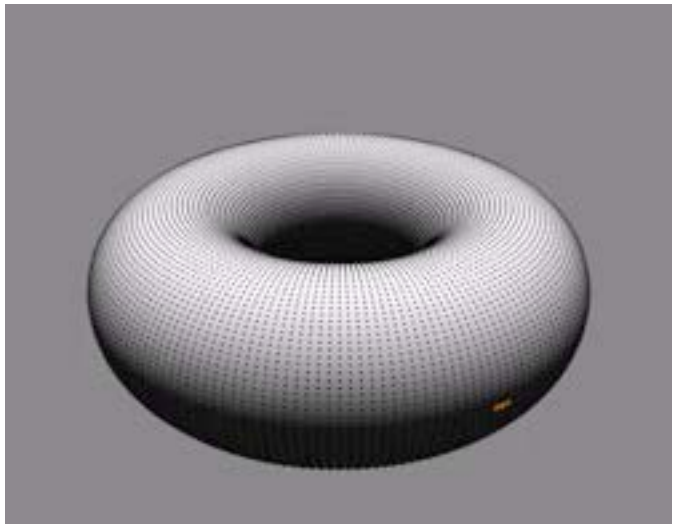
- Geometrical measures

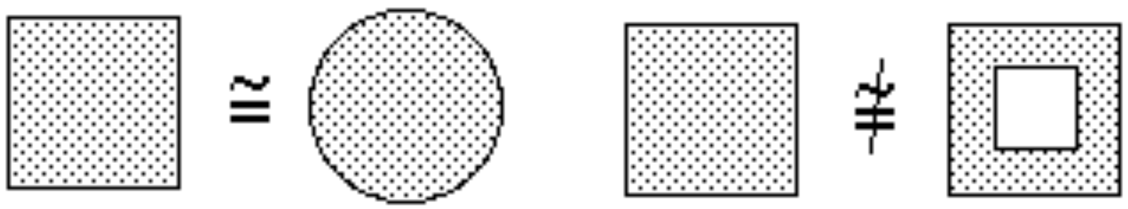
  - Crossing statistics (Riec 1944, 1945, Ryden 1988, Rahimitabar et. al., 2001-2012, S.M.Vaez Allaei et. al., (2014) in progress, S.M.S. Movahed et. al. (2014), in progress)
  - Peaks theory (BBBKS (1986), Matsubara (2003-2013), S.M.S. Movahed, Javanmardi, R.K. Sheth. MNRAS 2013)
  - Skeletons and saddles (Novikov et. al., 2006; Pogosyan et. al. 2012)
  - Contour analysis (Kondev et. al., PRE 2000; A.A Saberi et. al. PRL, 2008; S. Hosseinabadi et.al., PRE 2012)

A brief about Topology

Topology is (roughly) the study of properties invariant under "continuous transformation

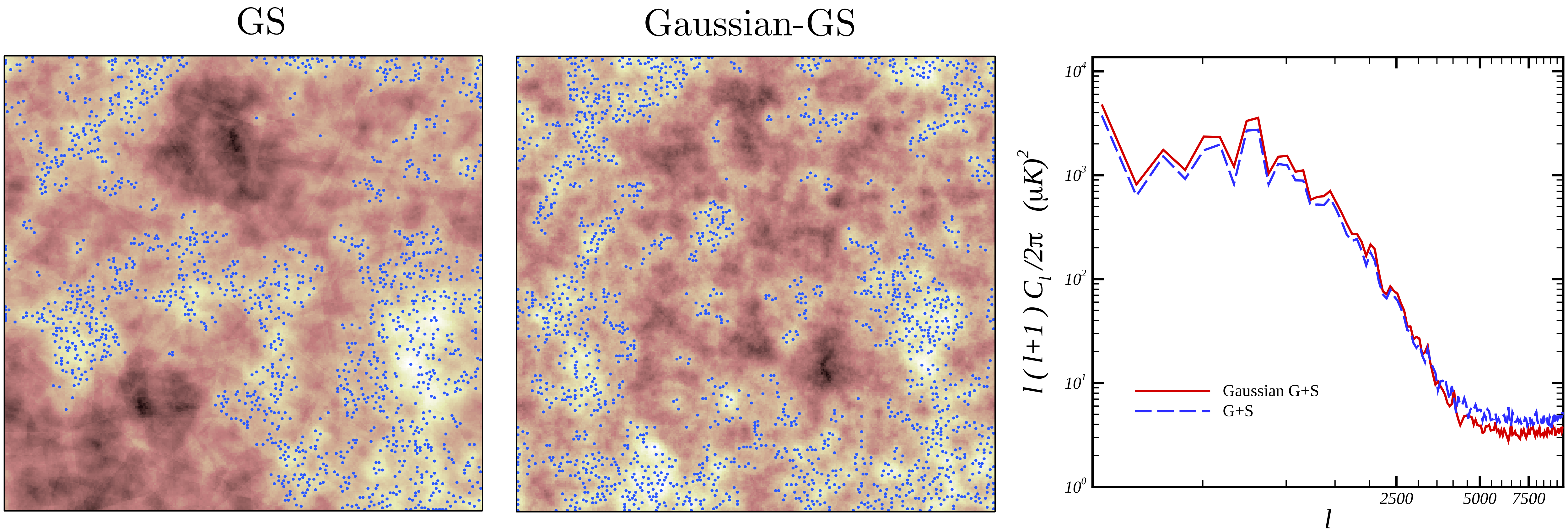
- Two shapes are topologically equivalent if and only if one shape can continuously deform to the other shape.
e.g. Sphere, cube, pyramid are all topologically equivalent.
On the other hands, Sphere and torus are different from topological point of view.





An important motivation:

Both of these fields have same power spectrum
But their textures are completely different



Movahed et.al, MNRAS 2013

Why is topology so important?

To answer to this question let me explain PDF and correlation function
- PDF shows the abundance of features

while

-correlation corresponds to probability of finding features with a condition

To distinguish between various stochastic fields mentioned tools are not enough

Probabilistic frameworks

- Beside the mathematical definition of some criteria, in principle, it is possible to derive them in probabilistic frameworks

- One most relevant motivation for such approaches is that, it facilitates to compare computational and theoretical predictions

Theoretical approach

One-point statistics

$$\langle f \rangle = \langle \text{Conditions correspond to feature} \rangle$$

$$= \langle f \rangle_{\text{Gaussian}} + \text{Perturbative Parts}\Big|_{\text{NG + Anisotropy}}$$

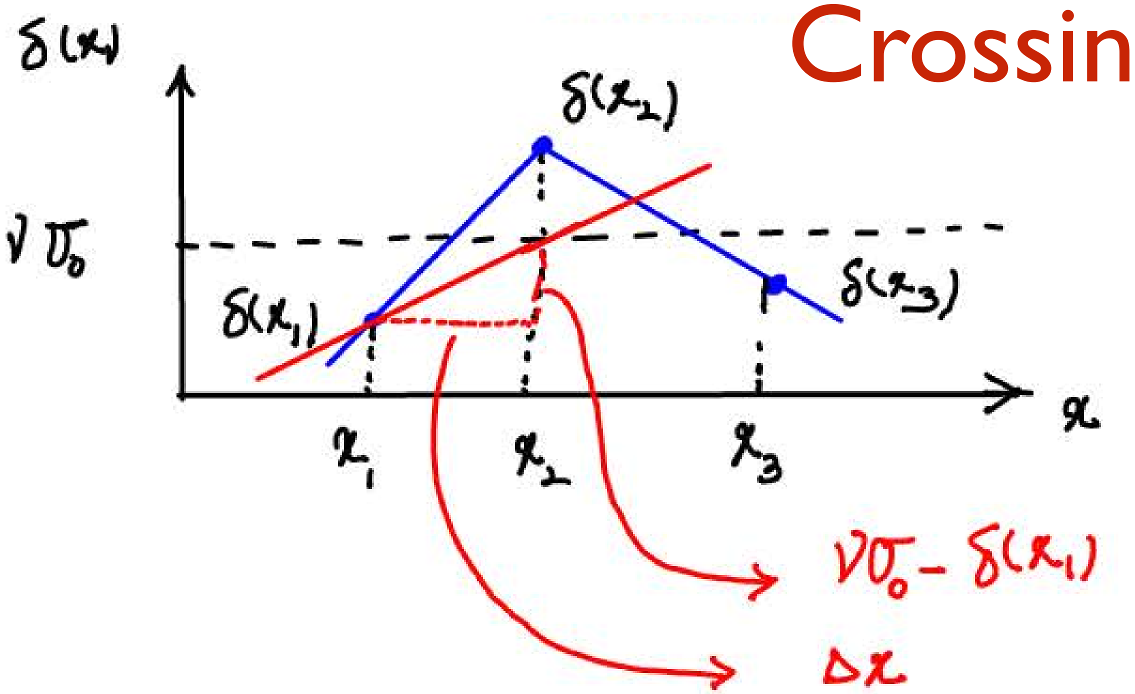
Two-point statistics

$$\langle f(r_1)\, g(r_2) \rangle = \int dA_1\, dA_2\, P(A_1, A_2)\, f(r_1)\, g(r_2)$$

$$P(\vec{A}_1, \vec{A}_2) = \left[\frac{1}{2\pi^N \mathrm{Det}(K)}\right]^{1/2} \exp\left(-\frac{A_1^{\dagger} \cdot K^{-1} \cdot A_2}{2}\right)$$

T. Matsubara, APJ 2003; S. Codis et. al., 1305.7402, Christophe Gay et. al., PRD 2012

Crossing from Mathematics



① $\delta(x_1) < \nu\sigma_0$

② $\frac{\nu\sigma_0 - \delta(x_1)}{\Delta x} < \eta_x \Rightarrow \delta(x_1) > \nu\sigma_0 - |\eta_x|\Delta x$

$$\Delta x\, N_1(\nu) = \lim_{\Delta x \to 0} \int d\eta_x \int_{\nu\sigma_0 - |\eta_x|}^{\nu\sigma_0} d\delta\; P(\eta_x, \delta)$$

1D= no. of crossing

$$= \int d\eta_x\; \Delta x\, |\eta_x|\, P(\eta_x, \delta = \nu\sigma_0) = \langle \delta_D(\alpha - \nu)\, |\eta_1|\, \Theta(\eta_1) \rangle$$

2D= mean length of iso-density contour

$$N_2(\nu) = \int d\eta_x d\eta_y\, (\eta_x^2 + \eta_y^2)^{1/2}\, P(\vec{\eta}, \nu\sigma_0) = \langle \delta_D(\alpha - \nu)\, |\eta_x^2 + \eta_y^2|^{1/2} \rangle$$

3D= mean surface of iso-density region

$$N_3(\nu) = \int d\eta_x d\eta_y d\eta_z\, |\vec{\eta}|\, P(\vec{\eta}, \nu\sigma_0) = \langle \delta_D(\alpha - \nu)\, |\eta_x^2 + \eta_y^2 + \eta_z^2|^{1/2} \rangle$$

Homogeneous and Isotropic field

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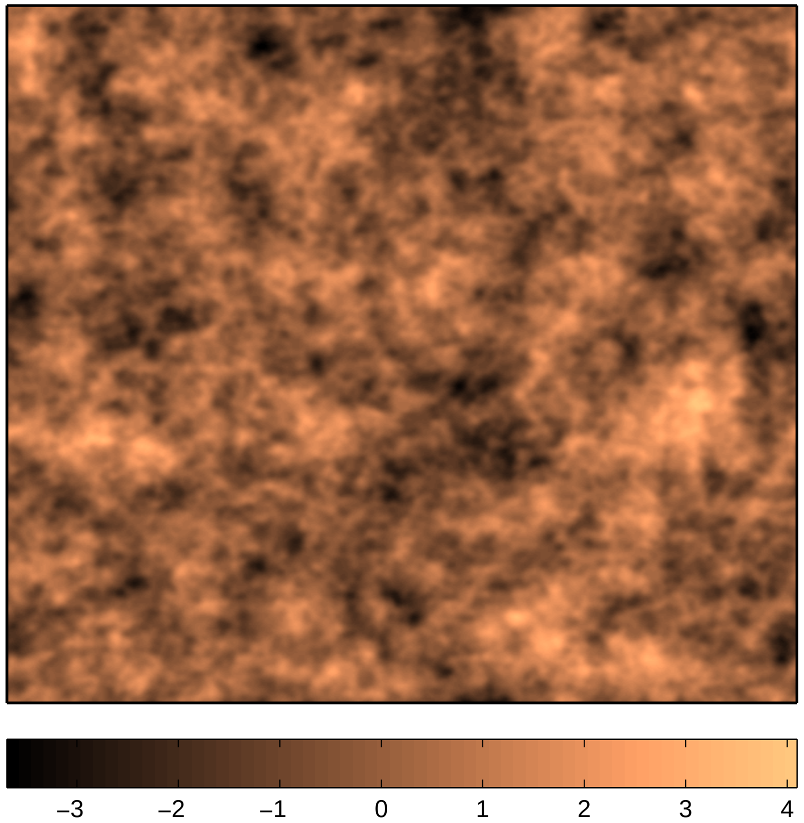
$$N_3(\nu) = \langle|\vec{\eta}|\rangle P_G(\nu\sigma_0) = \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\nu^2/2} \langle|\vec{\eta}|\rangle_{\nu\sigma_0}$$

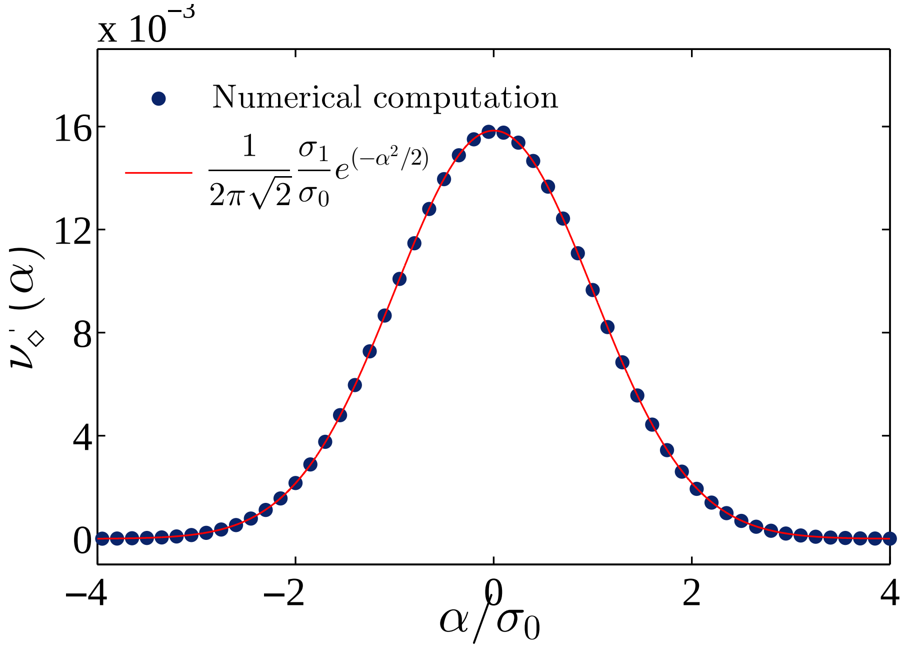
$$N_3(\nu) = \frac{4}{\pi} N_2(\nu) = 2 N_1(\nu)$$

$$\langle|\vec{\eta}|\rangle = 2\left(\frac{2}{3\pi}\right)^{1/2} \sigma_1$$

$$\sigma_0^2 \equiv \langle \delta^2(r) \rangle = \frac{L^D}{(2\pi)^D} \int d^D k \, S^D(\vec{k})$$

$$\sigma_n^2 \equiv \left\langle \left(\frac{\partial^n \delta}{\partial x^n}\right)^2 \right\rangle = \frac{L^D}{(2\pi)^D} \int d^D k \, k^{2n} S^D(\vec{k})$$





S.M.Vaez Alaei e. al., (in progress)

Application for anisotropic surface

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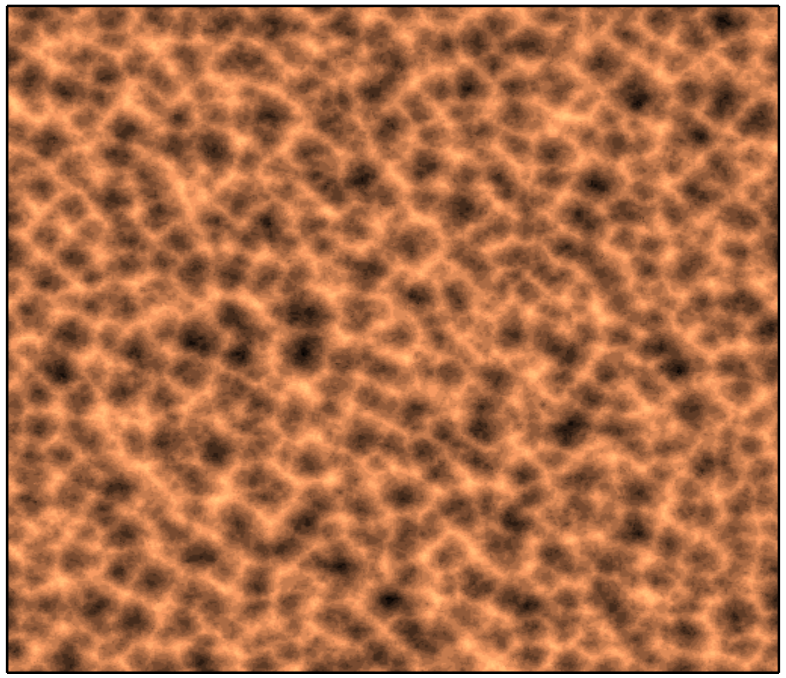
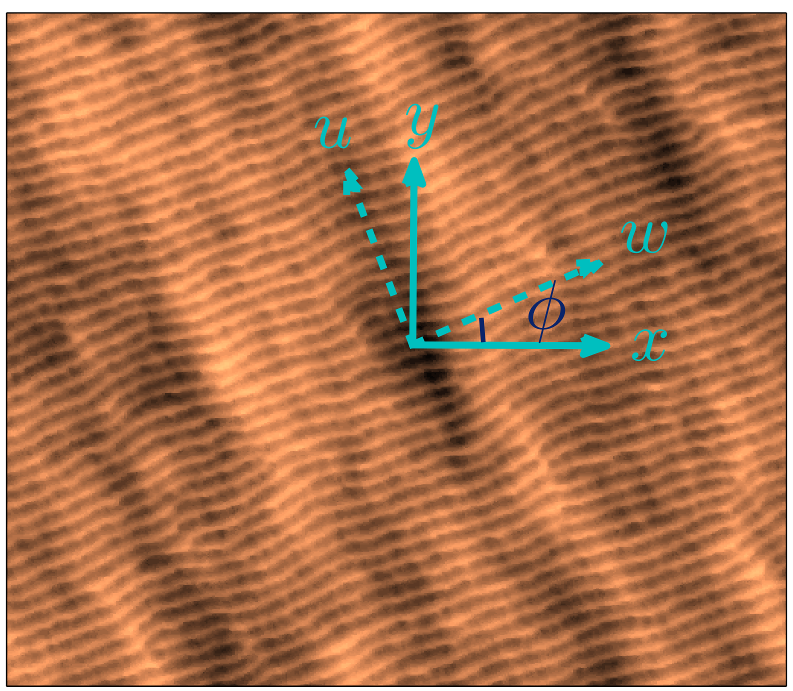
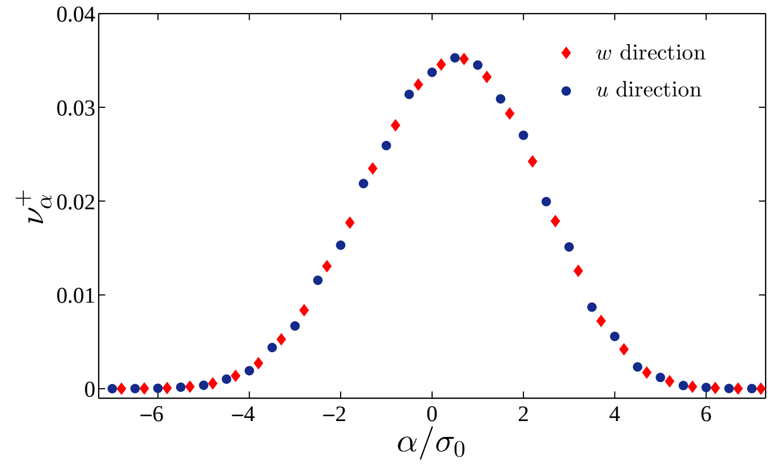
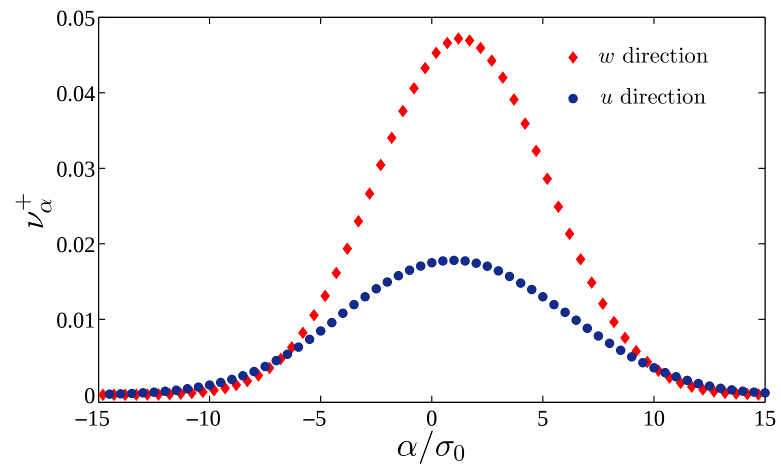


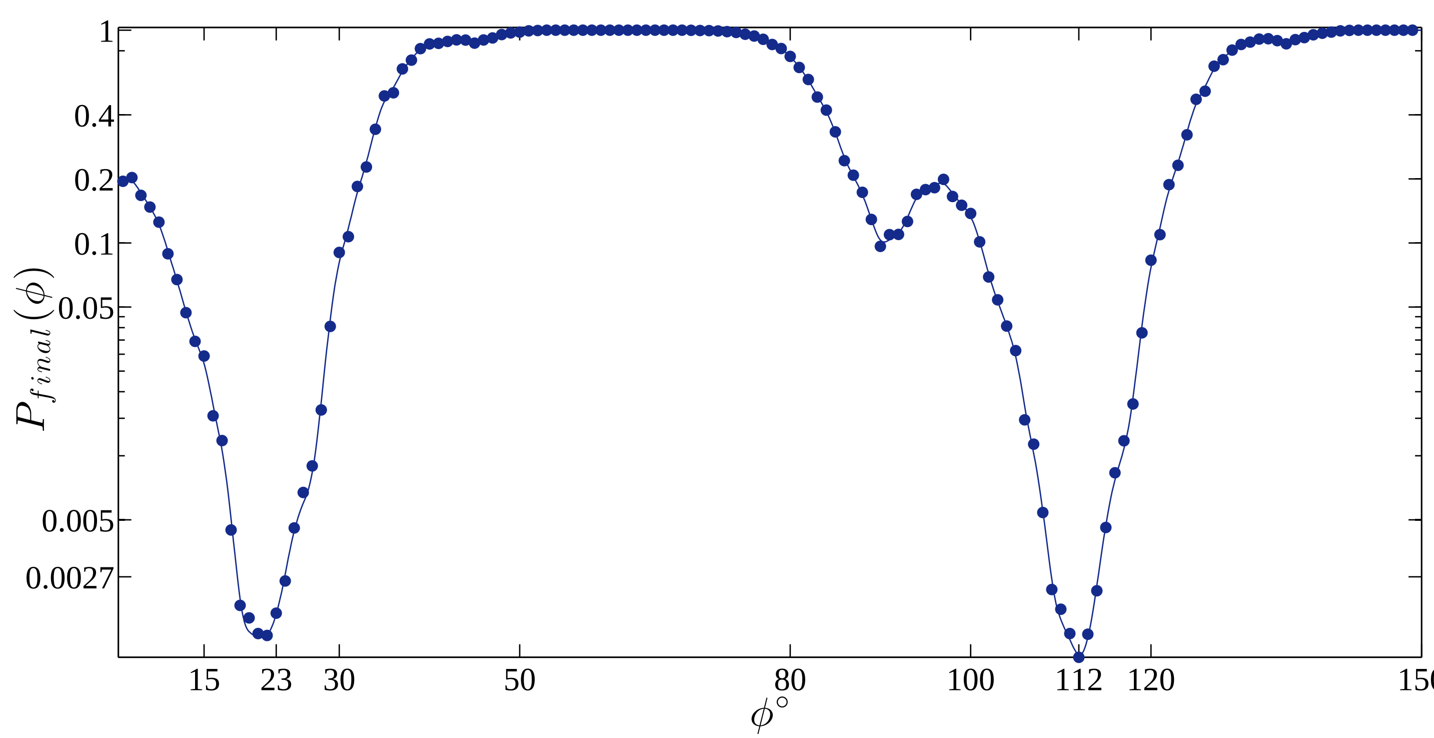






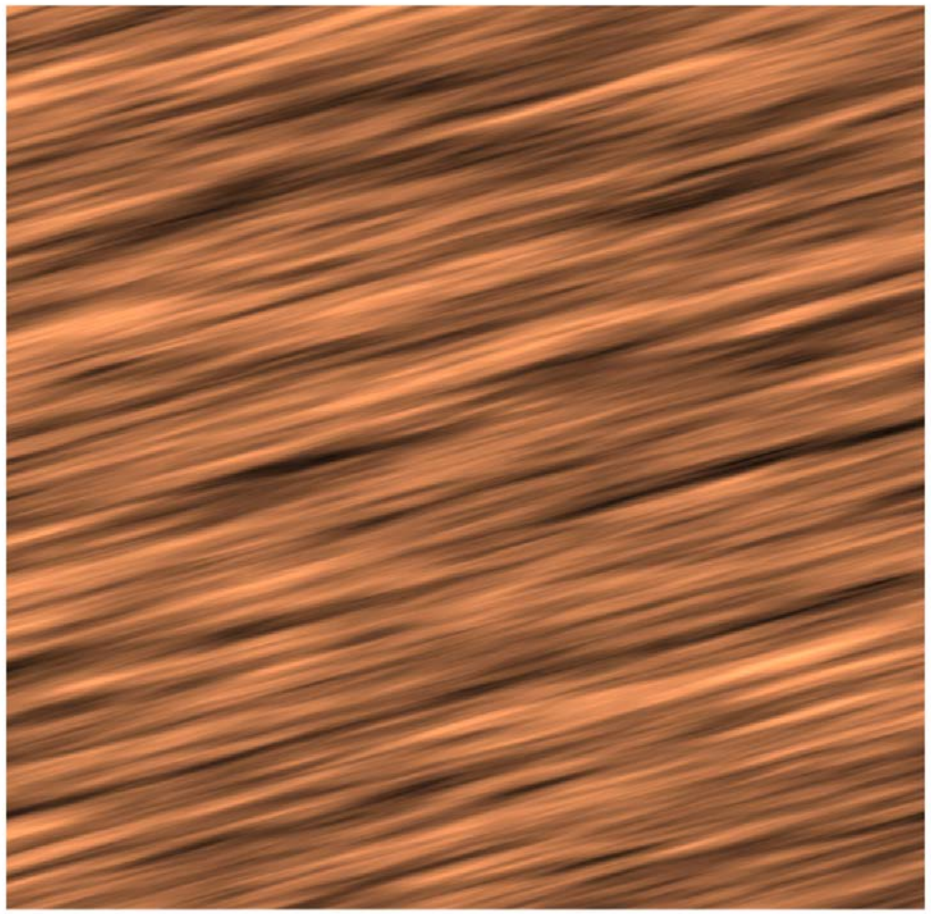
$$t(\phi, q) = [N^u_{tot}(\phi, q) - N^w_{tot}(\phi, q)] \times \sqrt{\frac{N_{run}}{\sigma^2_u(\phi, q) + \sigma^2_w(\phi, q)}}$$

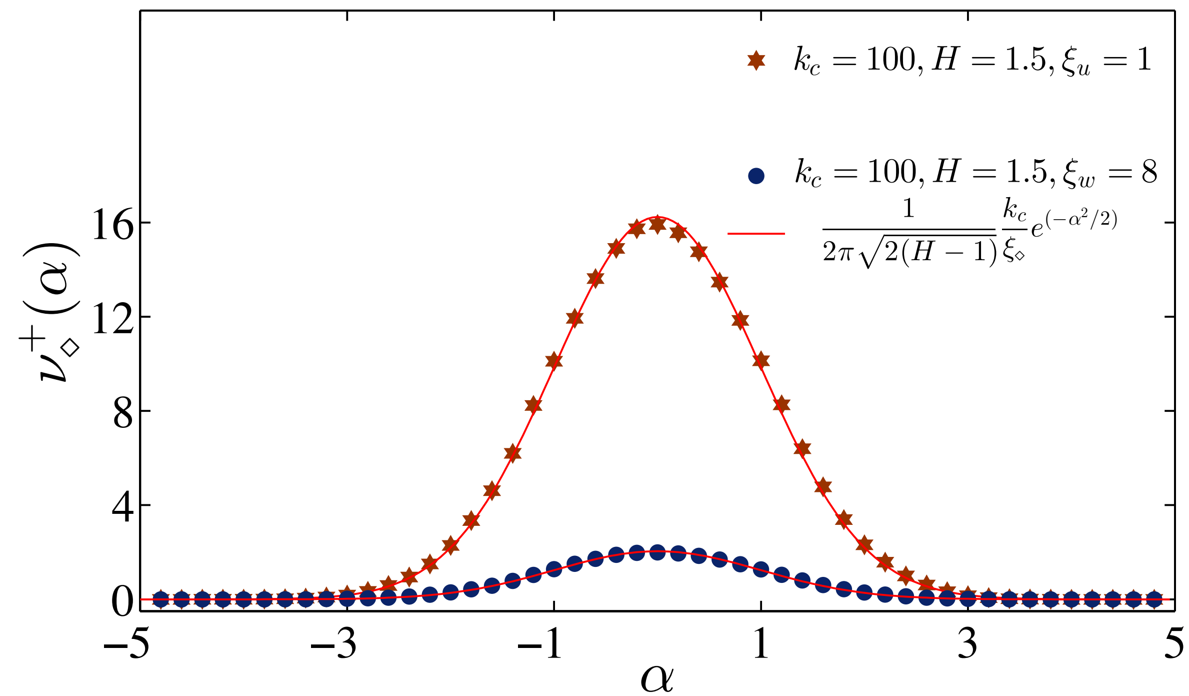
$$P_{final}(\phi) = 1 - \frac{1}{2^{\mu/2}\Gamma(\mu/2)} \int_0^{\chi^2(\phi)} e^{-x/2} x^{\mu/2-1} dx$$

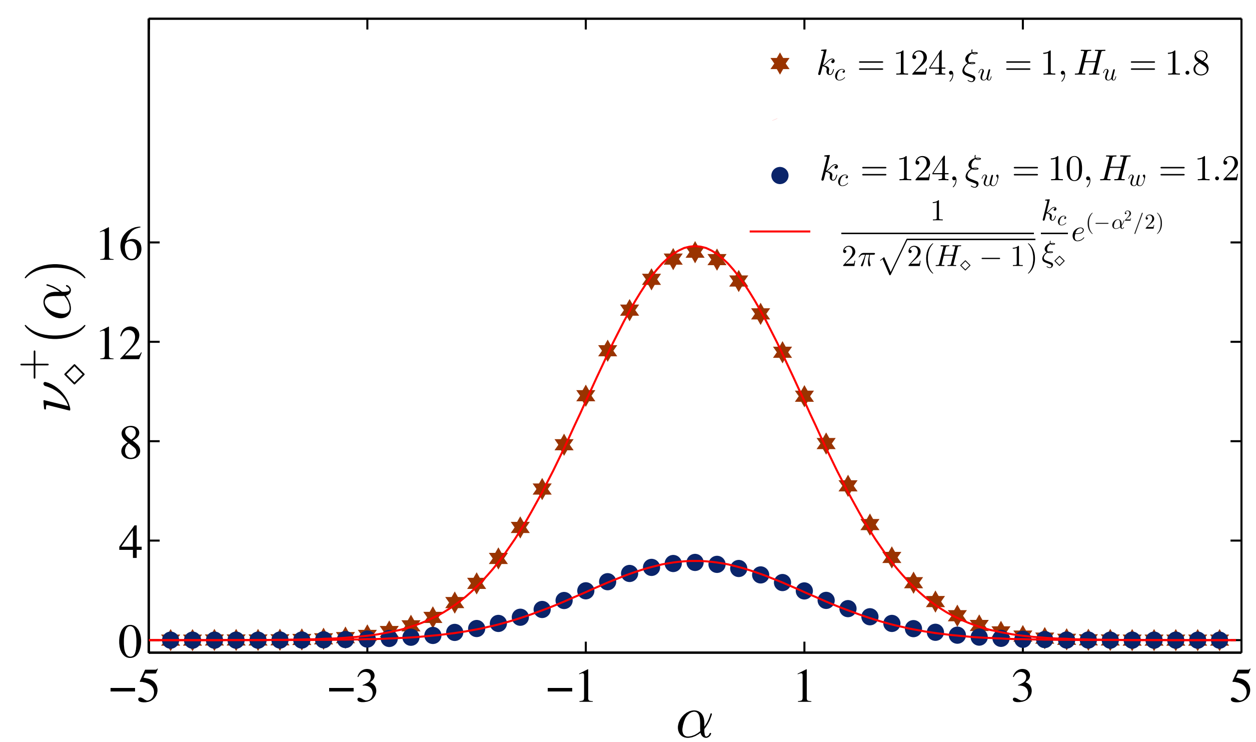


S.M.Vaez Alaei e. al., (in preparing)

Application for anisotropic surface







Model1: correlation length anisotropy

Model 2: Scaling anisotropy

S.M. Vaez Alaei e. al., (in preparing)

Perturbative parts in D-dimension Isotropic field

$$N_1(\nu) = \left\langle \delta_D(\alpha-\nu)\,|\eta_1|\,\Theta(\eta_1) \right\rangle$$

$$= N_1^{G}(\nu) + \text{Perturbative Parts}$$

$$= \frac{1}{\pi}\frac{\sigma_1}{\sqrt{D}\,\sigma_0}\, e^{-\nu^2/2} + N_1^{NG}(\nu)$$

$$= N_1^{G}(\nu)\left[1 + A\sigma_0 + B\sigma_0^2 + \mathcal{O}(\sigma_0^3)\right]$$

$$A = \frac{S}{6} H_3(\nu) + \frac{S^{(1)}}{3} H_1(\nu)$$

$$B = \frac{1}{24}\left(K - S S^{(1)}\right) H_4(\nu) - H_2(\nu)\left(\frac{1}{12}K^{(1)} + \frac{1}{96}S^{(1)2}\right) + \frac{1}{72} S^2 H_6(\nu) + \frac{1}{8}\left(-K^{(3)}\right)$$

$$S = \frac{\langle \alpha^3 \rangle}{\sigma_0}, \quad S^{(1)} = -\frac{3}{4}\frac{\langle \alpha^2 \nabla^2 \alpha \rangle}{\sigma_0^2 \sigma_1^2}, \quad K^{(3)} = \frac{\langle \nabla f^4 \rangle}{2\sigma_0^2 \sigma_1^4}$$

see S. Bazmi's poster

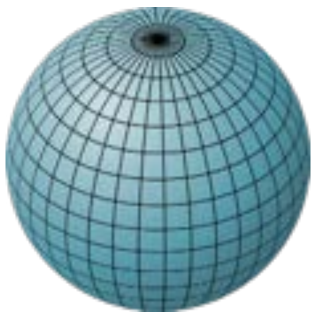
Genus from Mathematics

1) In principle Genus is

G3= # of handles to the surface - # of holes enclosed by the surface

G3= # of handles of contours - # of isolated contours

G2= # of contours around higher dense region - # of contours around lower dense region



G=0

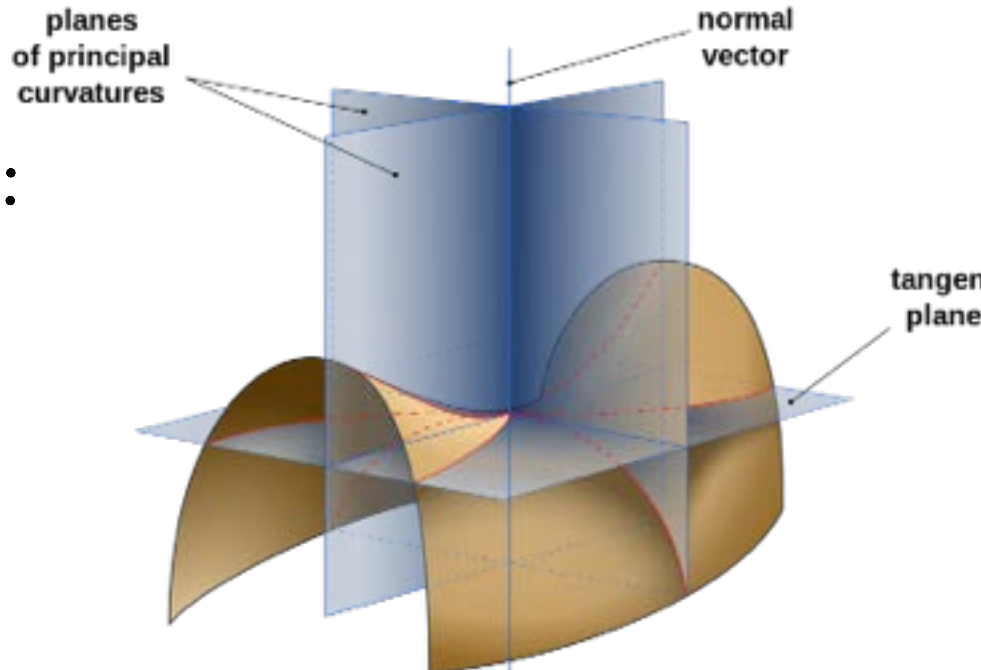
G=1

G=3

2) It is related to Gaussian curvature of surface as:

$$G = 1 - \frac{1}{4\pi}\int \frac{dA}{R_1 R_2}$$

Global Property (G) — local Property ($\frac{dA}{R_1 R_2}$)



Genus from Mathematics

3) If there is no information about Gaussian curvatures, Euler characteristic can be used for determining Genus as:

$$\chi \equiv \#\text{ of faces} + \#\text{ of vertices} - \#\text{ of edges}$$

$$\chi \equiv \#\text{ of Maxima} + \#\text{ of Minima} - \#\text{ of Saddle Points}$$

$$G = 1 - \frac{\chi}{2}$$

Multi-connected field has G>0

Euler characteristic from Statistics

Euler-Poincaré characteristic χ

$$\chi^{3D}(\nu) = \left\langle \delta_D(\alpha-\nu)\, \delta_D(\eta_1)\, \delta_D(\eta_2)\, |\eta_3|\, (\xi_{11}\xi_{22} - \xi_{12}^2) \right\rangle$$

$$\chi^{2D}(\nu) = \left\langle \delta_D(\alpha-\nu)\, \delta_D(\eta_1)\, |\eta_2|\, \xi_{11} \right\rangle$$

Intuitive definition

$$\chi^{1D}(\nu) = \left\langle \delta_D(\alpha-\nu)\, |\eta_1| \right\rangle = N_1(\nu)$$

$$G = 1 - \frac{\chi}{2}$$

$$\langle G(\nu) \rangle = \langle G(\nu) \rangle_G + Perturbation\ parts \Big|_{Non-Gaussian,\, Anisotropic}$$

T. Matsubara, APJ 2003; S. Codis et. al., 1305.7402

Genus from Statistics

$$\overset{3D}{G}(\nu) = \frac{2}{(2\pi)^2}\left(\frac{\sigma_1}{\sqrt{D}\,\sigma_0}\right)^3 e^{-\nu^2/2}\left[H_2(\nu) + \left(\frac{S^{0}}{6}H_5(\nu) + S^{(1)}H_3(\nu) + S^{(2)}H_1(\nu)\right)\sigma_0 + O(\sigma_0^2)\right]$$

$$\overset{2D}{G}(\nu) = \frac{-2}{(2\pi)^{3/2}}\left(\frac{\sigma_1}{\sqrt{D}\,\sigma_0}\right)^2 e^{-\nu^2/2}\left[H_1(\nu) + \left(\frac{S^{0}}{6}H_4(\nu) + \frac{2S^{(1)}}{3}H_2(\nu) + \frac{S^{(2)}}{3}\right)\sigma_0 + O(\sigma_0^2)\right]$$

$$\overset{1D}{G}(\nu) = \frac{N_1(\nu)}{2} = \frac{1}{\pi}\,\frac{\sigma_1}{\sqrt{D}\,\sigma_0}\,e^{-\nu^2/2}\left[1 + \left(\frac{S^{0}}{6}H_3(\nu) + \frac{S^{(1)}}{3}H_1(\nu)\right)\sigma_0 + O(\sigma_0^2)\right]$$

T. Matsubara, APJ 2003; S. Codis et. al., 1305.7402

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Euler characteristic from mathematics

Leonhard Euler (1707-1783)

1) The *Euler Characteristic* is something which generalises Euler's observation of 1751 (in fact already noted by Descartes in 1639) that on "triangulating" a sphere into F regions, E edges and V vertices one has $V - E + F = 2$.

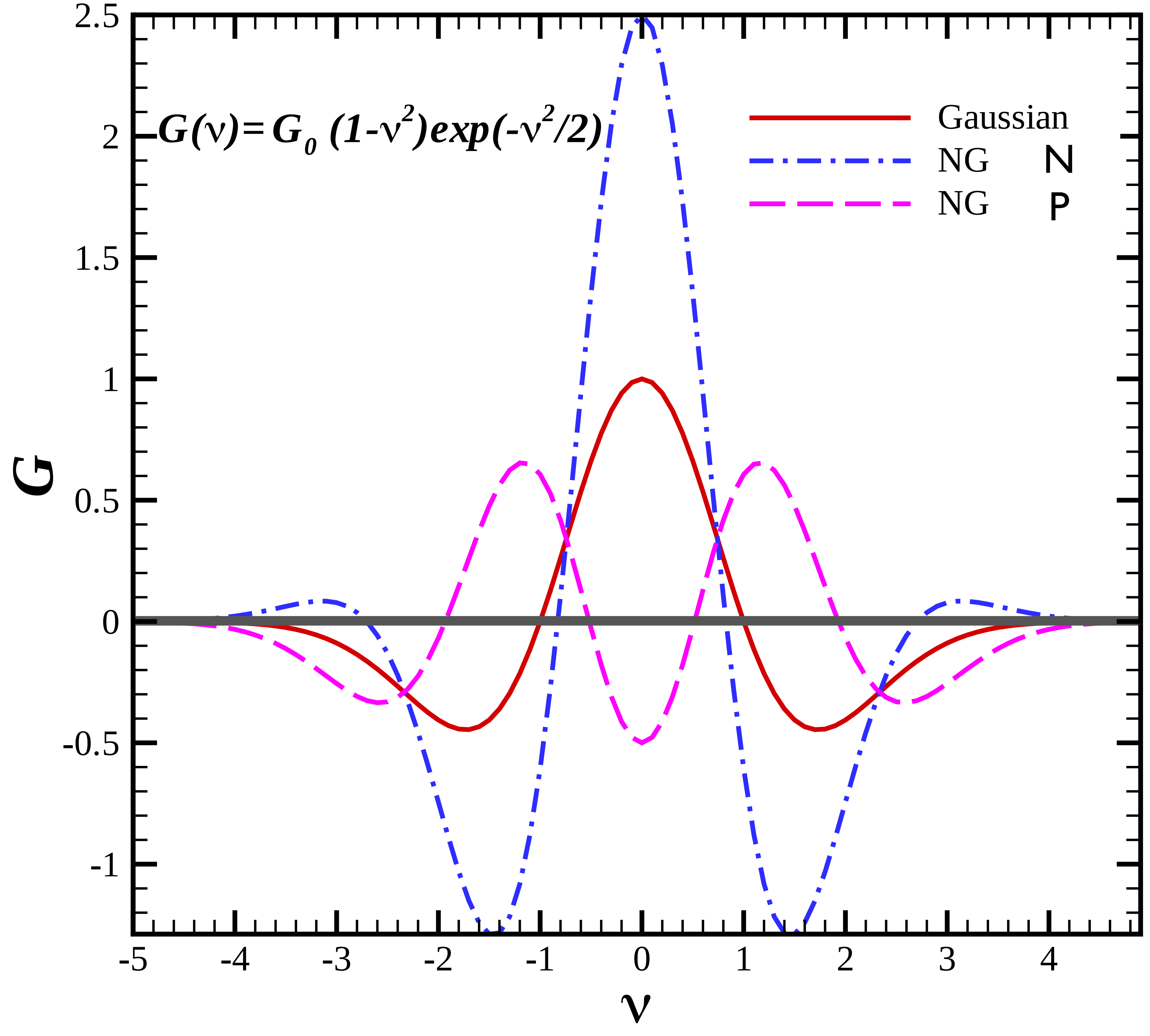
2) In addition the value of Euler characteristic does not depend on how tessellation is done

3) Euler for Sphere is equal to 2

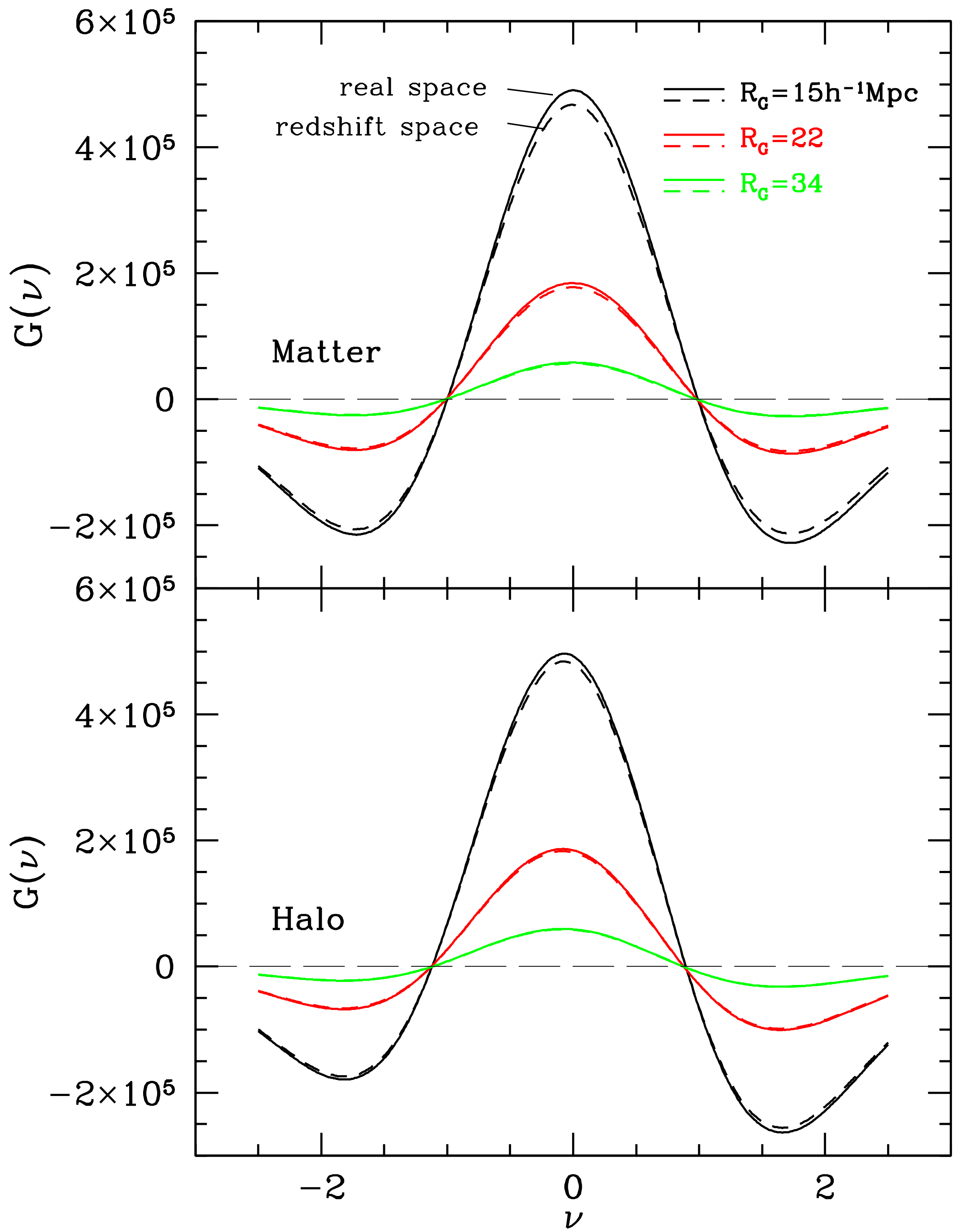
The Euler characteristic for convex polyhedra always equals 2

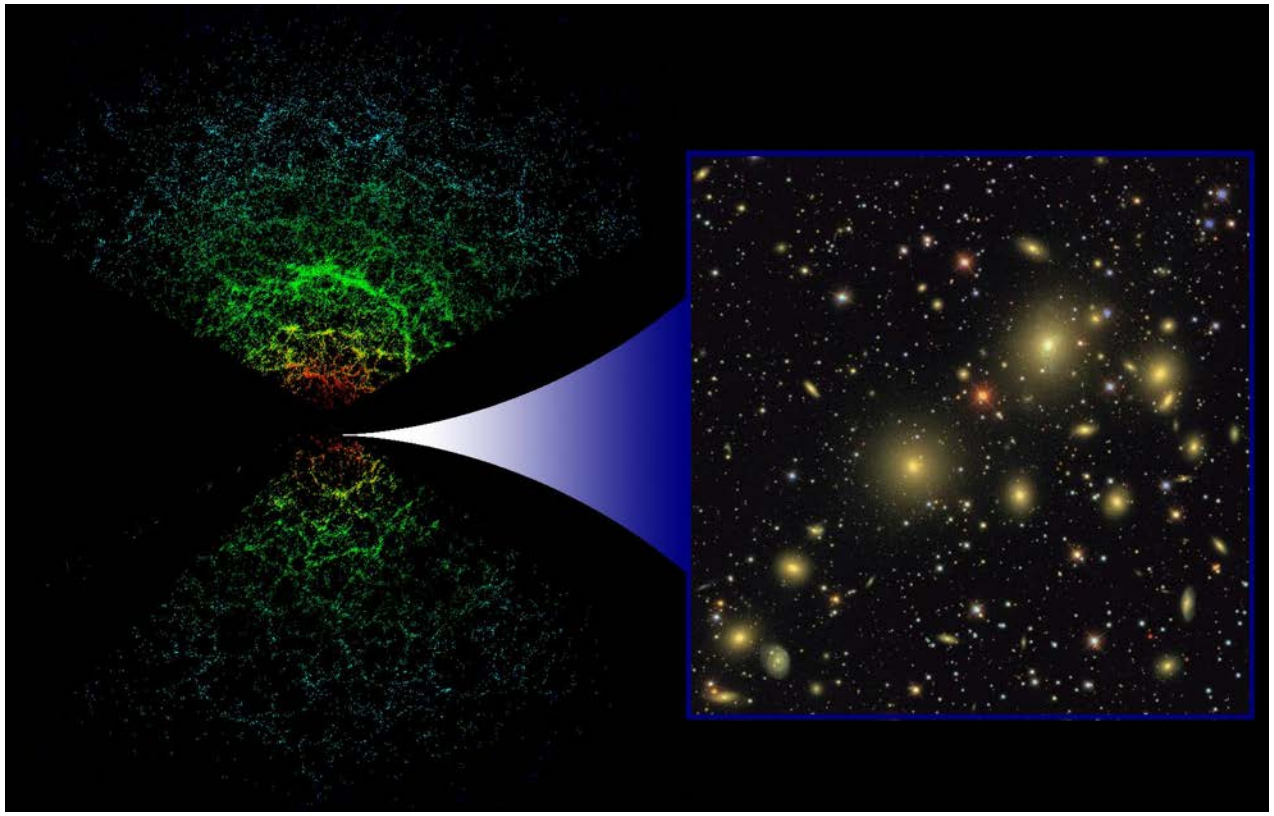
Name	Image	Vertices V	Edges E	Faces F	Euler characteristic $V - E + F$
Tetrahedron		4	6	4	2
Hexahedron or cube		8	12	6	2
Octahedron		6	12	8	2
Dodecahedron		20	30	12	2
Icosahedron		12	30	20	2

Genus from Gaussian and NG fields 3D



Topology of our Universe





1) Topological consideration is consistent with Gaussian field
2) At median density, topology of our local universe is swiss-cheese
3) At large threshold, the topology belongs to meatball

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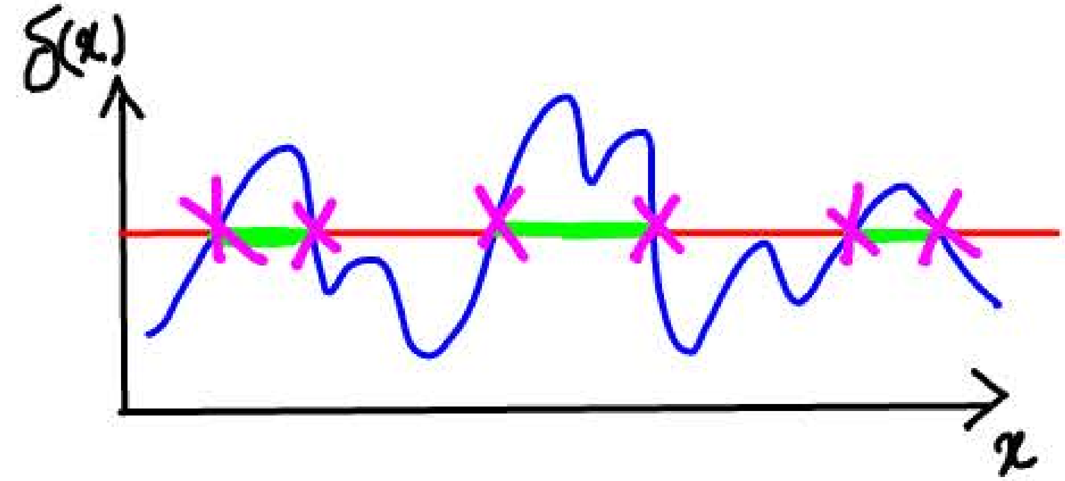
Minkowski Functionals

1D field



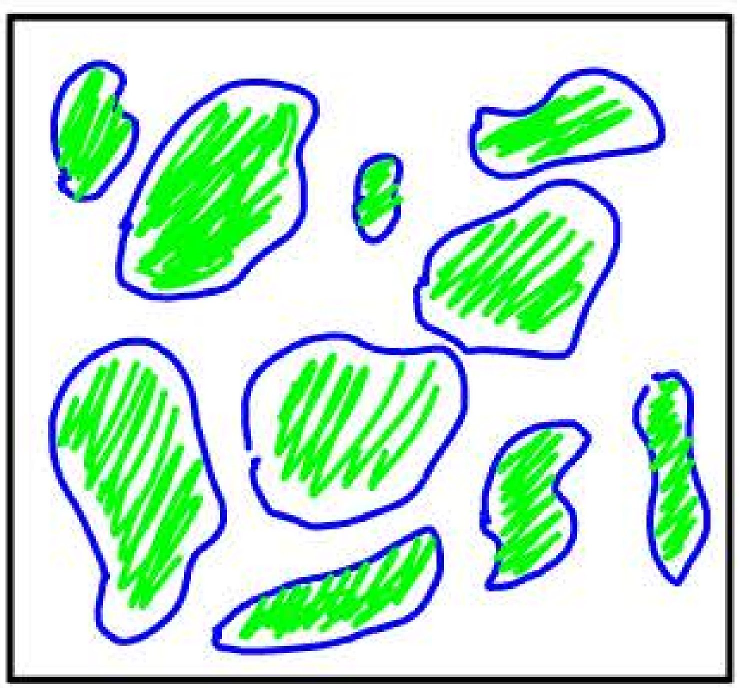
$$V_0(\nu) = \int_Q dl = \langle \Theta(\alpha-\nu) \rangle$$

$$V_1(\nu) = \int_{\partial Q} dl = \frac{1}{2}\langle \delta_D(\alpha-\nu)|\eta_1| \rangle \sim N_1(\nu)$$

2D field

- $$V_0(\nu) = \int_Q dA = \langle \Theta(\alpha-\nu) \rangle$$
- $$V_1(\nu) = \frac{1}{4}\int_{\partial Q} dl = \frac{\pi}{8}\langle \delta_D(\alpha-\nu)|\eta_1| \rangle \sim N_1(\nu)$$

$$V_2(\nu) = \frac{1}{2\pi}\int_{\partial Q} \kappa \, dl = -\frac{1}{2}\langle \delta_D(\alpha-\nu)\,\delta_D(\eta_1)\,|\eta_2|\,\xi_{11} \rangle$$



Minkowski Functionals
General case

Volume fraction of excursion set $\quad \bar{V}_0^{(D)}(\nu) = \frac{1}{V}\int_K dV = \langle \Theta(\nu-\alpha) \rangle$

2D

$$\bar{V}_j^{(2)}(\nu) = \frac{1}{V}\int_{\partial K} dL(x)\, v_j^{(2)}(\nu,x)$$

$$v_1^{(2)}(\nu,x) \equiv \frac{1}{4}$$

$$v_2^{(2)}(\nu,x) \equiv \frac{1}{2\pi}\frac{1}{R_1}$$

$\frac{1}{R_1} \equiv$ Principal curvature

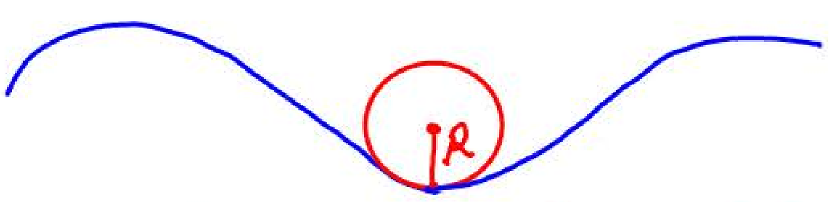
3D

$$\bar{V}_j^{(3)}(\nu) = \frac{1}{V}\int_{\partial K} d^2A(x)\, v_j^{(3)}(\nu,x)$$

$$v_1^{(3)}(\nu,x) \equiv \frac{1}{6}$$

$$v_2^{(3)}(\nu,x) \equiv \frac{1}{6\pi}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

$$v_3^{(3)}(\nu,x) \equiv \frac{1}{4\pi}\frac{1}{R_1 R_2}$$



R is radius of osculating circul

Minkowski functionals

Crofton (1868) :

$$V_3^{(D)}(\nu) = -\frac{\omega_D}{\omega_{D-3}\,\omega_3}\, G_3(\nu)$$

$$V_2^{(D)}(\nu) = \frac{\omega_D}{\omega_{D-2}\,\omega_2}\, G_2(\nu)$$

$$V_1^{(D)}(\nu) = \frac{\omega_D}{2\,\omega_{D-1}\,\omega_1}\, N_1(\nu)$$

$$\omega_D \equiv \frac{\pi^{D/2}}{\Gamma(D/2+1)}$$

Clustering of Peaks and crossing in 2D stochastic field

To this end we should introduce peaks

Theoretical approach for clustering (1+2D)
Number density of Peaks

- We are interested in investigating local extrema on the CMB as one of most famous stochastic field

- To this end we should evaluate number density of peaks:

$$\mathcal{F}(\mathbf{r}) \equiv [T(\mathbf{r}) - \langle T(\mathbf{r})\rangle]/\langle T(\mathbf{r})\rangle$$

$$p(\vec{\mathcal{W}}) = \sqrt{\frac{1}{(2\pi)^6 \det\mathcal{M}}}\, e^{-\frac{1}{2}(\mathcal{W}^T.\mathcal{M}^{-1}.\mathcal{W})} \quad \mathcal{M} = \langle \mathcal{W} \otimes \mathcal{W} \rangle$$

$$\vec{\mathcal{W}} = \{\mathcal{F}, \mathcal{F}_x, \mathcal{F}_y, \mathcal{F}_{xx}, \mathcal{F}_{xy}, \mathcal{F}_{yy}\} \quad \mathcal{F}_i \equiv \frac{\partial \mathcal{F}}{\partial x_i} \quad \mathcal{F}_{ij} \equiv \frac{\partial^2 \mathcal{F}}{\partial x_i \partial x_j}$$

$$n(\vartheta) = \int p(\vec{\mathcal{W}} | \mathcal{F}_i = 0)\, |\det \mathcal{F}_{ij}|\, d\vec{\mathcal{W}}$$

$$n(\vartheta) = \frac{1}{(2\pi)^{3/2}\gamma^2} e^{-\vartheta^2/2} \mathcal{G}(\Psi, \Psi\vartheta)$$

Bond J. R. and Efstathiou G., Mon. Not. Roy. Astron. Soc. **226**, 655-687, 1987

$$\mathcal{G}(\Psi, \Psi\vartheta) \equiv (\Psi^2\vartheta^2 - \Psi^2)\left\{1 - \frac{1}{2}\mathrm{erfc}\left[\frac{\Psi\vartheta}{\sqrt{2(1-\Psi^2)}}\right]\right\}$$

$$+\Psi\vartheta(1-\Psi^2)\frac{e^{-\frac{\Psi^2\vartheta^2}{2(1-\Psi^2)}}}{\sqrt{2\pi(1-\Psi^2)}}$$

$$+\frac{e^{-\frac{\Psi^2\vartheta^2}{3-2\Psi^2}}}{\sqrt{3-2\Psi^2}}\left\{1 - \frac{1}{2}\mathrm{erfc}\left[\frac{\Psi\vartheta}{\sqrt{2(1-\Psi^2)(3-2\Psi^2)}}\right]\right\}$$

$$\Psi \equiv \frac{\sigma_1^2}{\sigma_0\sigma_2} \qquad \gamma \equiv \sqrt{2}\frac{\sigma_1}{\sigma_2}$$

$$\sigma_0^2 \equiv \left\langle \mathcal{F}(\mathbf{r})^2 \right\rangle = \frac{1}{(2\pi)^2}\int S(|\mathbf{k}|)d\mathbf{k}$$

$$\sigma_n^2 \equiv \left\langle \left(\frac{\partial^n \mathcal{F}(\mathbf{r})}{\partial x^n}\right)^2 \right\rangle = \frac{1}{(2\pi)^2}\int k^{2n} S(|\mathbf{k}|)d\mathbf{k}$$

Weak non-Gaussian field

$$\frac{\partial n_{\rm ext}}{\partial x} = \int d^3 x_{ij} P(x, x_i = 0, x_{ij}) |x_{ij}|$$

$$\frac{\partial n_{\rm max/\,min}}{\partial \nu} = \frac{1}{\sqrt{2\pi} R_*{}^2} \exp\left(-\frac{\nu^2}{2}\right)\left[1 \pm \mathrm{erf}\left(\frac{\gamma\nu}{\sqrt{2(1-\gamma^2)}}\right)\right] K_1(\nu, \gamma) \pm \frac{1}{\sqrt{2\pi(1-\gamma^2)} R_*{}^2} \exp\left(-\frac{\nu^2}{2(1-\gamma^2)}\right) K_3(\nu, \gamma)$$

$$+ \frac{\sqrt{3}}{\sqrt{2\pi(3-2\gamma^2)} R_*{}^2} \exp\left(-\frac{3\nu^2}{6-4\gamma^2}\right)\left[1 \pm \mathrm{erf}\left(\frac{\gamma\nu}{\sqrt{2(1-\gamma^2)(3-2\gamma^2)}}\right)\right] K_2(\nu, \gamma),$$

$$\frac{\partial n_{\rm sad}}{\partial \nu} = \frac{2\sqrt{3}}{\sqrt{2\pi(3-2\gamma^2)} R_*{}^2} \exp\left(-\frac{3\nu^2}{6-4\gamma^2}\right) K_2(\nu, \gamma),$$

$$\zeta = (x + \gamma J_1)/\sqrt{1-\gamma^2}$$

$$\gamma = -\langle x J_1 \rangle$$

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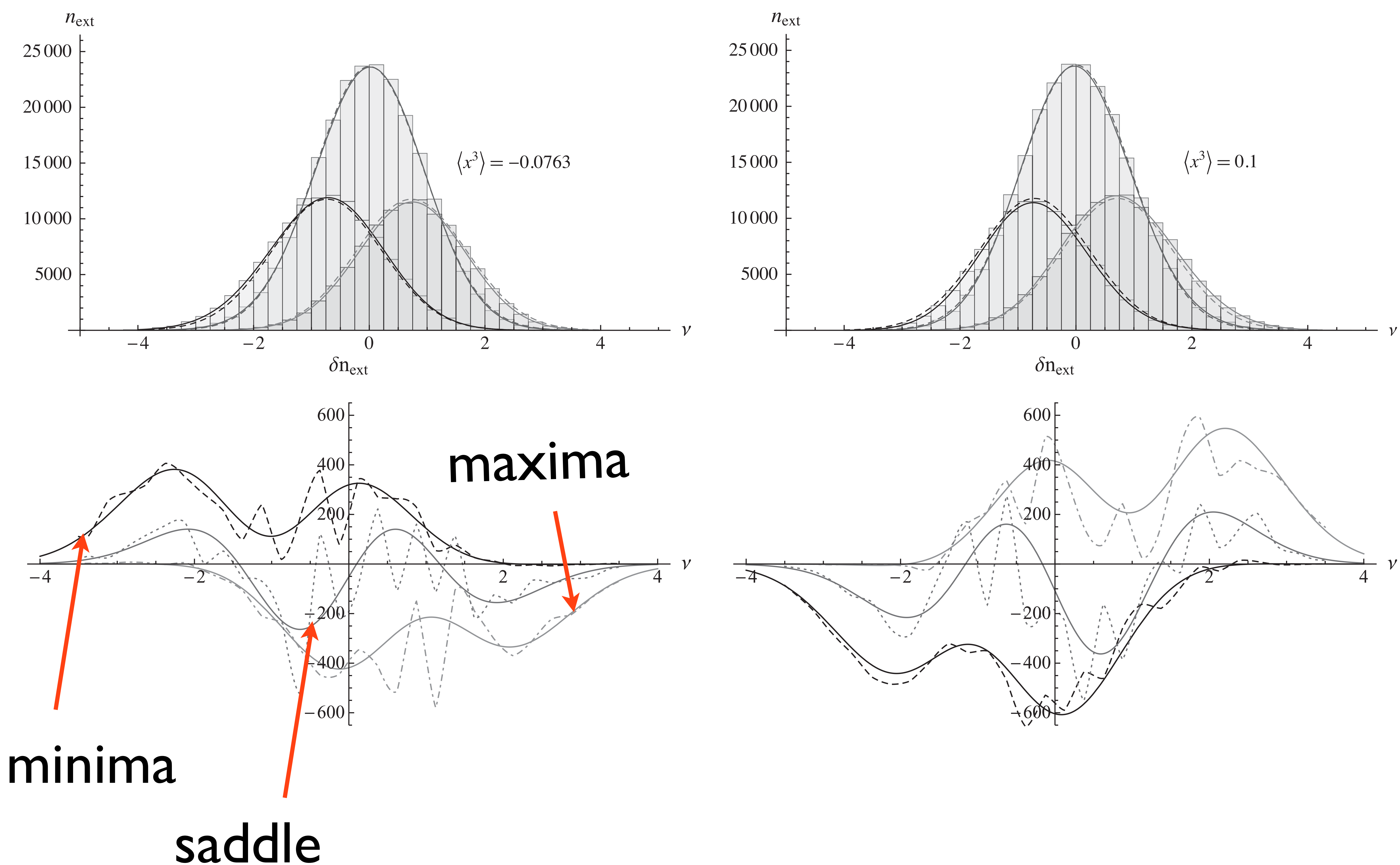
$$K_1 = \frac{\gamma^2}{8\pi}\left[H_2(\nu) + \left(\frac{2}{\gamma}\langle q^2 J_1\rangle + \frac{1}{\gamma^2}\langle x{J_1}^2\rangle - \frac{1}{\gamma^2}\langle xJ_2\rangle\right)H_1(\nu) - \left(\langle xq^2\rangle + \frac{1}{\gamma}\langle x^2 J_1\rangle\right)H_3(\nu) + \frac{1}{6}\langle x^3\rangle H_5(\nu)\right]$$

$$K_2 = \frac{1}{8\pi\sqrt{3}}\left[1 - \left(\langle xq^2\rangle + \frac{1}{3}\langle x{J_1}^2\rangle - \frac{4}{3}\langle xJ_2\rangle + \frac{2}{3}\gamma\langle q^2 J_1\rangle + \frac{2}{9}\gamma\langle {J_1}^3\rangle - \frac{2}{3}\gamma\langle J_1 J_2\rangle\right)\mathcal{H}_1^-\left(\nu, \sqrt{1-2/3\gamma^2}\right)\right.$$
$$\left. + \frac{1}{6}\left(\langle x^3\rangle + 2\gamma\langle x^2 J_1\rangle + \frac{4}{3}\gamma^2\langle x{J_1}^2\rangle + \frac{2}{3}\gamma^2\langle xJ_2\rangle + \frac{8}{27}\gamma^3\langle {J_1}^3\rangle + \frac{4}{9}\gamma^3\langle J_1 J_2\rangle\right)\mathcal{H}_3^-\left(\nu, \sqrt{1-2/3\gamma^2}\right)\right]$$

$$K_3 = \frac{(1-\gamma^2)}{2(2\pi)^{3/2}(3-2\gamma^2)^3}\left[\gamma(3-2\gamma^2)^3\mathcal{H}_1^+\left(\nu, \sqrt{1-\gamma^2}\right) + \left(\frac{1}{2}\gamma^3(1+\gamma^2-26\gamma^4+28\gamma^6-8\gamma^8)\langle x^3\rangle\right.\right.$$
$$- \gamma^4(26-28\gamma^2+8\gamma^4)\langle x^2 J_1\rangle + \gamma(1-\gamma^2)(1+2\gamma^2)(3-2\gamma^2)^2\langle xq^2\rangle - \gamma(24-26\gamma^2+8\gamma^4)\langle xJ_1^2\rangle$$
$$\left. + \gamma(15-23\gamma^2+8\gamma^4)\langle xJ_2\rangle + 4(1-\gamma^2)(3-2\gamma^2)^2\langle q^2 J_1\rangle - (10-12\gamma^2+4\gamma^4)\langle J_1^3\rangle + 6(1-\gamma^2)(2-\gamma^2)\langle J_1 J_2\rangle\right)$$
$$- \frac{1}{6}(\gamma(27+36\gamma^2-224\gamma^4+192\gamma^6-48\gamma^8)\langle x^3\rangle + (108-324\gamma^2+216\gamma^4-48\gamma^6)\langle x^2 J_1\rangle$$
$$\left. + 6\gamma(3-2\gamma^2)^3\langle xq^2\rangle - 36\gamma\langle xJ_1^2\rangle - 18\gamma\langle xJ_2\rangle - 8\gamma^2\langle J_1^3\rangle - 12\gamma^2\langle J_1 J_2\rangle)\mathcal{H}_2^+\left(\nu, \sqrt{1-\gamma^2}\right)\right]$$

$$n_{\max/\min} = \frac{1}{8\sqrt{3}\pi {R_*}^2} \pm \frac{18\langle q^2 J_1\rangle - 5\langle J_1^3\rangle + 6\langle J_1 J_2\rangle}{54\pi\sqrt{2\pi}{R_*}^2},$$

$$n_{\rm sad} = \frac{1}{4\sqrt{3}\pi {R_*}^2}, \qquad J_1 \equiv \lambda_1 + \lambda_2,\ J_2 \equiv (\lambda_1 - \lambda_2)^2$$



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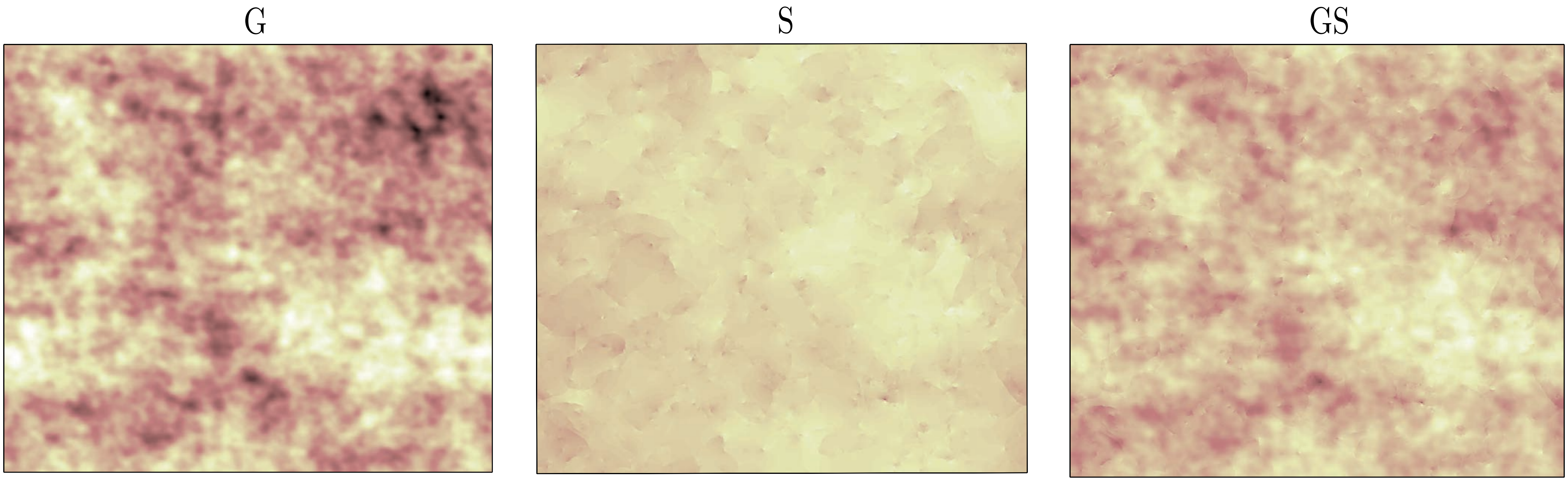
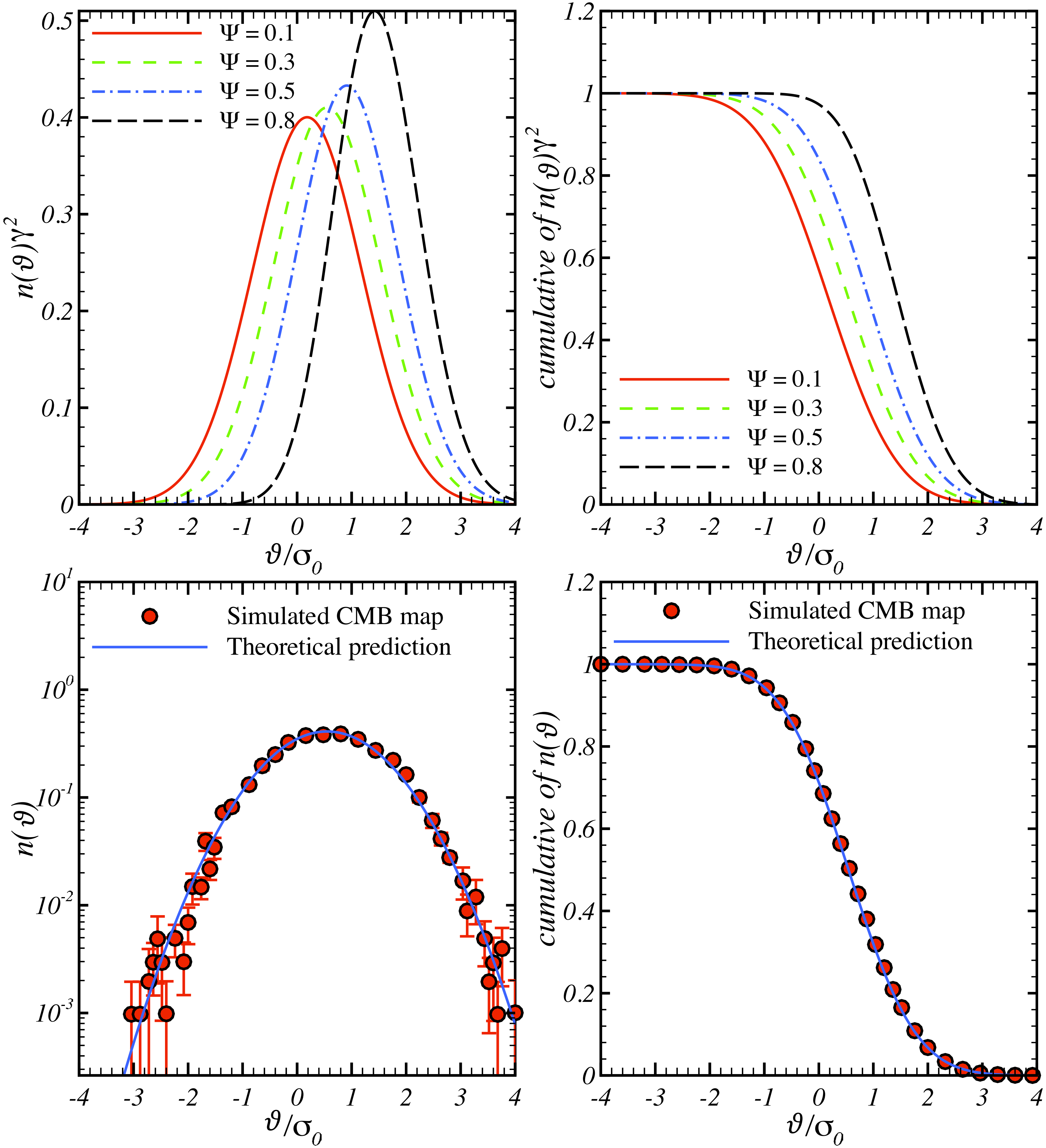
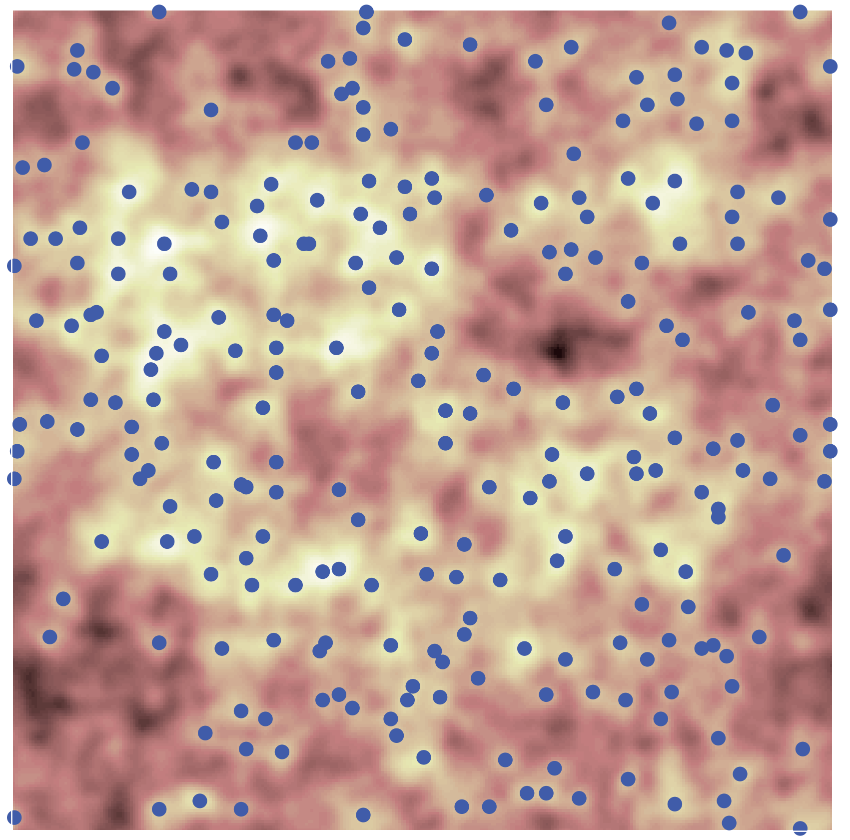
Our Simulation

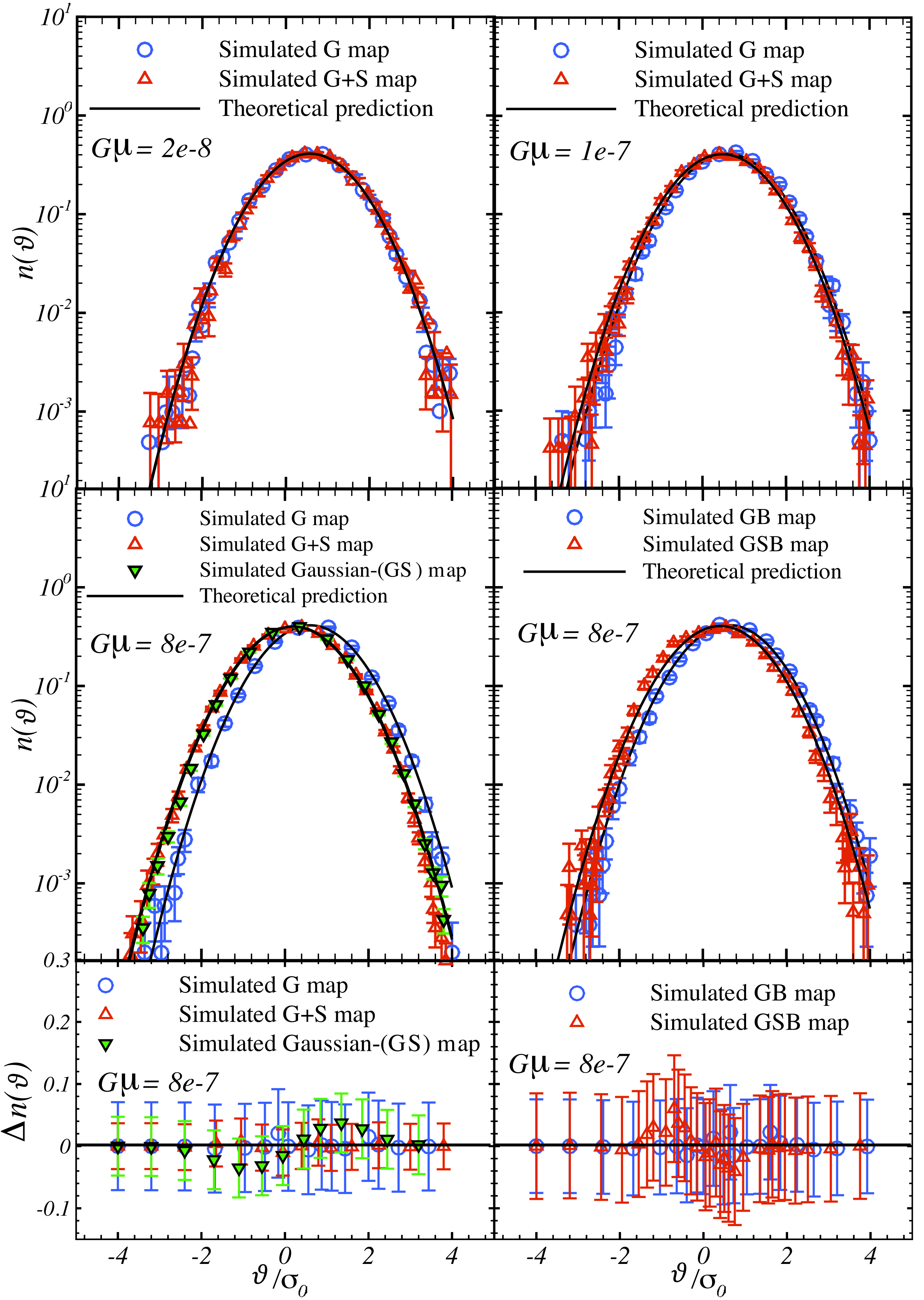




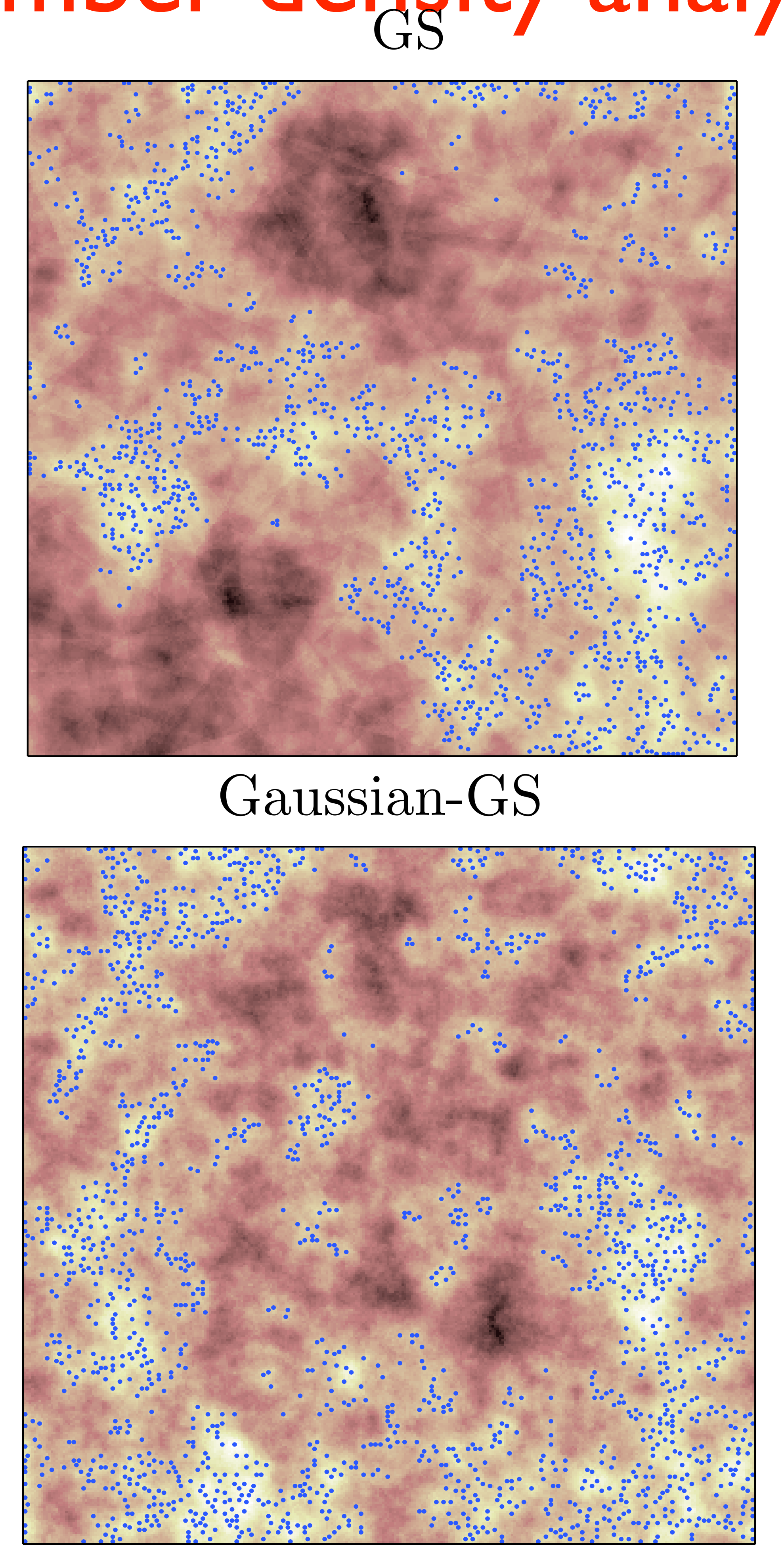


Number density of peaks

Number density analysis



$$\Delta n(\vartheta) \equiv n_{com.}(\vartheta) - n_{the.}(\vartheta)$$



Theoretical approach for clustering (1+2D)

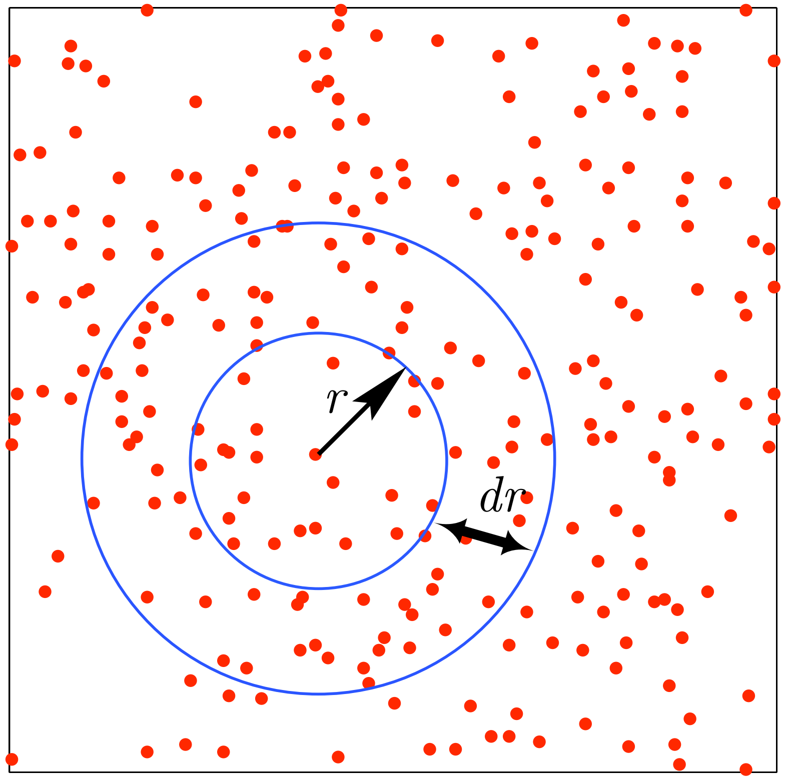
- Excess probability of finding a peak at a given distance from another relative to the probability for a uniform distribution of peaks (Peacock & Heavens and Peebles):

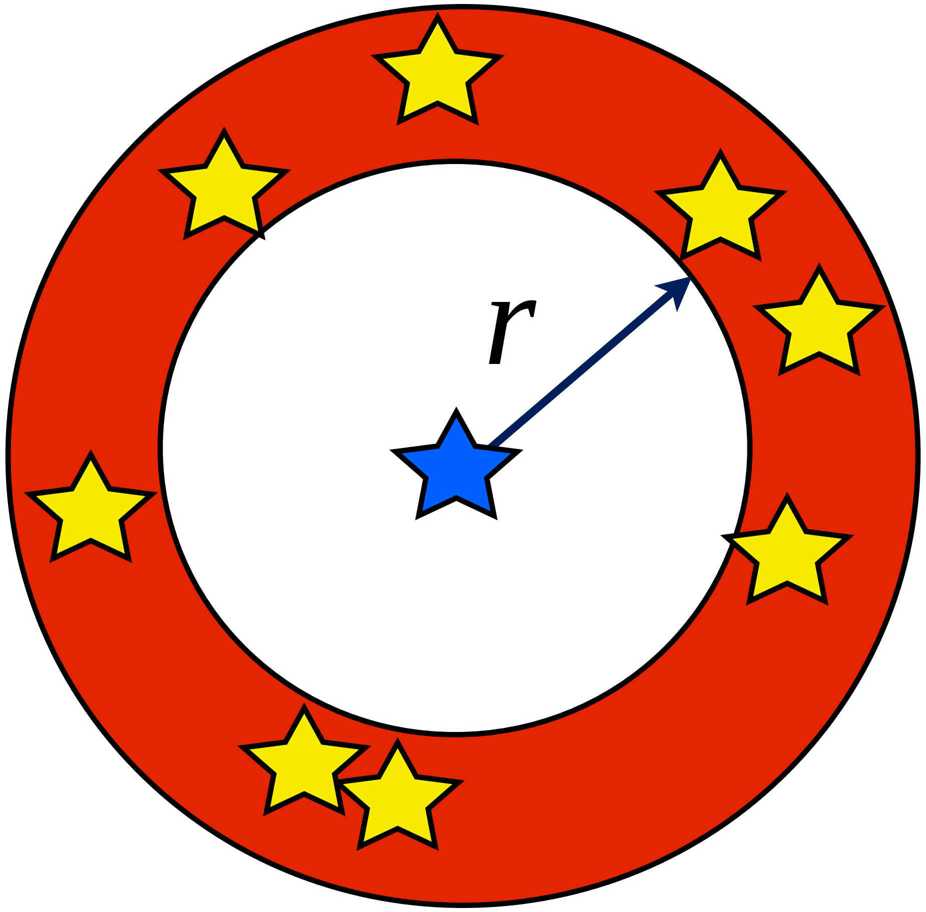
$$\Delta P_{12}(r) = n\Delta A_1 \times n\Delta A_2 \times (1+\xi(r))$$

$$\Delta P_{12}(r) = n_{pair}\Delta A(r) \times (1+\xi(r))$$

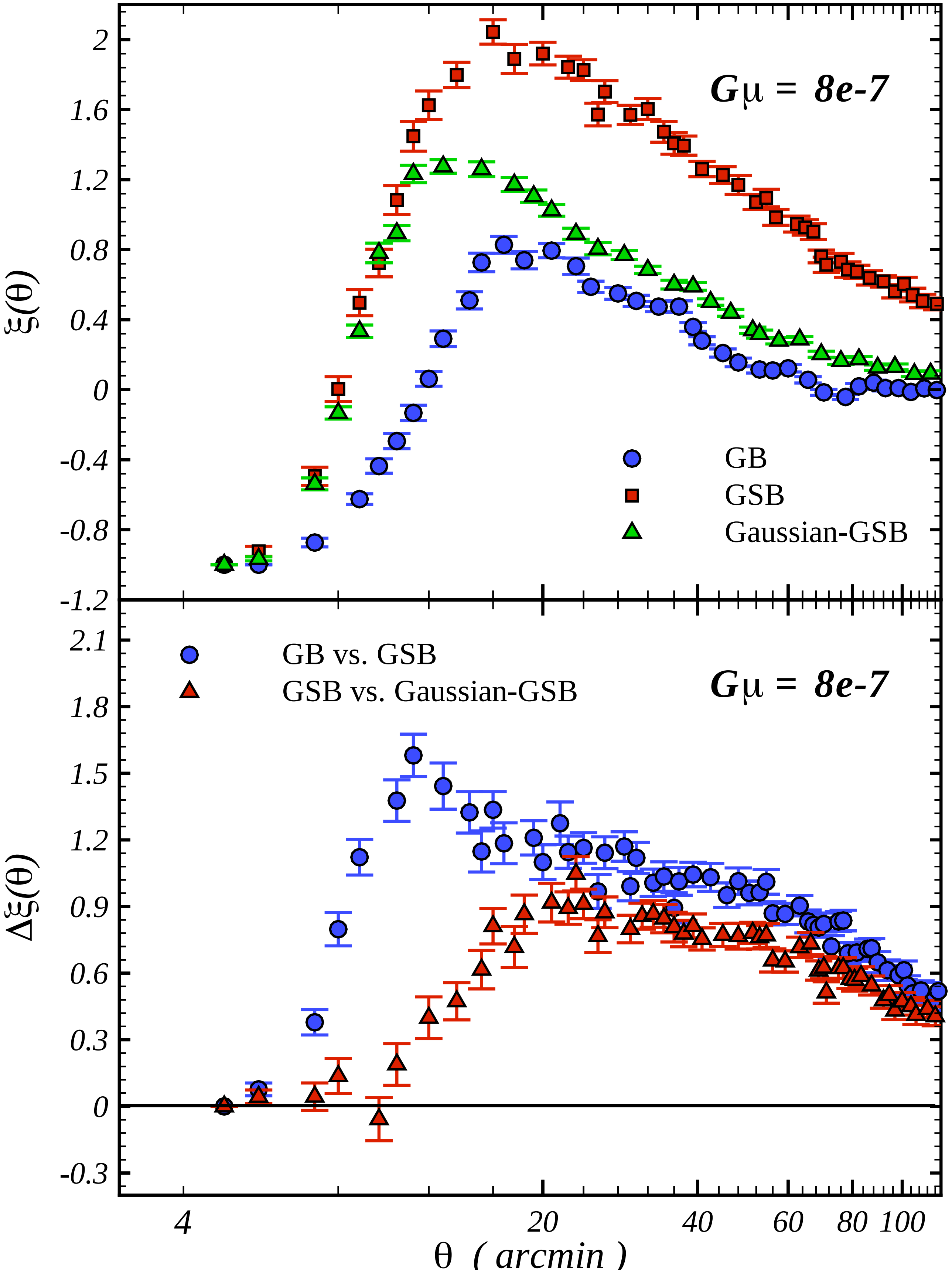
$$n_{pair} = \frac{M(M-1)}{2A} \sim \frac{M}{2}n$$

$$\xi(r) = \frac{\langle N \rangle_{pairs}}{n_{pair} 2\pi r \Delta r} - 1$$

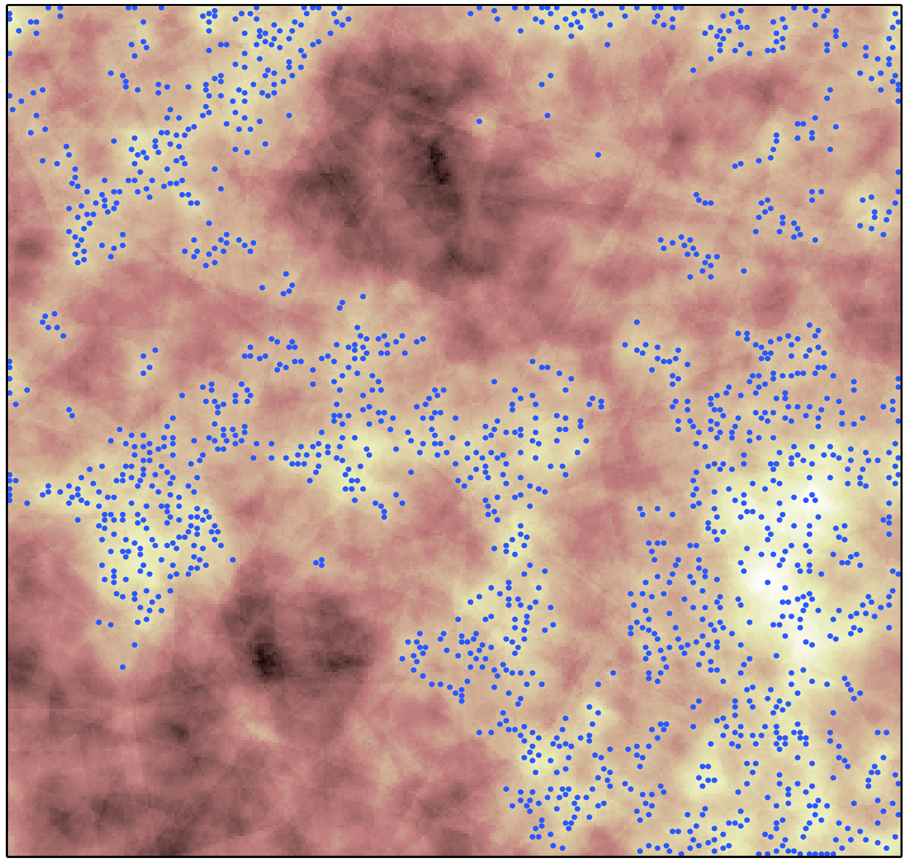




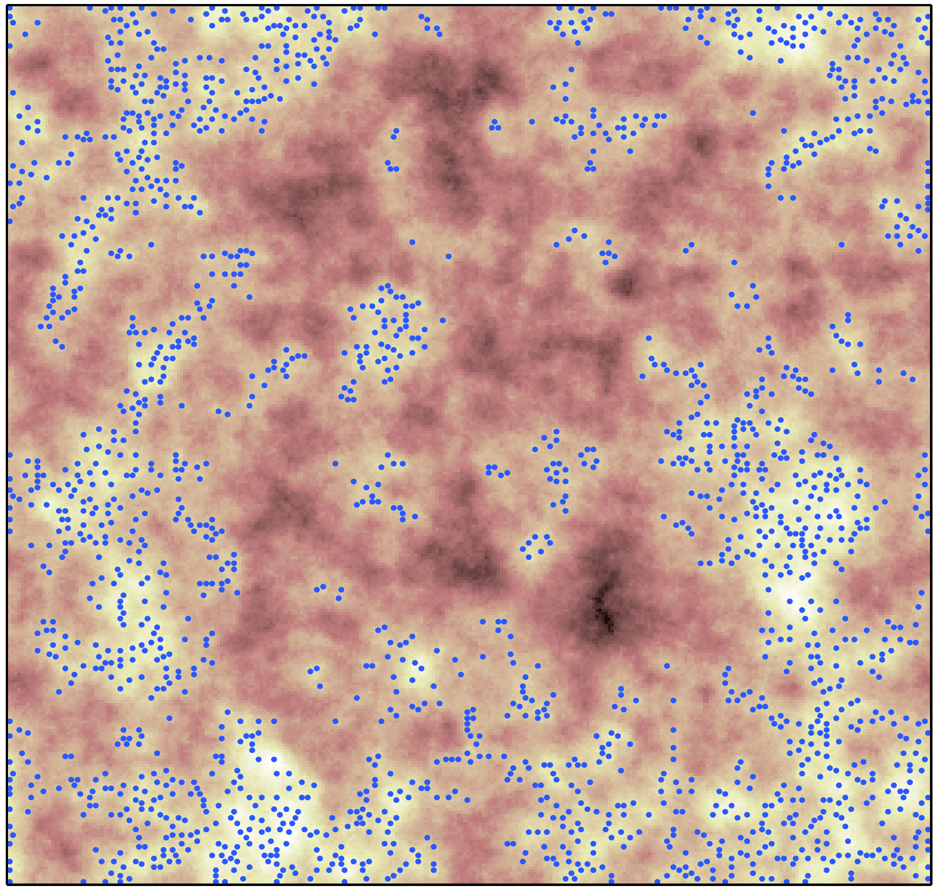
TPCF



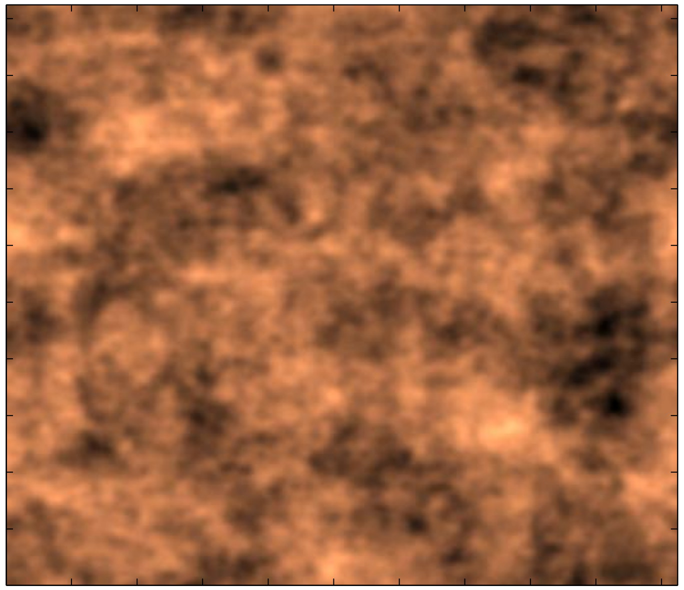
GS



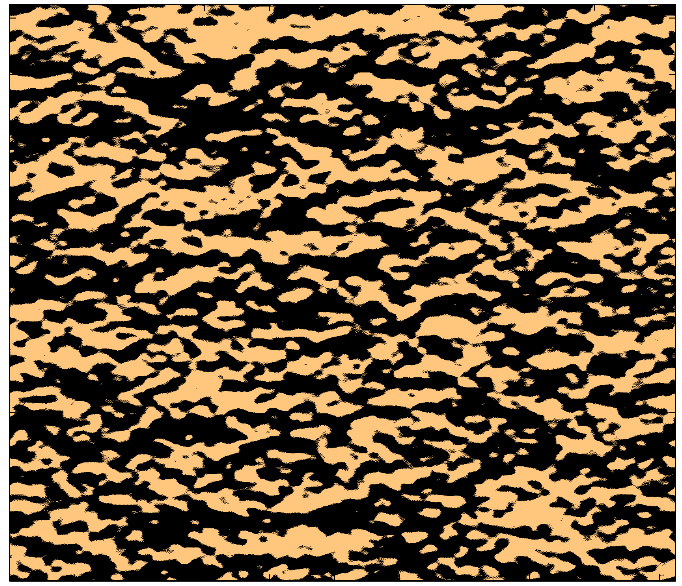
Gaussian-GS



Clustering of Up-Crossing

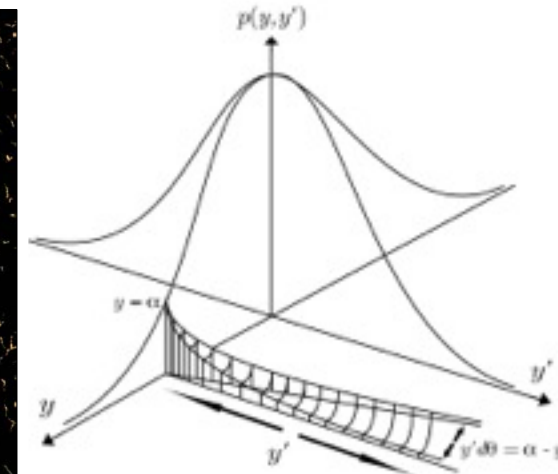


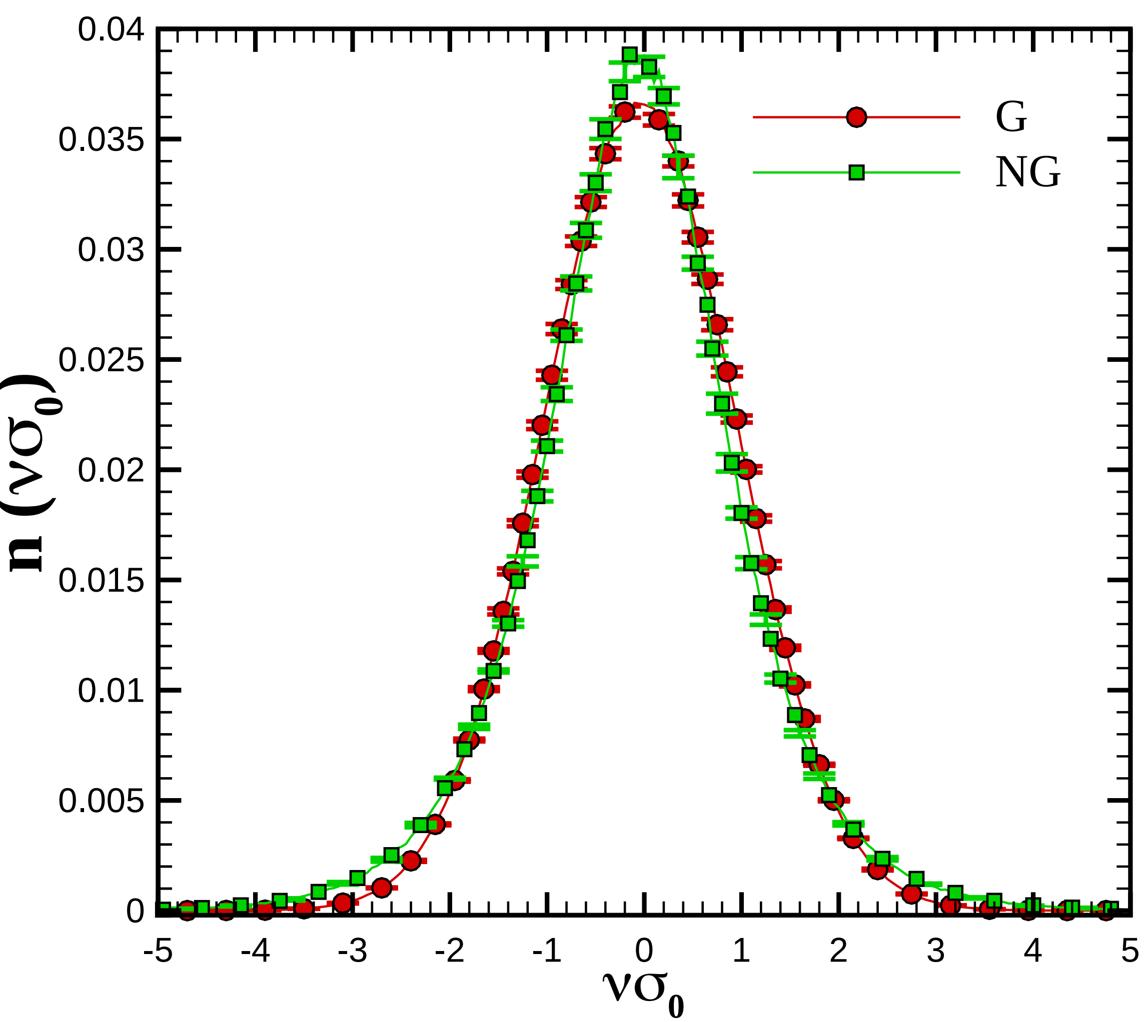
Gaussian field

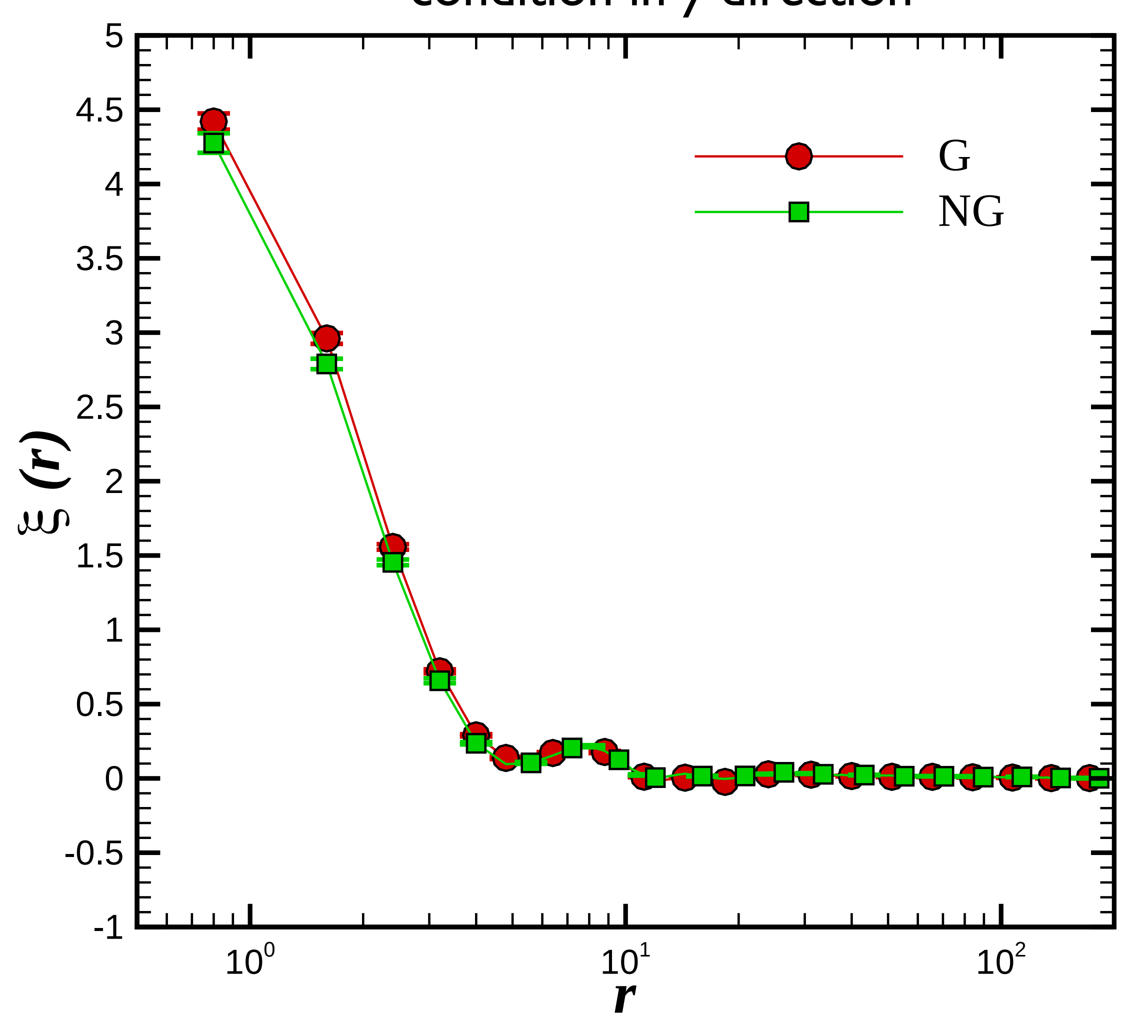


Cumulative of crossing in x

Cumulative of crossing in x with further condition in y direction

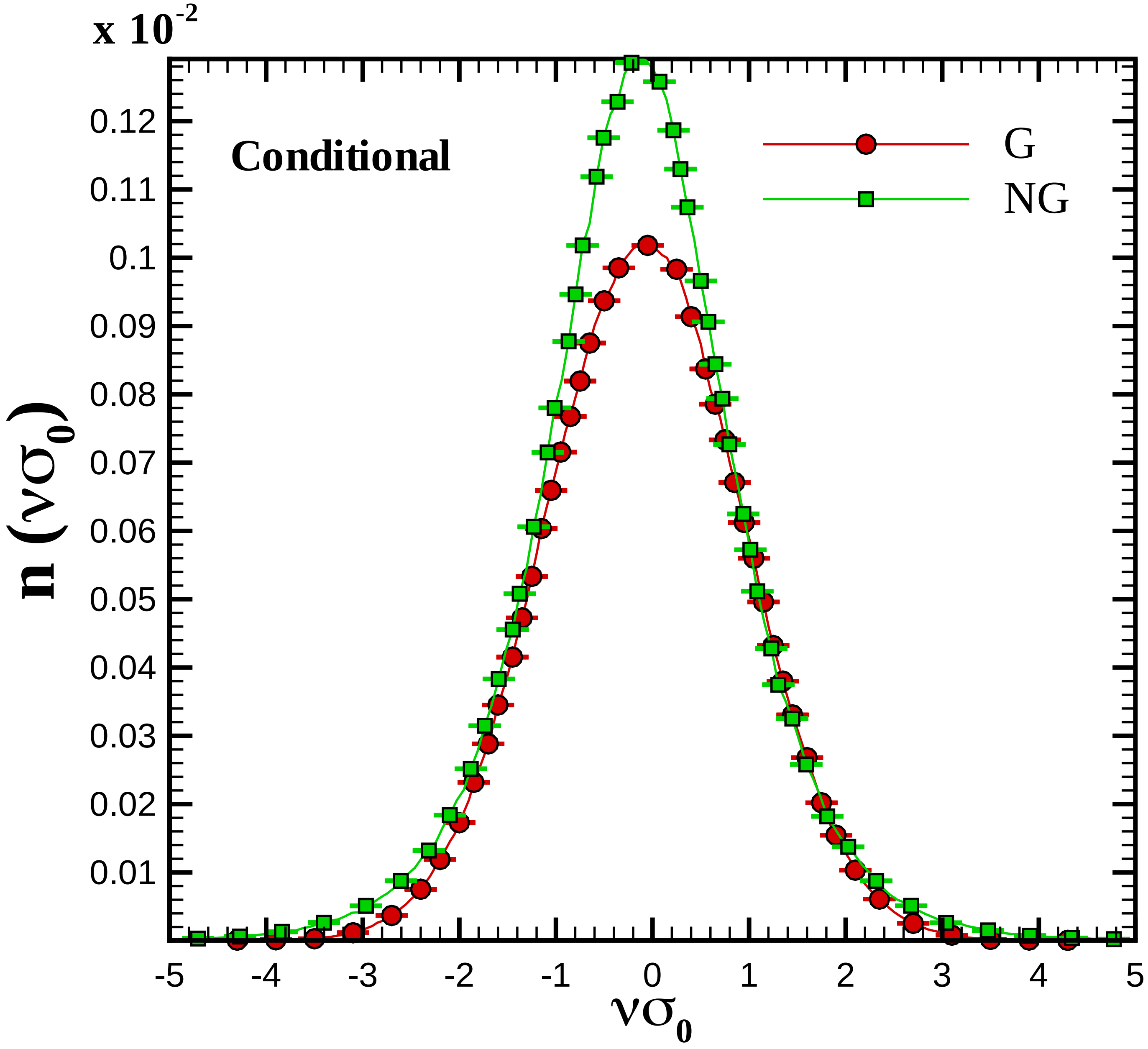


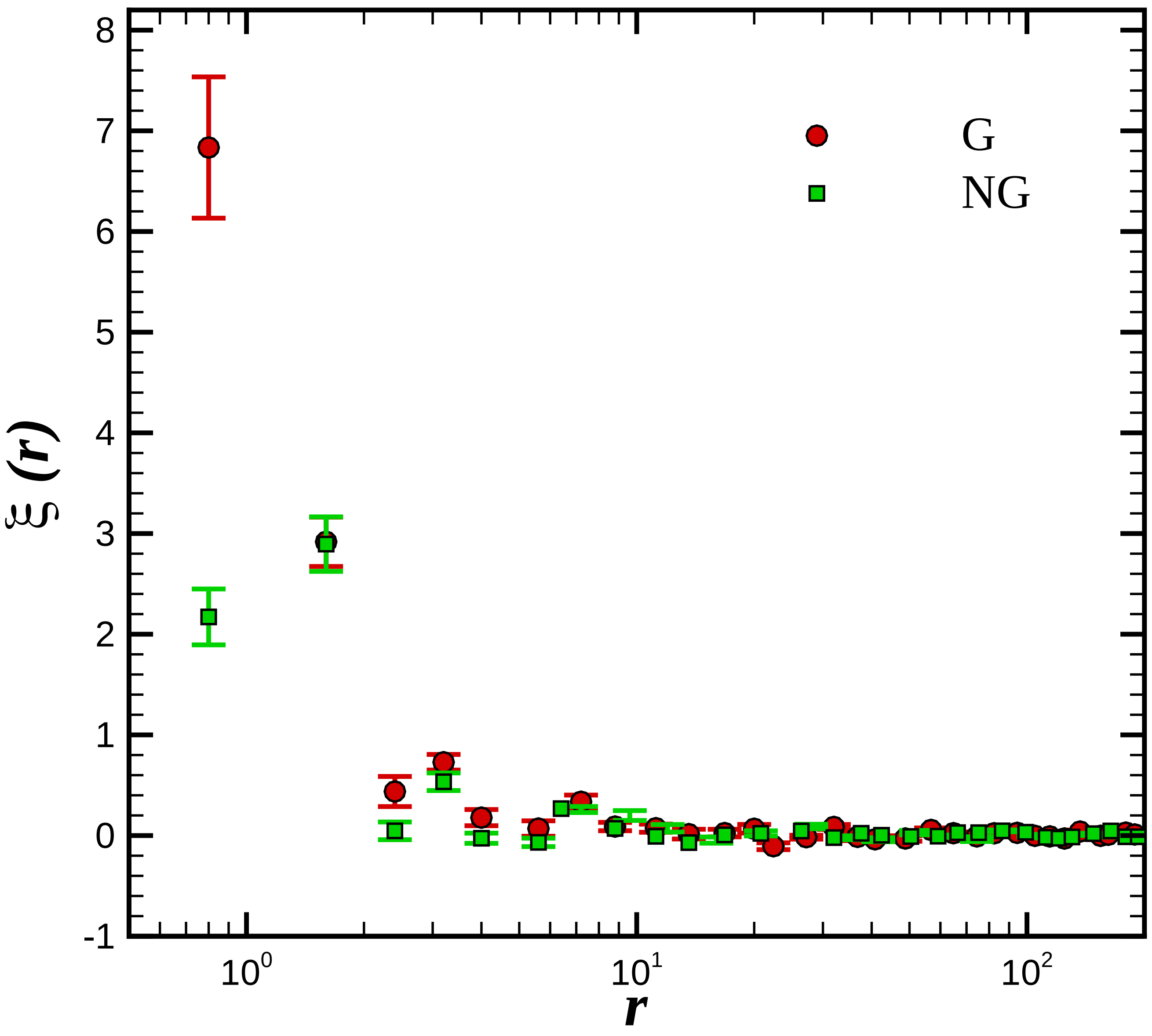




A. Vafaei Sadr, Ravi K. Sheth, S.M.S. Movahed, in progress

Clustering of Up-Crossing





Theoretical prediction

$$n_{up}(\nu) = \sum_{\mu} \delta_D(\vec{r} - \vec{r}_\mu)$$

$$\langle n_{up}(\nu) \rangle = \langle \delta_D(\vec{r} - \vec{r}_\mu) \rangle = \int dA_\mu \, P(A_\mu) \, \delta_D(\vec{r} - \vec{r}_\mu)$$

$$A_\mu = (F(\vec{r}), \eta_x(\vec{r}), \eta_y(\vec{r}), \xi_{xx}, \xi_{xy}, \xi_{yy})$$

$$F(\vec{r}) = F(r_\mu) + (r - r_\mu)_x \eta_x + \mathcal{O}(\delta r^2)$$

$$\eta_y(\vec{r}) \simeq (r - r_\mu)_y \xi_{xx}$$

$$\delta_D(F(\vec{r}) - F(r_\mu)) = \delta_D(x - x_\mu) |\eta_x|^{-1}$$

$$\delta_D(x - x_\mu) = |\eta_x| \, \delta_D(F(\vec{r}) - F(r_\mu))$$

$$\delta_D(y - y_\mu) = |\xi_{xx}| \, \delta_D(\eta_y)$$

$$\langle n_{up}(\nu) \rangle = \int dF \, d\eta_x \, d\eta_y \, d\vec{\xi} \; P(A_\mu) \, \delta_D(F - F(r_\mu)) |\eta_x| \delta_D(\eta_y) |\xi_{yy}|$$

$$= \langle \delta_D(F - \nu\sigma_0) |\eta_x| \delta_D(\eta_y) |\xi_{yy}| \rangle$$

Theoretical prediction

$$\left\langle n_{up}(r,\nu)\, n_{up}(r',\nu') \right\rangle \equiv \xi_{\nu\nu'}(\vec{r},\vec{r}')$$

$$1+\xi_{\nu\nu'}(\vec{r},\vec{r}') = \frac{1}{n_{up}(r,\nu)\, n_{up}(r',\nu')} \int dA_{\mu_1}\, dA_{\mu_2}\, P(A_{\mu_1},A_{\mu_2})\Big|_{\text{Condition}}$$

For 2D : A_μ : $F(r), \eta_x(r), \eta_y(r), \xi_{xx}(r), \xi_{xy}(r), \xi_{yy}(r)$
$F(r'), \eta_x(r'), \eta_y(r'), \xi_{xx}(r'), \xi_{xy}(r'), \xi_{yy}(r')$: 12-elements

To be Continued ...

Summary

1) Motivation regarding to Stochastic field

2) Perturbative expansion

3) Theoretical calculation of various features, e.g. extrema

4) Clustering and correlation function (Non-Gaussianity and Anisotropy)

Thank you