بسم الله الرحمن الرحیم

Operator Product Expansion
(OPE)

در فیزیک پدیده‌های بحرانی توابع نما را استخراج کنیم $\longleftarrow$ $[x_k]$

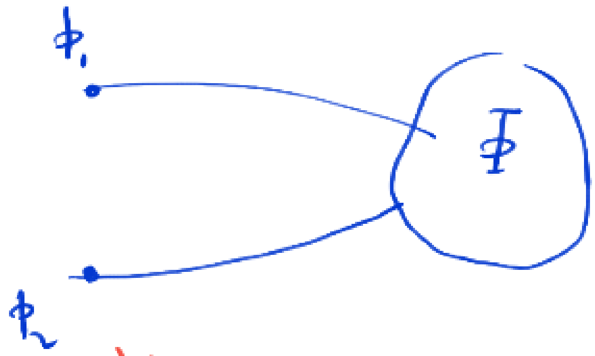
به عنوان یک ابزار برای محاسبه $[x_k]$ در این درس مورد توجه قرار می‌دهیم

$$\left\langle \phi_i(r_i)\, \phi_j(r_j)\quad \Phi \right\rangle = \; ?$$

$$\Phi = \prod_{\ell}^{N} \phi_\ell(r_\ell)$$

$$|r_1 - r_2| \ll |r_1 - r_\ell|$$

$$|r_1 - r_2| \ll |r_2 - r_\ell|$$



※ Theorem

$$\left\langle \phi_i(r_i)\, \phi_j(r_j)\quad \Phi \right\rangle = \sum_{k} \overbrace{C_{ijk}(|r_i - r_j|)}^{\text{ضریب ربط}} \left\langle \phi_k\left(\frac{r_i + r_j}{2}\right) \Phi \right\rangle$$

$$\rightarrow \; C_{ijk}(|r_i - r_j|) \equiv \frac{c_{ijk}}{|r_i - r_j|^{x_i + x_j - x_k}}$$

فرض $\phi_i(r_i) \sim \frac{1}{\;^{x_i}}$

5.2. Cardy

$$RG \overset{?}{\longleftrightarrow} OPE$$

$$Z = \mathrm{Tr}\, e^{-\mathcal{H}_0 - \sum g_i \sum_r a^{x_i} \phi_i(r)}$$

$$\boxed{\mathcal{H} = \mathcal{H}_0 + \sum_i g_i \sum_r a^{x_i} \phi_i(r)}$$ Effective $\mathcal{H}$

$$\sum_r \longrightarrow \int \frac{d^d r}{a^d}$$

$$\mathcal{H} = \mathcal{H}_0 + \sum_i g_i \int \frac{d^d r}{a^d} \underbrace{a^{x_i}}_{l^{x_i}} \phi_i(r)$$

field

Coupling Constant [K]

$$Z = Z_0 \Big[1 - \sum_i g_i \int \langle \phi_i(r) \rangle_0 \frac{d^d r}{a^{d-x_i}}$$

$$+ \frac{1}{2} \sum_{ij} g_i g_j \int \frac{d^d r_i \, d^d r_j}{a^{2d - x_i - x_j}} \langle \phi_i(r_i) \phi_j(r_j) \rangle_0$$

$$+ \cdots \Big]$$

اندیس ۰ در اینجا متوسط‌گیری بر روی نقطه‌ی بدون اختلال در نقطه بحرانی فرض می‌کنیم

$$\{ a \longrightarrow la, \quad l = (1 + \delta l) \}$$

RG : $\begin{cases} \text{Coarse grain} \\ \text{Rescale} \\ \text{Renormalization} \end{cases}$ $\rightarrow \int_0 \rightarrow \int_a \rightarrow \int_{al}$

$$Z \xrightarrow{RG} Z' \Big|_{\text{fixed par}} = Z \Big|_{\text{fixed par}} \rightarrow$$

β-function

$$\boxed{\beta_\ell = \frac{dg_k}{d\ell} = x_k g_k - \underbrace{\sum_{ij} C_{ijk}\, g_i\, g_j}_{} + \cdots}$$

Ex: کاربرد OPE در ترانزیشن‌های فاز

$\mathcal{H} = \mathcal{H}_0 + \mathcal{U}$

$\underset{d-4}{\overset{-\epsilon}{\sim}} \rightarrow$ chapter 7 Goldenfeld Chapter n

$$\mathcal{H} = \int d^d r \left[\frac{1}{2} (\nabla S)^2 + t\, a^{-2} S^2 + u\, a^{d-4} S^4 + h\, a^{-\frac{d}{2}-1} S \right] \tag{5.19}$$

$$t \equiv -\frac{(2\lambda + J)}{J R^2}, \quad u \equiv \frac{\lambda}{J^2 R^2}, \quad h \equiv -H / J^{1/2} R$$

$$g = \begin{cases} x_t = ? \\ x_h = ? \\ x_u = ? \end{cases}$$

At Zero-order using Dimensional Analysis

$$\text{RG}\begin{cases} a \to la \\ s \to s' = s/\upsilon \\ r \to r' = r/l \end{cases} \longrightarrow \quad d^dr = d^dr'\, l^d$$

$$\mathcal{H} \to \mathcal{H}' = \int d^dr'\, \underset{\uparrow}{l^d} \Big[\tfrac{1}{2} \underbrace{l^{-2}\upsilon^2}_{1} (\nabla s')^2 + \underbrace{t a^{-2} \upsilon^2}_{t' a^{-2}} s'^2 + \underbrace{u a^{d-4} \upsilon^4}_{u' a^{d-4}} s'^4 + \underbrace{h a^{-\frac{d}{2}-1} \upsilon}_{h' a^{-\frac{d}{2}-1}} s' \Big]$$

$$\mathcal{H} = \mathcal{H}'$$

شرط لازم و کافی اینکه

$$l^{d-2} \upsilon^2 = 1 \longrightarrow \upsilon = l^{1-d/2}$$

$$t' = l^d \upsilon^2 t = l^{d+2-d} t = l^2 t = l^{x_t} t \to \boxed{x_t = 2}$$

$$u' = l^{4-d} u = l^{x_u} u \longrightarrow \boxed{x_u = 4-d = \epsilon}$$

$$h' = l^{\frac{d}{2}+1} h = l^{x_h} h \longrightarrow \boxed{x_h = \tfrac{d}{2}+1}$$

* پس تا مرتبه صفر فقط با توجه به تحلیل ابعادی این نماها استخراج می شوند *

$\checkmark$ — RG — $\mathcal{H} = \mathcal{H}_0 + \langle V \rangle_0 + \frac{1}{2}\langle V^2 \rangle_0^c - \cdots$ ← روش قبلی

برای مراتب بالاتر

$? = \text{OPE}$

$$* \quad \frac{dg_k}{dl} = x_k g_k - \sum_{ij} \overset{?}{c_{ijk}} g_i g_j \qquad \{g\} = \{t, h, u\}$$

$$\boxed{x = d - x_i} \longleftarrow \int d^dx\, g_i\, \phi$$

$$\boxed{x_k \cdots p}$$

$$\int d^d x' \, g'_k \, \phi'$$

$$g'_k = \ell^{d} v^{-1} g_k$$

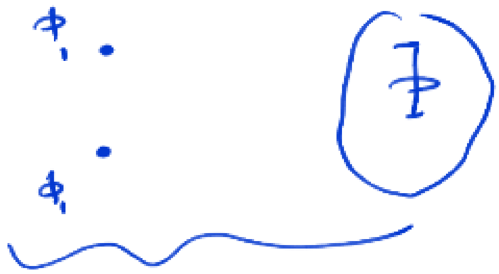
$$= \ell^{x_k} g_k$$

$$\boxed{x_k = d - x_\phi}$$

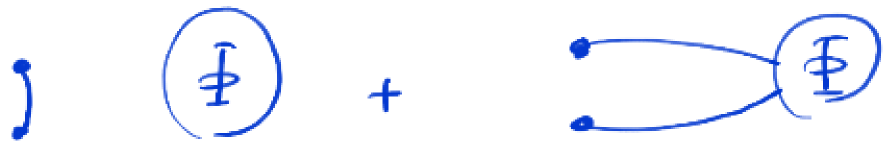
تعریف:

$$\begin{cases} \phi_n = :S^n: = \langle S^n \rangle_0^c \\ \phi_2 = :S^2: = \langle S^2 \rangle_0^c = \langle S^2 \rangle_0 - \langle S \rangle_0^2 \end{cases}$$

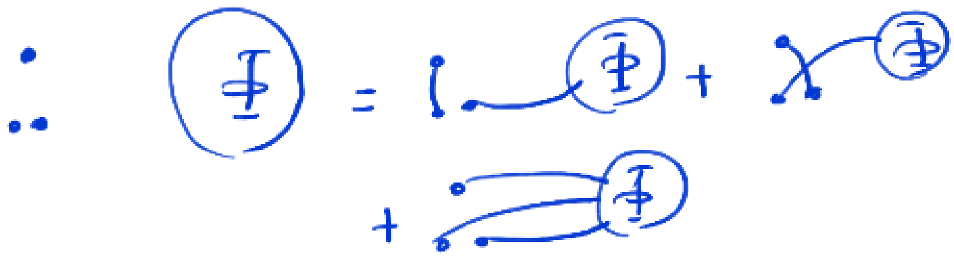
$$\phi_1 \cdot \phi_1 = \langle \phi_1 \phi_1 \; \Phi \rangle_0 =$$



$\phi_1 \cdot \phi_1 =$ 

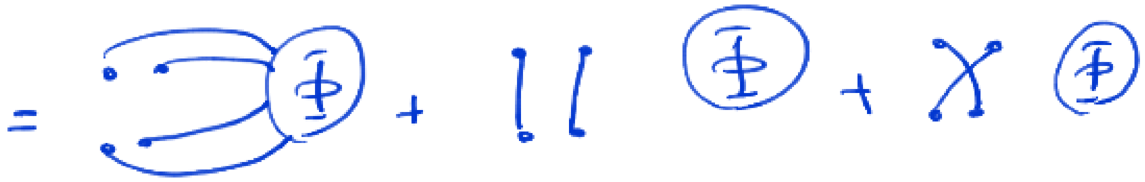
$=$ 

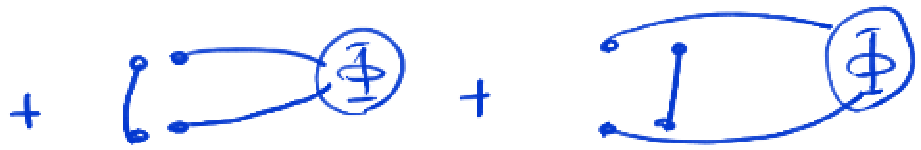
$$= 1 + \phi_2$$

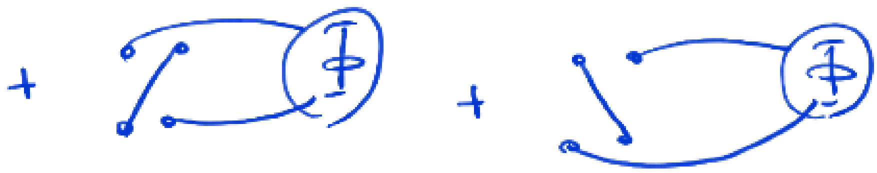
$\phi_1 \cdot \phi_2 =$ 

$$= \phi_1 + \phi_1 + \phi_3 = 2\phi_1 + \phi_3$$

$\phi_2 \cdot \phi_2 =$ 

$=$ 





$$= \phi_4 + 1 + 1 + \phi_2 + \phi_2 + \phi_2 + \phi_2$$

$$= 2 + \underset{\uparrow \binom{4}{2}}{4}\phi_2 + \phi_4$$

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① با توجه به شکل ها هامیلتونی داده شده ضرایب خنثی شدگی را مشخص می کنیم

$\{t, h, u\}$

② اسم میدان‌ها را در هامیلتونی مشخص می کنیم

$$\begin{array}{ccc} s & s^2 & s^4 \\ \downarrow & \downarrow & \downarrow \\ \phi_1 & \phi_2 & \phi_4 \\ \downarrow & \downarrow & \downarrow \end{array}$$

$h \qquad t \qquad u$

۴) گام ۴ تمام همبستگی‌های مجذوری میدان‌ها را دیاگرام کنیم:

$$\underbrace{s\,s}_{1},\ \underbrace{s\,s^2}_{2},\ \underbrace{s\,s^4}_{3},\ \underbrace{s^2 s^2}_{4},\ \underbrace{s^2 s^4}_{5},\ \underbrace{s^4 s^4}_{6}$$

① $\underset{h}{s}\ \underset{h}{s} \equiv (\phi_1 \cdot \phi_1) = \ \vdots \qquad \Phi = 1 + \phi_2$

② $\underset{h}{s}\ \underset{t}{s^2} \equiv (\phi_1 \cdot \phi_2) = \qquad \Phi = 2\phi_1 + \phi_3$

③ $\underset{h}{s}\ \underset{u}{s^4} \equiv (\phi_1 \cdot \phi_4) = \qquad \Phi = 4\phi_3 + \phi_5$

④ $\underset{t}{s^2}\ \underset{t}{s^2} \equiv (\phi_2 \cdot \phi_2) = \qquad \Phi = 2 + \phi_4 + 4\phi_2$

s^2

⑤ $\underset{t}{s^2}\ \underset{u}{s^4} \equiv (\phi_2 \cdot \phi_4) = \qquad \Phi = 12\phi_2 + 8\phi_4 + \phi_6$

⑥ $\underset{u}{s^4}\ \underset{u}{s^4} \equiv (\phi_4 \cdot \phi_4) = \qquad \Phi = \phi_8 + \binom{4}{1}\times 4\ \phi_6$

$$+ \binom{4}{2}\times 4\times 3\ \phi_4$$

$$+ \underbrace{\binom{4}{3}\times 4\times 3\times 2}_{96}\ \phi_2 + 4!\ 1$$

$$dt = \overset{2}{(\alpha_+)}\,t - \sum \overset{?}{C_{i,i+}}\ g_i\, g_i \qquad ④$$

$$\frac{dh}{dl} = \overset{\frac{d}{2}-1}{\lambda_h}\, h - \sum_{ij} \overset{?}{\tilde{C}}_{ijh}\, g_i g_j$$

$$\frac{du}{dl} = \overset{\epsilon}{\lambda_u}\, u - \sum_{ij} \overset{?}{\tilde{C}}_{iju}\, g_i g_j$$

⑤

$$\frac{dt}{dl} = 2t - \Big[\overset{=4}{C_{ttt}}\, t^2 + \overset{=0}{C_{tht}} \times 2\, t h + \overset{=12}{C_{tut}} \times 2\, t u + \overset{=1}{C_{hht}}\, h^2 + \overset{=0}{C_{hut}} \times 2\, h u + \overset{=96}{C_{uut}}\, u^2 \Big]$$

$C_{ttt} = 4$:

نگاه کردیم، چه $S^2 S^2$ (tt) ←
چه جملهٔ S^2 (ϕ_2) دارد.

$$\frac{dt}{dl} = 2t - \left[4t^2 + 24ut + 96u^2 + \dots \right]$$

$$\frac{dh}{dl} = \left(\frac{d}{2}-1\right) h - 4ht$$

$$\frac{du}{dl} = \epsilon u - \left[t^2 + 16tu - 72u^2 \right]$$

$\beta_l = \checkmark$

$t_* = h_* = u_* = 0$ Gaussian

$u_* = \epsilon/72$

$$x_t = \frac{\partial \beta_\ell^{(t)}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{dt}{d\ell}\right)\Big| = 2 - 8t - 24u$$

Fixed-Point

$$x_t = 2 - \frac{24}{72}\epsilon \rightarrow \nu = \frac{1}{x_t} = \frac{1}{2(1-\epsilon/6)}$$

$$\simeq \underbrace{\frac{1}{2} + \frac{\epsilon}{12}}_{\text{chapter}}$$

$\odot$

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