بسم الله الرحمن الرحيم

Reveiw on Perturbative RG (Goldenfeld)

Chapter 12

① $Z = \int DS\, e^{\mathcal{H}}$ while $Z = \int Dm\, e^{-\beta\mathcal{H}}$

$$-\mathcal{H} = \int d^d r \left[\frac{1}{2}(\nabla S)^2 + \frac{1}{2} r_0 S^2 + \frac{1}{4} u_0 S^4 - h_0 S \right] \quad (12.5)\text{ و }(12.61)$$

Kardar

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U} = \int d^d r \left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2} m^2 + \frac{L}{2}(\nabla^2 m)^2 + \dots + u(m\cdot m)^2 \right]$$

(5.1) Kardar

F.T.

$$-\tilde{\mathcal{H}} = \frac{1}{2}\int \frac{d^d K}{(2\pi)^d}\left(r_0 + K^2\right)|S(k)|^2 \quad (12.65)$$

$$+ \frac{1}{4}u_0 \int \frac{d^d k_1\, d^d k_2\, d^d k_3\, d^d k_4}{(2\pi)^{4d}} (2\pi)^d \delta_D(k_1+k_2+k_3+k_4)$$

$$\times S(k_1)\, S(k_2)\, S(k_3)\, S(k_4)$$

$$-\tilde{\mathcal{H}} = -\mathcal{H}_0 - V(\{S'\},\{\sigma\}) = -\mathcal{H}_0(\{\sigma\}) - \mathcal{H}_0(\{S'\}) - V$$

F.T.

$$\beta\tilde{\mathcal{H}} = \int \frac{d^d q}{(2\pi)^d} \frac{t + Kq^2 + Lq^4 + \dots}{2} |m(q)|^2$$

$$+ u \int \frac{d^d q_1\, d^d q_2\, d^d q_3\, d^d q_4}{(2\pi)^{4d}} (2\pi)^d \delta_D(q_1+q_2+q_3+q_4)$$

$$m_\alpha(q_1)\, m_\alpha(q_2)\, m_\beta(q_3)\, m_\beta(q_4)$$

$\alpha, \beta = 1, \dots, n$

n-vector field

② Green's function

$$\langle S(k_1)\, S(k_2)\rangle_0 = \frac{(2\pi)^d \delta_D(k_1+k_2)}{r_0 + k_1^2} \quad (12.96)$$

$$\langle m_\alpha(q) m_\beta(q')\rangle_0 = \frac{(2\pi)^d \delta_D(q+q')\,\delta_{\alpha\beta}}{t + Kq^2 + Lq^4 + \dots} \quad (5.6)$$

Kardar

③ RG: Coarse-graining and rescaling and renormalizing.

$$K' = lK \qquad S' = \frac{S}{z}$$

$$q' = lq \qquad \tilde{m}' = \tilde{m}/z$$

$$S \equiv \{S'(K)\} \oplus \{\sigma(K)\} \qquad m \equiv \{\tilde{m}(q)\} \oplus \{\sigma(q)\}$$

Kardar

$$0 < K < \Lambda/l \qquad \Lambda/l < K < \Lambda$$

$$\underbrace{la < r < \infty}_{\text{Singular Part}} \qquad \underbrace{a < r < la}_{\text{Regular Part (Integrate out)}}$$

④ $$Z = Z_\sigma \int DS'\, e^{\mathcal{H}_{s'}} \langle e^{-V} \rangle_\sigma \quad (12.74)$$

$$= Z_\sigma \int Ds'\, e^{\mathcal{H}_{s'}}\, e^{\langle V\rangle_\sigma + \frac{1}{2}[\langle V^2\rangle_\sigma - \langle V\rangle_\sigma^2] + \mathcal{O}(V^3)}$$

Combination of Coarse-graining + Perturbation

$$\mathcal{O}(V^2) \qquad \mathcal{O}(V^3)$$

(12.77) $$\mathcal{H} = \mathcal{H}_\sigma + \mathcal{H}_{s'} + \langle V\rangle_\sigma + \frac{1}{2}[\langle V\rangle_\sigma^2 - \langle V\rangle_\sigma^2] + \mathcal{O}(V^3)$$

(5.32) $$\beta\tilde{\mathcal{H}} = V\delta f_l^0 + \int_0^{\Lambda/l} \frac{d^d q}{(2\pi)^d} \left(\frac{t + Kq^2}{2}\right)|\tilde{m}(q)|^2 - \ln\langle e^{-\mathcal{U}}\rangle_\sigma$$

(5.33) $$\ln\langle e^{-\mathcal{U}}\rangle_0 = -\langle \mathcal{U}\rangle_\sigma + \frac{1}{2}[\langle \mathcal{U}^2\rangle_\sigma - \langle \mathcal{U}\rangle_\sigma^2] + \mathcal{O}(\mathcal{U}^3)$$

$$(12.80) \qquad V = \frac{1}{4} u_0 \int \frac{d^d k_1\, d^d k_2\, d^d k_3\, d^d k_4}{(2\pi)^{4d}} (2\pi)^d \delta_D(k_1+k_2+k_3+k_4)\, \underbrace{S(k_1) S(k_2) S(k_3) S(k_4)}$$

$\left\{ \begin{array}{l} n=1 \text{ with respect} \\ \text{to Kardar} \end{array} \right\}$

(5) $\langle V \rangle_\sigma = ?$ Section (12.3.4)

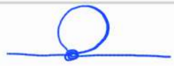
Recall $\frac{1}{2} r_0 S^2$

$u^{(1)} \rightarrow \mathcal{O}(V^2)$

$u_{2,l}$

S^2

$$t' = t_l = z^2 l^{-d} \left[r_0 l^2 + K_l^2 + \overset{12}{2\times 6} \right] \qquad (12.112)$$

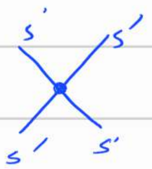


SSSS $\longrightarrow$ S^2

$u^{(1)}_{4,l} \rightarrow \mathcal{O}(V^2)$

S^4

$$u_0' = z^4 l^{-3d} \left[\quad \right] \qquad (12.113)$$



$\mathcal{O}(V^2)$ $\qquad y_t = x_t = 2$

$y_u = x_u = 4-d = \epsilon$

$\underbrace{(t^*, u^*) = (0,0)}$

Gaussian Fixed Point

For $d > 4 \longrightarrow y_u < 0$ — Irrelevant

Compatible with Perturbation

For $d < 4 \longrightarrow y_u > 0 \longrightarrow$ Relevant $\longrightarrow$ break down Perturbation

To mitigate this challenge we should compute up to $\mathcal{O}(V^3)$

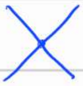
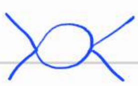
$\frac{1}{2!}\left[\langle V^2 \rangle_\sigma - \langle V \rangle_\sigma^2 \right] = ?$

⑥ $\frac{1}{2!}\left[\langle V^2\rangle_\sigma - \langle V\rangle_\sigma^2\right] = ?$ Section 12.5

$\mathcal{O}(V^2) \qquad \mathcal{O}(V^3)$

$u^{(1)}_{2,\ell} \longrightarrow u^{(2)}_{2,\ell} = z^2 \ell^{-d}\left[(r_0+K^2) + 12\right.$  $\left.\right]$ $\left\{\begin{matrix}(12.128)\\ \shortparallel \\ (12.112)\end{matrix}\right\}$?

$\mathcal{O}(V^2) \qquad \mathcal{O}(V^3)$

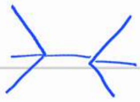
$u^{(1)}_{4,\ell} \longrightarrow u^{(2)}_{4,\ell} = z^4 \ell^{-3d}\left[\right.$  $- \; 36$  $\left.\right]$ $\left\{\begin{matrix}(12.129)\\ (12.113)\end{matrix}\right\}$

$\underbrace{\qquad}_{\frac{1}{4}u_0}$

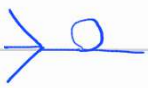
$\downarrow$?

[illegible]

s^6 ← B–B; D–D → r_0

$\left[\langle V^2\rangle_\sigma - \langle V\rangle_\sigma^2\right] = 16$  $+ \; 96$  $+ \; 144$ 

B–D, C–C, C–E

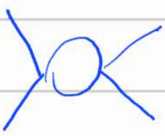
$+ \; 96$  $+ \; 72$  $+ \; 144$ 

[illegible] ← (96); SSSS, u_0 (72)

$+ \; 72$  $+ \; 24$ 

غیرمتصل ← E–E

$\frac{1}{2}\left[\langle V^2\rangle_\sigma - \langle V\rangle_\sigma^2\right] \longrightarrow \frac{72}{2} \longrightarrow 36$ ✓

⑦  we have computed before. (12.130)

$u^{(2)}_{2,\ell} = r' = \ell^2\left[r_0 + 3u_0 I_1\right] \longrightarrow \mathcal{O}(u_0^2) \to \mathcal{O}(\epsilon^2)$ (12.131) ☆

$u^{(2)}_{4,\ell} = u' = u_0 \ell^{\epsilon}\left[1 - 9u_0 I_2\right] \longrightarrow \mathcal{O}(u_0^3) \to \mathcal{O}(\epsilon^3)$ (12.132) ☆

$$? = I_1 \equiv \int_{\Lambda/\ell}^{\Lambda} \frac{d^d K}{(2\pi)^d} \frac{1}{r_0 + K^2}$$

$$I_2 \equiv \int_{\Lambda/\ell}^{\Lambda} \frac{d^d K}{(2\pi)^d} \frac{1}{(r_0 + K^2)} \frac{1}{(r_0 + (K_1 + K_2 - K)^2)} \Bigg|_{K_1 = K_2 = 0} \qquad (12.130)$$

$$? = I_2 \equiv \int_{\Lambda/\ell}^{\Lambda} \frac{d^d K}{(2\pi)^d} \frac{1}{(r_0 + K^2)^2}$$

$\rightarrow$ Long-mode

1974 Wilson

up to leading order near critical point

Symmetric Integral

$S_d\, q^{d-1} dq$

\# Hint:

$d^3q = d\varphi\, d\theta \sin\theta\, q^2 dq$

$(|q|, \varphi, \theta)$

$$I = \int \frac{d^d q}{(2\pi)^d} f(q^2) = \frac{S_d}{(2\pi)^d} \int dq\, q^{d-1} f(q^2)$$

Surface area

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$\boxed{\epsilon = 4 - d}$ —— $\boxed{d = 4 - \epsilon}$

Using (12.136) and (12.138) $\longrightarrow$ (12.131) and (12.132)

$\Downarrow$

$$r' = \ell^2 \left[r_0 + \frac{3}{4\pi^2} \frac{u_0}{4} \Lambda^2 \left(1 - \frac{1}{\ell^2}\right) - \frac{3}{2\pi^2} \frac{u_0}{4} r_0 \ln \ell \right] \qquad (12.147)$$

$$\frac{u'}{4} = \frac{u_0}{4} + \frac{u_0}{4} \left[\epsilon - \frac{9}{2\pi^2} \frac{u_0}{4} \right] \ln \ell \qquad (12.148)$$

$\ell = (1 + \delta\ell) = e^{\delta\ell}$

Fixed Points ? Section (12.5-3)

$\ell = 1 + \delta\ell \longrightarrow \beta_\ell = 0 \Rightarrow r^*, u^* = ?$

☆ First Fixed point $(r^*, u^*) = (0, 0)$ Gaussian fixed point

☆ Second Fixed point $(r^*, u^*) = \left(-\epsilon\frac{\Lambda^2}{6}, \frac{8\pi^2}{9}\epsilon\right) \sim O(\epsilon)$

We obtain that Second fixed Point is related to Gaussian fixed Point up to $O(\epsilon)$

$$\boxed{\text{Wilson\_Fisher Fixed Point}}$$

$[K]_\ell = R_\ell [K] \rightarrow [K]_\ell = [K]_* + k\frac{\partial R_\ell}{\partial k} + \cdots$

The Linearized RG transformat

$R' = T R$

T

$Mv = \Lambda v$

eigen vector

Eigen-value

$$M \equiv \begin{pmatrix} \partial r'/\partial r & \partial r'/\partial u \\ \partial u'/\partial r & \partial u'/\partial u \end{pmatrix}\Bigg|_{\text{Fixed-Pnt}}$$

Fixed-Pnt: Gau, W.F.

$$(12.152) \quad M\Big|_{Gau} = \begin{pmatrix} \ell^2 & \frac{3\Lambda^2(\ell^2-1)}{4\pi^2} \\ 0 & \ell^{\epsilon} \end{pmatrix} \quad (12.152)$$

$$\boxed{\Lambda = 1/a}$$

$$M_{W.F.} = \begin{pmatrix} \ell^{2-\frac{\epsilon}{3}} & \frac{3\Lambda^2(\ell^2-1)}{4\pi^2} \\ 0 & \ell^{-\epsilon} \end{pmatrix} \quad (12.153)$$

☆ At Gaussian Fixed Point $(r^*, u^*) = (0,0)$

Eigen-Value $\begin{cases} \Lambda_r = \Lambda_t = \ell^2 \Rightarrow y_t = \underline{y_r} = 2 \\ \Lambda_u = \Lambda_2 = \ell^{\epsilon} \Rightarrow y_u = y_2 = \epsilon \end{cases}$

$\boxed{\text{for Ism } d_c = 4}$

$\boxed{\epsilon = 4 - d}$

Same Universality Class

$d > d_c$

Eigen Vector $\begin{cases} V_1 = (1,0) = (1r, 0 \times u) \\ V_2 = \left(-\frac{3\Lambda^2}{4\pi^2}, 1\right) \end{cases}$

(6.17) $\boxed{d > \frac{2\beta + \gamma}{\nu} \equiv d_c}$

Upper critical Dimension

Page 171 Goldenfeld

* According to Ginzburg Criterion at $d > 4$ is always satisfied — MF. $O(\delta s^2)$

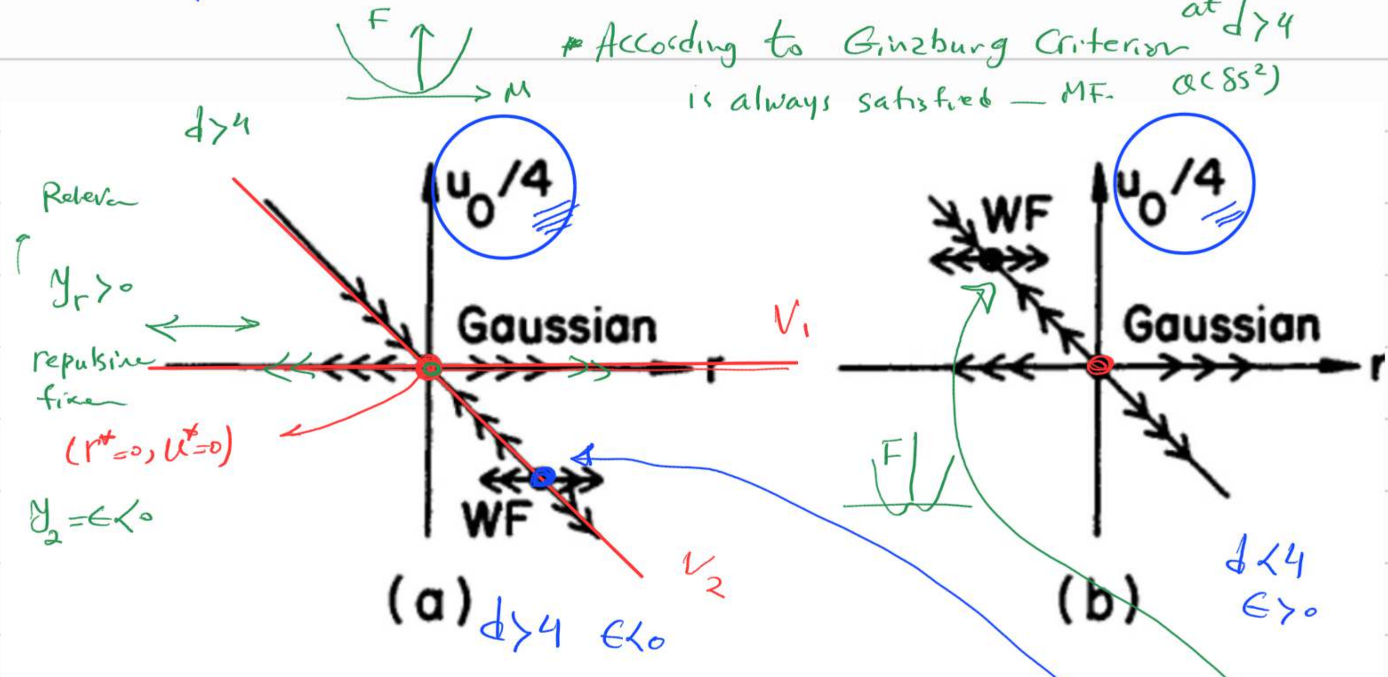


Figure 12.2 RG flows near four dimensions: (a) $d > 4$. (b) $d < 4$.

$$WF = (r^*, u^*) = \left(-\frac{\epsilon\Lambda^2}{6}, \frac{8\pi^2}{9}\epsilon\right)$$

$$\left(r^*, \frac{u^*}{4}\right) = \left(-\frac{\epsilon\Lambda^2}{6}, \frac{2\pi^2}{9}\epsilon\right)$$

For $d > 4$ $(\epsilon < 0)$ $\longrightarrow$ $\left(r^*, \frac{u^*}{4}\right) = \left(+\frac{|\epsilon|\Lambda^2}{6}, -\frac{2\pi^2}{9}|\epsilon|\right)$

For $d < 4$ $(\epsilon > 0)$ $\longrightarrow$ $\left(r^*, \frac{u^*}{4}\right) = \left(-\frac{|\epsilon|\Lambda^2}{6}, +\frac{2\pi^2}{9}|\epsilon|\right)$

For $d > 4$ $(\epsilon < 0)$ $y_u < 0$ $\longrightarrow$ Irrelevant

$$\text{W.F.}\quad \left(r^*, \frac{u^*}{4}\right) = \left(+\frac{|\epsilon|\Lambda^2}{6},\ -\frac{2\pi^2}{9}|\epsilon|\right)$$

$$\frac{u^*}{4} = -\frac{2\pi^2}{9}|\epsilon| < 0$$

$\boxed{\text{Not acceptable}}$

$u^* < 0 \longrightarrow M = \sqrt{-\frac{2a_2}{a_4}}$

$a_4 > 0 \longrightarrow$ physical soln

No physical soln $\longrightarrow$ only Gau fixed pnt is accepted.

W.F. is rejected

For $d < 4$ $(\epsilon > 0)$ $\quad$ W.F. $\left(r^*, \frac{u^*}{4}\right) = \left(-\frac{|\epsilon|\Lambda^2}{6},\ +\frac{2\pi^2|\epsilon|}{9}\right)$

$$u^* = \frac{8\pi^2|\epsilon|}{9} > 0 \longrightarrow a_4 > 0$$

physical soln

at W.F. $\quad y_2 = -\epsilon \longrightarrow$ Irrelevant.

$d < 0$

At W.F.

$\epsilon = 4 - d$

$$\begin{cases} \Lambda_t = l^{(2-\epsilon/3)} = l^{2(1-\epsilon/6)} \longrightarrow y_t = 2(1-\epsilon/6) \\ \Lambda_2 = l^{-\epsilon} \Rightarrow y_2 = -\epsilon \end{cases}$$

For $d > 4$ $(\epsilon < 0)$ $\quad y_t = 2 + \frac{|\epsilon|}{3} > 2 \longrightarrow$ Relevant.

$y_2 = +|\epsilon| \longrightarrow$ Relevant $\times$

$$\text{W.F.}: \left(-\frac{\epsilon\Lambda^2}{6},\ \frac{2\pi^2\epsilon}{9}\right) = \left(+\frac{|\epsilon|\Lambda^2}{6},\ -\frac{2\pi^2|\epsilon|}{9}\right)$$

No physical Solutn

For $d<4$ $(\epsilon>0)$ $\qquad y_t = 2 - \frac{|\epsilon|}{3}$

$y_2 = -|\epsilon| < 0 \longrightarrow$ Irrelevant attractive Fixed Pon

$$W.F. \left(-\frac{|\epsilon|\Lambda^2}{6}, +\frac{2\pi^2|\epsilon|}{9} \right) \quad \text{physical soln}$$

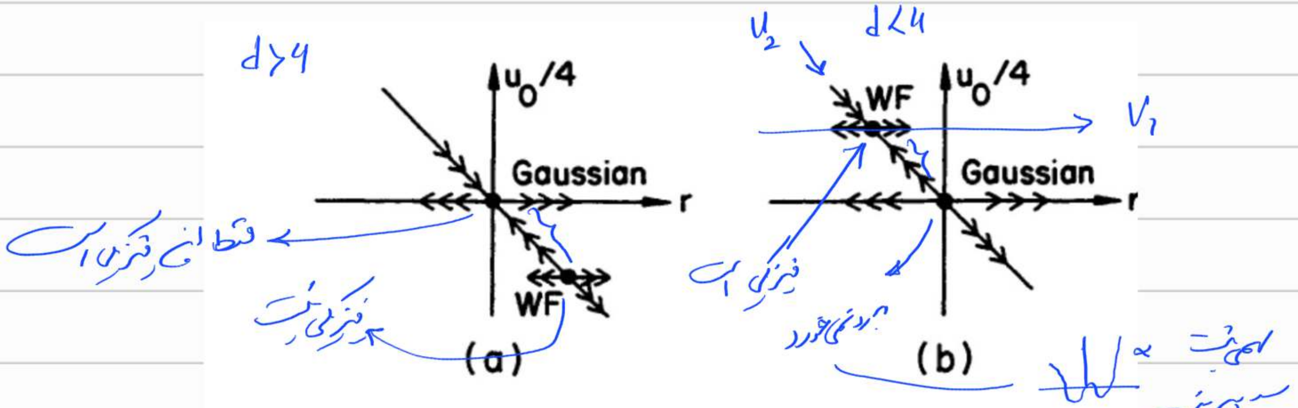


Figure 12.2 RG flows near four dimensions: (a) $d > 4$. (b) $d < 4$.

Notice that

$$\left(r^*_{W.F.}, \frac{u^*_{W.F.}}{4} \right) = \left(-\frac{\epsilon\Lambda^2}{6}, \frac{2\pi^2\epsilon}{9} \right), \quad \vec{V}_2 = \left(-\frac{3\Lambda^2}{4\pi^2}, 1 \right)$$

$$= \frac{2\pi^2\epsilon}{9} \vec{V}_2 = \frac{2\pi\epsilon}{9} \left(-\frac{3\Lambda^2}{4\pi^2}, 1 \right)$$

For $\epsilon \to 0$ W.F. $\longrightarrow$ Gaussian fixed-Point

For $d>4$ $(\epsilon<0)$ W.F. has Negative value of Quartic coupling Constn which is unphysical.

For $d<4$ ($\epsilon>0$) W. F. has positive value of quartic coupling const which is physical.

(8) $\nu = \frac{1}{y_t} = \frac{1}{2}\left(1+\frac{\epsilon}{6}\right) + O(\epsilon^2)$

$$\left.\begin{aligned} \gamma &= 1+\epsilon/6 \\ \eta &= 0 \\ \alpha &= \epsilon/6 \\ \beta &= \frac{1}{2}-\epsilon/6 \end{aligned}\right\} O(\epsilon^2)$$

$\delta = 3+\epsilon$

Table 12.1 CRITICAL EXPONENTS FOR THE ISING UNIVERSALITY CLASS IN THREE DIMENSIONS

Exponent	ϵ-expansion to $O(\epsilon)$	Mean Field	Experiment	Ising ($d=3$)
α	0.167	0 (disc.)	0.110 − 0.116	0.110(5)
β	0.333	1/2	0.316 − 0.327	0.325±0.0015
γ	1.167	1	1.23 − 1.25	1.2405±0.0015
δ	4.0	3	4.6 − 4.9	4.82(4)
ν	0.583	1/2	0.625±0.010	0.630(2)
η	0	0	0.016 − 0.06	0.032±0.003

ϵ-Expan

Meanfield

$d<4 \rightarrow$ Ginzburg criterion shows that $d>4$ for mean field

For α

	$n=1$	$n=2$	$n=3$	$n=4$
$\mathcal{O}(\epsilon)$ at $\epsilon=1$	0.17	0.11	0.06	0
Experiments in $d=3$	0.11	−0.01	−0.12	–

From Page 90
Karden

$$\nu = \frac{1}{y_t} = \frac{1}{2} + \frac{1}{4}\,\frac{n+2}{n+8}\,\epsilon + \mathcal{O}(\epsilon^2) \qquad (5.60)$$

$$\alpha = 2 - d\nu = \frac{4-n}{2(n+8)}\,\epsilon + \mathcal{O}(\epsilon^2) \qquad (5.61)$$

$$\beta = \frac{1}{2} - \frac{3}{2(n+8)}\,\epsilon + \mathcal{O}(\epsilon^2) \qquad (5.63)$$

$$\gamma = \frac{2y_h - d}{y_t} = 1 + \frac{n+2}{2(n+8)}\,\epsilon + \mathcal{O}(\epsilon^2) \qquad (5.64)$$

Sec 4.7 Kardar — براساس ادبیات مطالعه مطابق، تحلیل ابعادی مطابق؛

Eq. (4.64) $Z = Z_0 Z_m$

Coarse-grain قطعه‌بندی

Regular

{rescale, renormalize} (4.67)

$$\beta\mathcal{H} = \left[\int_0^{\Lambda/\ell} \frac{d^d q}{(2\pi)^d} \left(\frac{t + Kq^2 + Lq^4 + \cdots}{2}\right) |m(q)|^2\right] - h \cdot \tilde{m}(q=0)$$

$\downarrow$

$$\left.\begin{array}{ll} \text{Rescaling} \longrightarrow & q' = q\ell \\ \text{Renormalize} - & m' = m/z \end{array}\right\} \longrightarrow (4.67)$$