25/07/2024
۱۴۰۳/۵/۴

بسم الله الرحمن الرحیم

Fisher Forecast (Continuation)

① A brief Reveiw on χ^2 and its Relation to Fisher Information Matrix

→ ② Estimation of Confidence Interval for model's Free parameters

→ ③ Ellipsoid Error Curve equation

④ Goodness of fit.

① A main Purpose of Fisher Forecast is Quantifying the Significance of constraint on model's free Parameters

according to
- → a pipeline for doing an Exp. (obs.)
- → Based on a given measures.

② In order to remind you.

Ⓐ From observational approach: the Likelihood can be written as (Considering Centeral limit theorem)

$$\mathcal{L}^{(D)} \equiv \frac{1}{\sqrt{(2\pi)^N \mathrm{Det}(C)}} e^{-\frac{\chi^2}{2}}$$

$$\chi^2 \equiv \delta y^T \cdot C^{-1} \cdot \delta y$$

$\delta y \equiv y_{obs} - y_{theor}(\theta, ..)$

$C \equiv$ Covariance Matrix of Observed Data

$$\mathcal{L}^{(D)}_{Rel} \equiv \frac{\mathcal{L}^{(D)}}{\mathcal{L}^{(0)}_{max}} \longrightarrow \chi^2 \longrightarrow \chi^2_{min}$$

$$\mathcal{L}^{(D)}_{Rel} = e^{-\frac{\Delta\chi^2}{2}}$$

$$\boxed{\Delta\chi^2 = \chi^2 - \chi^2_{min}}$$

(B) From Model's Free Parameter

$$\mathcal{L}^{(P)} = \sqrt{\frac{\mathrm{Det}(F)}{(2\pi)^M}}\, e^{-\frac{\Delta\theta^T \cdot F \cdot \Delta\theta}{2}}$$

$$\Delta\theta = \theta - \theta_{Best}$$

$F =$ Fisher Matrix $\longrightarrow \boxed{F = C_\theta^{-1}}$

(C) $$\mathcal{L}^{(D)}_{Rel} \triangleq \mathcal{L}^{(P)}$$

(D) $$F_{ij} = -\left\langle \frac{\partial^2 \mathcal{L}^{(P)}}{\partial\theta_i\, \partial\theta_j} \right\rangle \Bigg|_{\{\theta\}=\{\theta\}_{Best}}$$

(E) $$F_{ij} = \frac{1}{2} \left\langle \frac{\partial^2 \chi^2}{\partial\theta_i \cdot \partial\theta_j} \right\rangle_{ens} \Bigg|_{\{\theta\}=\{\theta\}_{Best}}$$

$$[F] = \frac{1}{2} \begin{bmatrix} \frac{\partial^2}{\partial\theta_1^2} & \frac{\partial^2}{\partial\theta_1\partial\theta_2} & \cdots & \frac{\partial^2}{\partial\theta_1\partial\theta_M} \\ \vdots & & & \\ \vdots & & & \\ \frac{\partial^2}{\partial\theta_M\partial\theta_1} & \cdots & \cdots & \frac{\partial^2}{\partial\theta_M^2} \end{bmatrix} \chi^2$$

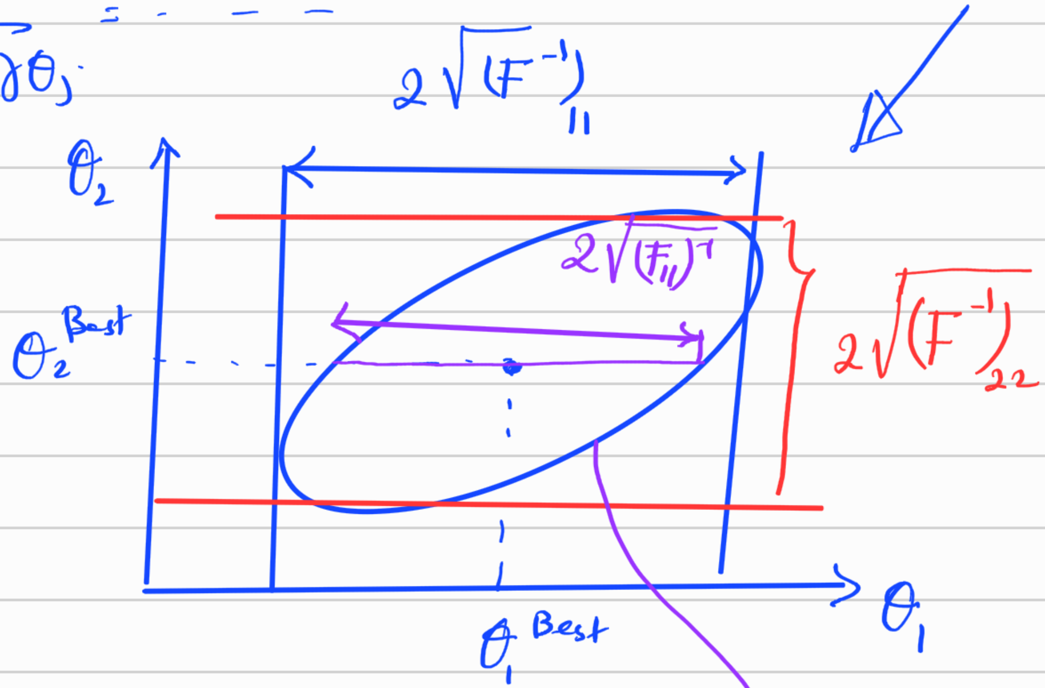
e.g. $$\underbrace{F_{11} = \frac{1}{2}\frac{\partial^2\chi^2}{\partial\theta_1^2}}\Bigg|_{\{\theta\}=\{\theta\}_{Best}}$$

From numerical approach.

$$\frac{\partial^2\chi^2}{\partial\theta_1^2} \simeq \underbrace{\lim_{\Delta\theta_1\to 0}}_{\Delta\theta_1\to 0} \underbrace{\frac{\chi^2(\theta_1+\Delta\theta_1)+\chi^2(\theta_1-\Delta\theta_1)-2\chi^2(\theta_1)}{(\Delta\theta_1)^2}}\Bigg|_{\theta_1=\theta_1^{Best}}$$

$$\frac{\partial^2\chi^2}{\partial\theta_i\,\partial\theta_j} = \dots$$

F



$$\sqrt{(F_{11})^{-1}} \neq \sqrt{(F^{-1})_{11}}$$

1D
2D
3D

Equation for Plotting such Ellipsoid

③ Ellipsoid Error Curve !

Suppose that we have $M=2$, $\{\theta\} = \{\theta_1, \theta_2\}$

$$[F] = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}_{2\times 2\,(M\times M)} = C_\theta^{-1}$$

$$\boxed{F_{12} = F_{21}}$$

$$C = F^{-1} = \begin{bmatrix} \dfrac{F_{22}}{F_{11}F_{22} - F_{12}^2} & -\dfrac{F_{12}}{F_{11}F_{22} - F_{12}^2} \\ -\dfrac{F_{21}}{F_{11}F_{22} - F_{12}^2} & \dfrac{F_{11}}{F_{11}F_{22} - F_{12}^2} \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_2^2 \end{bmatrix}$$

$(F^{-1})_{11}$

$$(F_{11})^{-1} \neq \sigma_1^2 = \frac{F_{22}}{F_{11}F_{22} - F_{12}^2} \xrightarrow{F_{12} = F_{21} = 0} \sigma_1^2 = \frac{1}{F_{11}} = (F_{11})^{-1}$$

$$(F_{22})^{-1} \neq \sigma_2^2 = \frac{F_{11}}{F_{11}F_{22} - F_{12}^2} \xrightarrow{F_{12} = F_{21} = 0} \sigma_2^2 = \frac{1}{F_{22}} = (F_{22})^{-1}$$

$(F^{-1})_{22}$

$$\mathcal{L}^{(P)} \sim e^{-\frac{\Delta\theta^T \cdot F \cdot \Delta\theta}{2}}$$

$$\Delta\theta^T \cdot F \cdot \Delta\theta$$

$$\boxed{\Delta\theta^T \cdot F \cdot \Delta\theta = \Delta\chi^2}$$

For $M=2$ (θ_1, θ_2)

$$\Delta\chi^2 = \frac{\delta\theta_1^2}{\sigma_{11}^2(1-\rho^2)} + \frac{\delta\theta_2^2}{\sigma_{22}^2(1-\rho^2)} - \frac{2\rho}{(1-\rho^2)}\frac{\delta\theta_1}{\sigma_{11}}\frac{\delta\theta_2}{\sigma_{22}}$$

$$\rho \equiv \frac{\sigma_{12}^2}{\sigma_{11}^2\sigma_{22}^2}$$

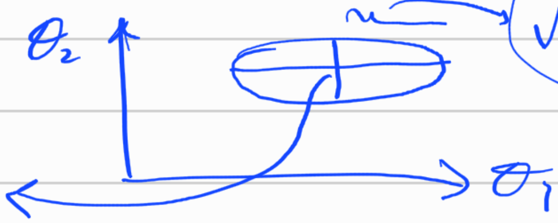
$$\sigma_{12}^2 \equiv \rho\,\sigma_{11}^2\sigma_{22}^2$$

$$-1 < \rho < +1$$

$\rho = 0 \rightarrow$

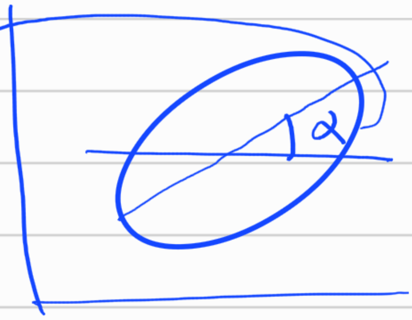
$$\boxed{\frac{\delta\theta_1^2}{\sigma_{11}^2} + \frac{\delta\theta_2^2}{\sigma_{22}^2} = \Delta\chi^2}$$

$\sim \sqrt{\Delta\chi^2\sigma_{11}^2}$



$\sqrt{\Delta\chi^2\sigma_{22}^2}$

$$\tan 2\alpha = \frac{2\rho\,\sigma_{11}\,\sigma_{22}}{\sigma_{22}^2 - \sigma_{11}^2}$$



$\Delta\chi^2 = ?$

$$P_{\Delta\chi^2}\% = \int_0^{\Delta\chi^2} d\Delta\chi^2 \; P_\nu(\Delta\chi^2)$$

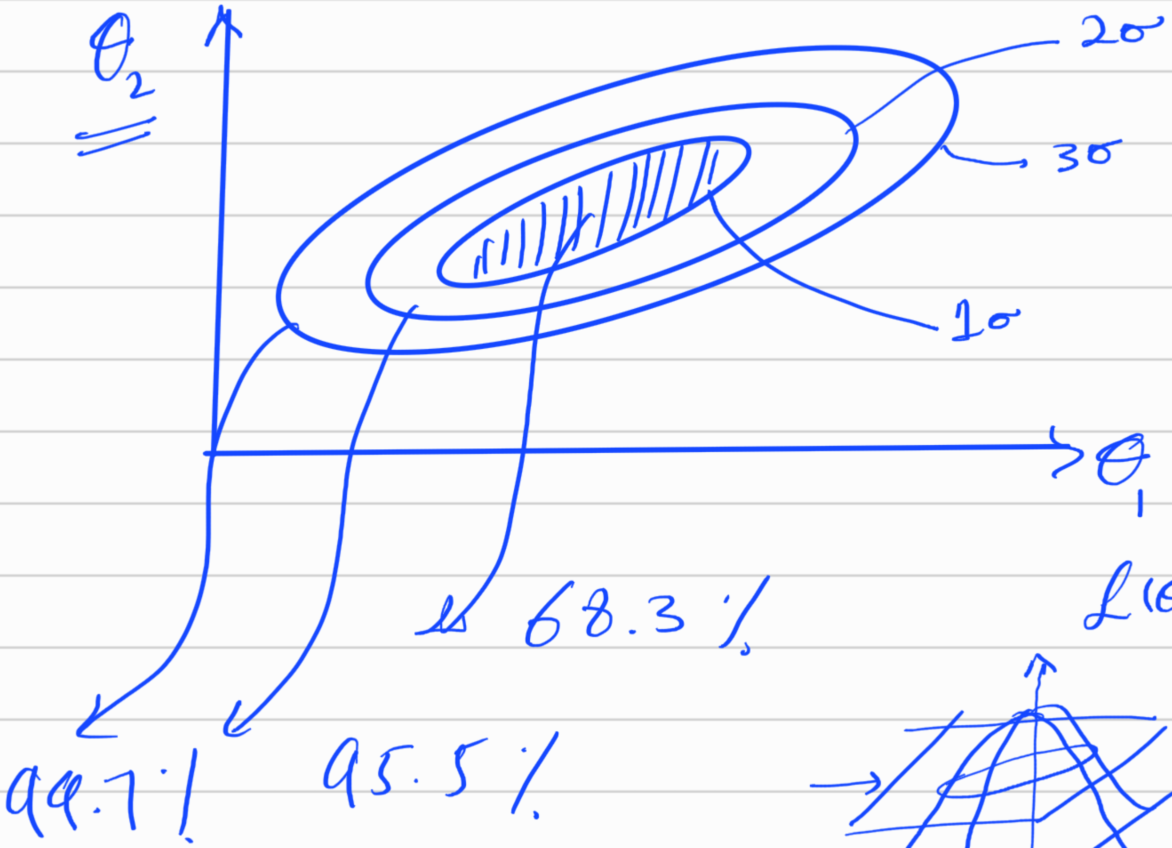
ν = Degree of freedom = $N - M$

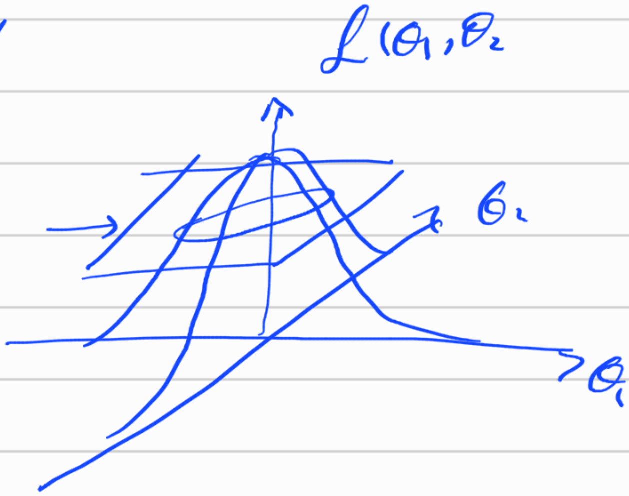
$$= 1 - \frac{1}{2}\, \underbrace{Q\left(-\frac{(\nu-2)}{2}, \frac{\Delta\chi^2}{2}\right)}_{\text{InComplete Beta Funct}} \quad *$$

Ex: 95.5% (2σ)

$$\underline{\underline{95.5}} = P_{\Delta\chi^2} = \int_0^{\Delta\chi^2} d\Delta\chi^2 \; P(\Delta\chi^2)$$

$$\left\{\begin{array}{lll} M=2, & (2\sigma) & \Delta\chi^2 = 6.17 \\ M=2, & (1\sigma) & \Delta\chi^2 = 2.3 \\ M=2 & (3\sigma) & \Delta\chi^2 = 11.8 \end{array}\right\}$$





$[F] \longrightarrow [F]^{-1} = [C] \longrightarrow$ Ellipsoid Curve Eq

$\Delta\chi^2$

$P\% \longrightarrow \Delta\chi^2$

$\nu = N - M$

(4) $$F_{ij} = \frac{1}{2}\left\langle \frac{\partial^2 \chi}{\partial\theta_i \partial\theta_j} \right\rangle_{ens} \Bigg|_{\{\theta\}=\{\theta\}_{Best}}$$

N

$$\Delta\chi^2 = \delta y^T \cdot C^{-1} \cdot \delta y - \chi_{min}$$

$N=1$ $$\Delta\chi^2 = \chi^2 - \chi^2_{min} =$$

$$\chi^2 = \delta y^T \cdot C^{-1} \cdot \delta y$$

$$= \frac{(y_{obs} - y_{the}(\theta))^2}{\sigma_{obs}^2}$$

$$\frac{\partial \chi^2}{\partial\theta} = \frac{2}{\sigma_{obs}^2}(y_{obs} - y_{the}(\theta)) \frac{\partial y_{Theo}}{\partial\theta}$$

$$\frac{\partial^2 \chi^2}{\partial\theta^2}\Bigg|_{\theta=\theta_{Best}} = \frac{2}{\sigma_{obs}^2}(y_{obs} - y_{th}(\theta)) \frac{\partial^2 y_{The}}{\partial\theta^2}$$

$$+ \frac{2}{\sigma_{obs}^2}\left(-\frac{\partial y_{Th}}{\partial\theta}\right)\left(\frac{\partial y_{Th}}{\partial\theta}\right)$$

$M=1,\ N=1$

$$F_{11} = \frac{1}{2}\left\langle \frac{\partial^2 \chi^2}{\partial\theta^2} \right\rangle_{ens} \Bigg|_{\theta=\theta_{Best}}$$

$$F_{11} = \frac{1}{2} \left\langle \frac{2}{\sigma_{obs}^2} (y_{obs} - y_{th}) \underbrace{\frac{\partial^2 y_{th}}{\partial \theta^2}} \right\rangle_{em} \Bigg|_{\theta = \theta_{Best}}$$

$$- \frac{1}{2} \left\langle \frac{2}{\sigma_{obs}^2} \underbrace{\left(\frac{\partial y_{th}}{\partial \theta} \right)^2} \right\rangle_{e} \Bigg|_{\theta = \theta_B}$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{2}}{\sigma_{obs}^2} \frac{\partial^2 y_{th}}{\partial \theta^2} \left\langle y_{obs} - y_{th} \right\rangle_{ens, \theta = \theta_{Best}}$$

$$\underbrace{- \frac{1}{\sigma_{obs}^2} \underbrace{\left(\frac{\partial y_{th}}{\partial \theta} \right)^2}} \Bigg|_{\theta = \theta_{Best}}$$

y_{obs}

$\frac{\partial^2 C}{\partial \theta^2}$

$$C = \begin{bmatrix} \langle \delta y_1 \delta y_1 \rangle & \langle \delta y_1 \delta y_2 \rangle \\ \vdots & \end{bmatrix}_{N \times N}$$

[Expected Covariance for observation Exp. .]

(5) General Formalism : Multivariate Gaussian Form.

$$\mathcal{L} = \frac{1}{\sqrt{(2\pi)^N \, \mathrm{Det}(C)}} \, e^{-\frac{\delta y^T \cdot C_y^{-1} \cdot \delta y}{2}}$$

(Multivariate Gaussian function)

$$F_{ij} = -\left\langle \frac{\partial^2 \ln \mathcal{L}}{\partial \theta_i \, \partial \theta_j} \right\rangle$$

$$\ln \mathcal{L} = -\frac{1}{2} \ln(\mathrm{Det}\, C_y) - \frac{N}{2} \ln(2\pi) - \frac{1}{2} \delta y^T \cdot C_y^{-1} \cdot \delta y$$

$$C_y \equiv \langle \delta y^T \delta y \rangle = \langle \delta y \otimes \delta y \rangle$$

$$\begin{cases} \ln(\mathrm{Det}\, C_y) = \mathrm{Tr}(\ln C_y) \\ \delta y^T \cdot C_y^{-1} \cdot \delta y = \mathrm{Tr}(\delta y^T \cdot C^{-1} \cdot \delta y) \end{cases}$$

$$F_{ij} = \mathrm{Tr}\left[C^{-1} C_{,i} C^{-1} C_{,j} + C^{-1} \langle \delta y \, \delta y \rangle_{,ij} \right]$$

$$\boxed{F_{ij} = \mathrm{Tr}\left[C^{-1} C_{,i} C^{-1} C_{,j} \right] + \delta y^T_{,i} \cdot C^{-1} \cdot \delta y_{,j}}$$

⑥ General Properties of Fisher Matrix

$$F = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1M} \\ F_{21} & F_{22} & & F_{2M} \\ & & \ddots & \\ F_{M1} & & \cdots & F_{MM} \end{bmatrix}_{M\times M}$$

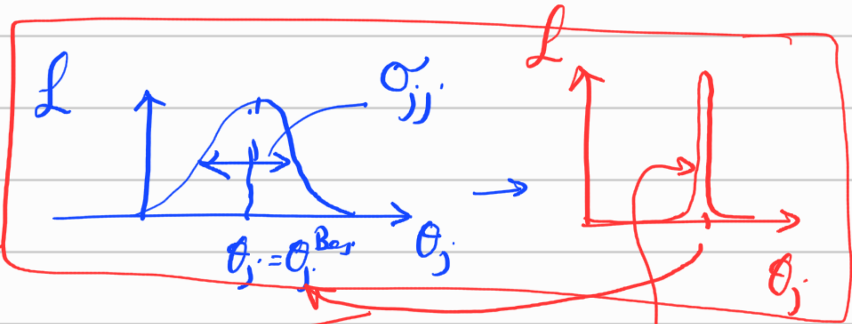
Ⓐ Marginalization $\boxed{p(x) = \int P(x,y)\,dy}$

jth

$$\mathcal{L}(\theta_1 \cdots \cancel{\theta_j} \cdots \theta_M) = \int d\theta_j\, \mathcal{L}(\theta_1 \cdots \theta_j \cdots \theta_M)$$

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{1j} & \cdots & F_{1M} \\ F_{21} & F_{22} & F_{2j} & & F_{2M} \\ \vdots & & & & \\ F_{j1} & & & & F_{jM} \\ \vdots & & & & \\ F_{M1} & & F_{Mj} & & F_{MM} \end{bmatrix} = F\begin{bmatrix} & \\ & \end{bmatrix}_{(M-1)\times(M-1)}$$

Ⓑ $\boxed{\theta_j = \text{fixed}}$



$F_{jj} \longrightarrow \infty$

Fixing $\theta_j \longrightarrow \sigma_{jj} = 0 \longrightarrow \boxed{F_{jj} = \infty}$

(c) [F] without any Prior information

Recall that $P(\theta|D) = \dfrac{\mathcal{L}(D|\theta)\,\tilde{P}(\theta)}{\int d\theta\, \mathcal{L}(D|\theta) P(\theta)}$ Prior Inf

assumption

Without prior $\chi^2 = \delta y^T \cdot C^{-1} \cdot \delta y$

With Prior $\chi^2 = \delta y^2 \cdot C^{-1} \cdot \delta y + \dfrac{(\theta_j - \bar{\theta}_j)^2}{\sigma_{\theta_j}^2}$

jth θ_j

Gaussian

$[F] \overset{?}{\longleftrightarrow} [F]$

Without Prior With Prior

$$F_{jj} \longrightarrow \boxed{F_{jj} + \frac{1}{\sigma_{\theta_j}^2}}$$

(7) Goodness of fit.

(A) $\nu = N - M \longrightarrow \chi^2 \rightarrow \chi_\nu^2 = \dfrac{\chi^2}{\nu}$

$\chi_\nu^2 = 1 \pm \sqrt{\chi_\nu^2}$

(B) $AIC = 2M - 2\ln \mathcal{L}_{max}$ 　　 M = # of model Free Param

(C) $BIC = M\ln(N) - 2\ln \mathcal{L}_{max}$

↓

(SIC, SBC, SBIC)

N = # of Data.