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⑥ Bayesian Model averaging (BMA)

⑦ Analytical Algorithm

$\boxed{\text{⑧ Computational Algorithm.}}$ * برای جلسه آینده

Cosmo MC

Monte Carlo → $\boxed{\text{Random Generater}}$

Monte Python ...

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① A brief Review.

Bayesian approach.

☆ According to the Bayes theorem

☆ $\{D\} : \{D_1 \ldots \; D_N\}$ $\boxed{N \equiv \# \text{ of observations}}$

☆ $\{\theta\} : \{\theta_1 \ldots \; \theta_M\}$ $M \equiv \#$ of model's free parameters.

$$P(\{\theta\}|D) = \frac{\mathcal{L}(D|\{\theta\}) \times P(\{\theta\})}{\int d\{\theta\}\, \mathcal{L}(D|\{\theta\})\, P(\{\theta\})}$$

Posterior Probability Distribution funct — تابع احتمال پسینی

→ likelihood → تابع احتمال درست نمایی

→ Prior Probability Distribut — تابع احتمال پیشینی

Prior Distribution func → Cts.

$$P(\theta|D) \sim \mathcal{L}(D|\theta)$$

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⑥ Bayesian Model Averaging (BMA)

→ Model 1 → $\{\theta_1^{(1)}, \theta_2^{(1)}, \theta_3^{(1)}\}$ → $\{\theta_1 = \theta_1^{Best}, \theta_2 = \theta_2^{Best}, \theta_3 = \theta_3^{Best}\}$

→ Model 2

→ Model 3 → $\{\theta_1^{(2)}, \theta_2^{(2)}\}$ → $\{\theta_1^{(2)}(Best). -\}$

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$\Theta_1^{(1)}$ (Best) → Model 1, Model 2, Model 3

7 Analytical Algorithm.

بیشینه سازی تابع Posterior

Maximizing Posterior function

$D:\{x_i, y_i\} \quad i = 1, \ldots, N$

$\Theta:\{\theta_j\} \qquad Y(x;\{\Theta\}) = \sum_{k=1}^{M} \theta_k f_k(x)$ ← $\boxed{\text{Generic form}}$

$$p(\Theta \mid D) = \frac{p(D \mid \Theta)\, p(\Theta)}{\int d\Theta\; p(D \mid \Theta)\, p(\Theta)}$$

$if \quad \boxed{p(\Theta) = cts}$

در صورتی که تابع prior وجود نداشته باشد در نتیجه تحلیل posterior به تحلیل Likelihood تبدیل می شود

$p(\Theta \mid D) \sim p(D \mid \Theta)$

$$p(D \mid \Theta) \equiv \prod_{i=1}^{N} P_i = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{[y_i - Y(x_i;\{\Theta\})]^2}{2\sigma_i^2}\right) = e^{-\frac{\chi^2}{2}}$$

N-Joint

۱) با فرض اینکه اندازه گیری ها از یکدیگر مستقل باشند

۲) با توجه به قضیه حد مرکزی می توان در نظر گرفت که هر نقطه اندازه گیری حول مقدار بهینه اش به صورت گوسی توزیع شده است

Centeral limit theorem

$$\chi^2(\{\Theta\}) \equiv \sum_{i=1}^{N} \frac{[y_i - Y(x_i;\{\Theta\})]^2}{\sigma_i^2} = \sum_{i=1}^{N} \frac{\left[y_i - \sum_{k=1}^{M} \theta_k f_k(x_i)\right]^2}{\sigma_i^2}$$

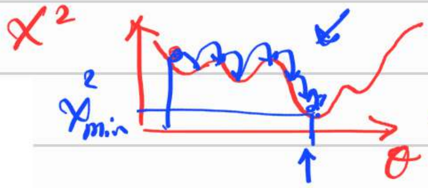
$$p(D \mid \Theta) \sim e^{-\frac{\chi^2(\{\Theta\})}{2}}$$

بیشینه شدن Likelihood معادل با کمینه شدن chi^2 است

23

$\mathcal{L} \sim e^{-\chi^2/2} \rightarrow \boxed{\mathcal{L}_{Max} \sim e^{-\chi^2_{min}/2}}$

$\boxed{Y(x,\{\theta\}) = \sum \theta f}$

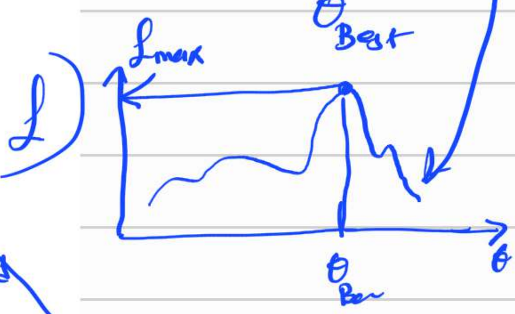


$$\chi^2(\{\Theta\}) \equiv \sum_{i=1}^{N} \frac{\left[y_i - \sum_{k=1}^{M} \theta_k f_k(x_i)\right]^2}{\sigma_i^2}$$

$$b_i \equiv \frac{y_i}{\sigma_i}$$

$$A_{ik} \equiv \frac{f_k(x_i)}{\sigma_i}$$

$$\boxed{Min\|b - A\Theta\| \rightarrow \Theta_{Best}}$$



Coupled Eq

$$\boxed{\frac{\partial \chi^2}{\partial \theta_j} = 0 \quad j = 1, \ldots, M}$$

Normal Equations

Singular Value Decomposition(SVD)

24

$Y(x,\{\theta\})$ chi^2 کمینه کردن

$$\chi^2(\{\Theta\}) = \sum_{i=1}^{N} \frac{\left[y_i - \sum_{k=1}^{M}\theta_k f_k(x_i)\right]^2}{\sigma_i^2} \quad \rightarrow \quad \chi^2(\{\Theta\} = \{\Theta\}_{Best}) = \chi^2_{\min}$$

$$\left.\frac{\partial \chi^2(\{\Theta\})}{\partial \theta_j}\right|_{\{\Theta\}=\{\Theta_{Best}\}} = 0$$

$$= \sum_{i=1}^{N}\left[y_i - \sum_{k=1}^{M}\theta_k f_k(x_i)\right] \times \frac{f_j(x_i)}{\sigma_i^2}$$

$$\sum_{i=1}^{N}\frac{y_i f_j(x_i)}{\sigma_i^2} = \sum_{i=1}^{N}\sum_{k=1}^{M}\frac{\theta_k f_k(x_i) f_j(x_i)}{\sigma_i^2}$$

$$\beta_j \equiv \sum_{i=1}^{N}\frac{y_i f_j(x_i)}{\sigma_i^2}$$

$$\alpha_{jk} \equiv \sum_{i=1}^{N}\frac{f_j(x_i) f_k(x_i)}{\sigma_i^2}$$

$$\beta_j = \sum_{k=1}^{M}\alpha_{jk}\theta_k$$

$$[\theta]_{Best} = [\Theta] = [\alpha]^{-1}[\beta]$$

25

اکنون سوال این است که حوزه اعتبار کمیت Θ چقدر است؟

$Y(x,\{\theta\})$

$$\beta_j \equiv \sum_{i=1}^{N}\frac{y_i f_j(x_i)}{\sigma_i^2}$$

$$\alpha_{jk} \equiv \sum_{i=1}^{N}\frac{f_j(x_i) f_k(x_i)}{\sigma_i^2}$$

$f_i \times f_j = f_{ij}$

$$[\Theta] = [\alpha]^{-1}[\beta]$$

Best

$$\sigma^2(\theta_i) = \sum_{j=1}^{N}\sigma_j^2\left(\frac{\partial\theta_i}{\partial y_j}\right)^2 + \sum_{j=1}^{N}\sum_{l\neq j}^{N}\left(\frac{\partial\theta_i}{\partial y_j}\right)\left(\frac{\partial\theta_i}{\partial y_l}\right)Cov(y_j y_l) \quad \rightarrow \quad \left(\frac{\partial\theta_i}{\partial y_j}\right) = \sum_{k=1}^{M}[\alpha_{ik}]^{-1}\frac{\partial\beta_k}{\partial y_j}$$

$$\left(\frac{\partial\theta_i}{\partial y_j}\right)^2 = \sum_{l=1}^{M}\sum_{k=1}^{M}[\alpha_{ik}]^{-1}[\alpha_{il}]^{-1}\frac{\partial\beta_k}{\partial y_j}\frac{\partial\beta_l}{\partial y_j}$$

$$\sigma^2(\theta_i) = \sum_{l=1}^{M}\sum_{k=1}^{M}[\alpha_{ik}]^{-1}[\alpha_{il}]^{-1}\sum_{j=1}^{N}\sigma_j^2\frac{\partial\beta_k}{\partial y_j}\frac{\partial\beta_l}{\partial y_j} \quad \rightarrow \quad \boxed{\sigma^2(\theta_i) = \sum_{l=1}^{M}\sum_{k=1}^{M}[\alpha_{ik}]^{-1}[\alpha_{il}]^{-1}[\alpha_{kl}]}$$

$$\rightarrow \sum_{k=1}^{M}[\alpha_{il}]^{-1}[\alpha_{kl}] = \delta_{ik}$$

$$\boxed{\sigma^2(\theta_i) = [\alpha]^{-1}_{ii}}$$

$[\alpha] \rightarrow [\alpha]^{-1}_{ii} \rightarrow \sigma^2 \rightarrow 1\sigma$ confidence

$\boxed{68.3\%}$

26

polynomial function

$$D:\{x_i,y_i\}\rightarrow \quad i=1,\dots,N$$

$$\Theta:\{\theta_k\} \quad \rightarrow \quad k=1,\dots,M \quad \rightarrow \quad Y(x;\Theta)=\theta_1+\theta_2 x^1+\dots+\theta_M x^{M-1}$$

$$P(D\mid\Theta)=\prod_{i=1}^{N}P(D_i\mid\Theta)=\prod_{i=1}^{N}\left(\frac{1}{\sigma_i\sqrt{2\pi}}\right)\exp\left(-\frac{1}{2}\sum_{i=1}^{N}\frac{\left[y_i-\sum_{k=1}^{M}\theta_k x_i^{k-1}\right]^2}{\sigma_i^2}\right)$$

$$\text{for} \quad M=3 \quad \chi^2(\theta_1,\theta_2,\theta_3)=\sum_{i=1}^{N}\frac{\left[y_i-\sum_{k=1}^{3}\theta_k x_i^{k-1}\right]^2}{\sigma_i^2}$$

$$0=\frac{\partial\chi^2(\theta_1,\theta_2,\theta_3)}{\partial\theta_1}=-2\sum_{i=1}^{N}\frac{\left[y_i-\sum_{k=1}^{3}\theta_k x_i^{k-1}\right]}{\sigma_i^2} \quad (1)$$

$$0=\frac{\partial\chi^2(\theta_1,\theta_2,\theta_3)}{\partial\theta_2}=-2\sum_{i=1}^{N}x_i\frac{\left[y_i-\sum_{k=1}^{3}\theta_k x_i^{k-1}\right]}{\sigma_i^2} \quad (2)$$

$$0=\frac{\partial\chi^2(\theta_1,\theta_2,\theta_3)}{\partial\theta_3}=-2\sum_{i=1}^{N}x_i^2\frac{\left[y_i-\sum_{k=1}^{3}\theta_k x_i^{k-1}\right]}{\sigma_i^2} \quad (3)$$

27

y

$$Y_{th}=\theta_1+\theta_2 x+\theta_3 x^2$$

$$\left\{\begin{array}{l} ax+by=c \\ dx+ey=f \end{array}\right.$$

x, y

$$\theta_1=\frac{1}{\Delta}\begin{vmatrix}\sum_{i=1}^{N}\frac{y_i}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^2}{\sigma_i^2}\\ \sum_{i=1}^{N}y_i\frac{x_i}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^2}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^3}{\sigma_i^2}\\ \sum_{i=1}^{N}y_i\frac{x_i^2}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^3}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^4}{\sigma_i^2}\end{vmatrix} \qquad \theta_2=\frac{1}{\Delta}\begin{vmatrix}\sum_{i=1}^{N}\frac{1}{\sigma_i^2} & \sum_{i=1}^{N}\frac{y_i}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^2}{\sigma_i^2}\\ \sum_{i=1}^{N}\frac{x_i}{\sigma_i^2} & \sum_{i=1}^{N}y_i\frac{x_i}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^3}{\sigma_i^2}\\ \sum_{i=1}^{N}\frac{x_i^2}{\sigma_i^2} & \sum_{i=1}^{N}y_i\frac{x_i^2}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^4}{\sigma_i^2}\end{vmatrix}$$

3×3

$$\theta_3=\frac{1}{\Delta}\begin{vmatrix}\sum_{i=1}^{N}\frac{1}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i}{\sigma_i^2} & \sum_{i=1}^{N}\frac{y_i}{\sigma_i^2}\\ \sum_{i=1}^{N}\frac{x_i}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^2}{\sigma_i^2} & \sum_{i=1}^{N}y_i\frac{x_i}{\sigma_i^2}\\ \sum_{i=1}^{N}\frac{x_i^2}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^3}{\sigma_i^2} & \sum_{i=1}^{N}y_i\frac{x_i^2}{\sigma_i^2}\end{vmatrix} \qquad \Delta=\begin{vmatrix}\sum_{i=1}^{N}\frac{1}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^2}{\sigma_i^2}\\ \sum_{i=1}^{N}\frac{x_i}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^2}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^3}{\sigma_i^2}\\ \sum_{i=1}^{N}\frac{x_i^2}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^3}{\sigma_i^2} & \sum_{i=1}^{N}\frac{x_i^4}{\sigma_i^2}\end{vmatrix}$$

84