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Section 5.6, 5.7 statistical physics of field

& Chapter 12 Goldenfeld

Summary so far!

① Scaling behaviour around and at Critical point

critical phase Transition point

$$\phi(x) \longrightarrow x \to \lambda x \quad \& \quad \underbrace{\phi(\lambda x) = \lambda^{a} \phi(x)}_{\text{Widom Hypothesis}}$$

scaling function

② Thermodynamical potentials → Helmholtz free Energy.

$$F \to f = \frac{F}{\Omega} = \frac{F}{N} = \frac{F}{V}$$

$$f = \underbrace{f^{\text{Singular}}} + f^{\text{Regular}}$$

This Part is relevant to capture Singularity @ critical Point

$$f([K]) \longrightarrow f(R_\ell[K]) = \ell^{d} f([K])$$

$$[K] \xrightarrow{R_\ell} \underbrace{[K'] = R_\ell[K]}_{\text{Recursive Relation}}$$

③ Self-similarity @ critical Point

Underlying System is invariant under

a typical $\underbrace{\text{length transformation}}$

↑

Mathematical description is given by

RG method.

$\frac{\Lambda}{\ell} \leqslant q \leqslant \Lambda$ $\qquad$ $0 \leqslant q \leqslant \frac{\Lambda}{\ell}$

$$\vec{m}_i(\vec{q}) \equiv \underbrace{\{\bar{\sigma}(q_>)\}}_{\text{Regular}} \oplus \underbrace{\{\tilde{m}(q_<)\}}_{\text{Singular}}$$

RoRoI

Integration out

Rescaling $\qquad x' = \frac{x}{\ell}$

Renormalizing.

$$m'(x') = \frac{m(x)}{\vartheta}$$

$$\tilde{m}'(q) = \frac{\tilde{m}(q)}{z}$$

$N_\ell = \ell^{-d} N$

$\xi_\ell = \ell^{-1} \xi$

$\left.\begin{array}{l} t_\ell = \ell^{x_t} t \\ h_\ell = \ell^{x_h} t \end{array}\right\}$ $\quad K_\ell = \ell^{x_K} K$

$x_K > 0 \quad \leftarrow$ Relevant

$x_K < 0 \quad \leftarrow$ Irrelevant

$x_K = 0 \quad \leftarrow$ Marginal

Dangerous Irrelevant variable.

12.2.5 Goldenfeld

④ RG in Configuration Space $\longrightarrow$ Ising model 1D, 2D

⑤ $[K^*] = R_\ell [K^*] \leftarrow$ Fixed point Definition

$$[K'] = [K^*] + \frac{\partial R_\ell [K]}{\partial K}\bigg|_{K=K^*} (K-K^*) + \mathcal{O}(\Delta K^2)$$

$$k' = k \left.\frac{\partial R_\ell}{\partial K}\right|_{K=K^*} + \mathcal{O}(k^2)$$

$$T = \frac{\partial R_\ell}{\partial K} \quad - \quad T\phi_i = \lambda_i \phi_i$$

$$k = \sum u_i \phi_i$$

$$k' = \sum u_i' \phi_i$$

$$u_i' = \lambda_i u_i$$

$$\lambda_i = \ell^{x_i}$$

$$[K'] = [K] + \frac{\partial R_\ell}{\partial \ell}\bigg|_{K=K^*} \delta\ell \qquad * \; B_\ell = \frac{\partial R_\ell}{\partial \ell}\bigg|_{K=K^*} \; *$$

$$= [K] - B_\ell \, \delta\ell.$$

$$x_i = -\frac{\partial \beta_\ell^{(i)}}{\partial K}\Big|_{K=K^*}$$

⑥ $\beta \mathcal{H} = \beta \mathcal{H}_0 + \mathcal{U}$ ← Perturbation part

$$\begin{cases} t' = \ell^2 \left[t + 4u(n+2) \displaystyle\int_{\Lambda/\ell}^{\Lambda} \frac{d^d q'}{(2\pi)^d} \frac{1}{t + Kq'^2} \right] \\ u' = \ell^{4-d}\, u \end{cases}$$

$$\frac{dt}{d\ell} = 2t + \frac{4u(n+2)}{t + K\Lambda^2} S_D \Lambda^d$$

$$\boxed{\epsilon \equiv 4-d}$$

$$\frac{du}{d\ell} = (4-d)u = \epsilon u$$

$t' = \ell^{x_t} t$ $\qquad x_t \big|_{\text{Gaussian fixed point}} = 2$

$u' = \ell^{x_u} u$ $\qquad x_u \big|_{\text{Gaussian fixed point}} = \epsilon$

Gaussian fixed point
$t^* = 0$
$u^* = 0$

For $d > 4 \longrightarrow \epsilon < 0 \longrightarrow x_u < 0 \longrightarrow u$ is irrelevant Coupling coef.

For $d < 4 \longrightarrow \epsilon > 0 \longrightarrow x_u > 0 \longrightarrow u$ is relevant coupling coeficient

The Perturbation is no longer valid ?!?

$$\beta \mathcal{H} = \beta \mathcal{H}_0 + \mathcal{U}$$

Perturbation part

ϵ- Expansion $\to$ $\mathcal{O}(\mathcal{U}^2)$ *

$\mathcal{O}(\mathcal{U}^3)$

$\mathcal{O}(\mathcal{U}^4)$

$\vdots$

Recall: $Z = \int Dm(q)\, e^{-\beta \tilde{\mathcal{H}}[m]}$

$$= \int D\tilde{m}(q)\, D\sigma(q)\, e^{-\beta \mathcal{H}_0 - \tilde{\mathcal{U}}}$$

$$= Z_\sigma \int D\tilde{m}(q)\, e^{-\beta \mathcal{H}_0[\tilde{m}(q)]} \left\langle e^{-\tilde{u}} \right\rangle_\sigma$$

$$\ln \langle e^{-\tilde{u}} \rangle_\sigma = -\langle \tilde{u} \rangle_\sigma + \frac{1}{2} \underbrace{\left[\langle \tilde{u}^2 \rangle_\sigma - \langle \tilde{u} \rangle_\sigma^2 \right]}_{} + \mathcal{O}(u^3)$$

we have considered this part

Now we are going to take into account this part.

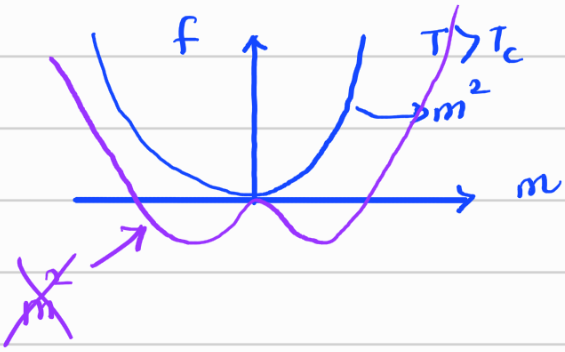
5.6. 5.7. . .

⑦ Critical Phenomena near Four Dimension

$$\begin{Bmatrix} \epsilon = 4-d \\ d \to 4^+ \\ d \to 4^- \end{Bmatrix}$$

☆ Recall that : The Gaussian model does not work for

$t<0$!
$T<T_c$



$\beta \mathcal{H} = a_0 + a_2 m^2 + a_4 m^4 - hm$

$m = \sqrt{\frac{-a_2}{a_4}}$ $\quad a_2<0$ $\quad a_4>0$

$a_4 \neq 0 \longrightarrow$ Non-Gaussian form
$\downarrow$
u

Dimensionless

☆ chapter 7 $\quad Z = \int D\eta \, e^{-\tilde{\beta L[\eta]}} = \int Dm \, e^{-\tilde{\beta \mathcal{H}}}$

$$L[\eta] = \int d^d r \left\{ \frac{1}{2}\gamma(\nabla\eta)^2 + at\eta^2 + \frac{1}{2}b\eta^4 - h\eta \right\}$$

Perterbation Part

$\beta L[\eta] = ?$

$$\phi \equiv (\beta\gamma)^{1/2}\eta$$

$$r_0 \equiv \frac{2at}{\gamma} \longrightarrow r_0 \equiv \bar{a}t$$

$$u_0 \equiv \frac{2b}{\beta \gamma^2} \quad \text{and} \quad h=0$$

$$\beta \mathcal{H} = \beta L[\eta] = \int d^d r \left[\underbrace{\frac{1}{2}(\nabla \phi)^2} + \underbrace{\frac{1}{2} r_0 \phi^2} + \underbrace{\frac{1}{4} u_0 \phi^4} \right]$$

$$Z = \int D\phi \, e^{-\beta L}$$

Dimensional Analysis :

$$\left[\int d^d r \, \frac{(\nabla \phi)^2}{2} \right] = 1 \implies L^d \, L^{-2} \, [\phi]^2 = 1$$

$$\boxed{[\phi] = L^{1-\frac{d}{2}}}$$

$$\left[\int d^d r \, \frac{1}{2} r_0 \phi^2 \right] = 1 \implies [r_0] = L^{-2} \longrightarrow \xi^{-2} \longrightarrow \nu = 1/2$$

$$\boxed{[u_0] = L^{d-4} = L^{-\epsilon}} \qquad \boxed{\epsilon = 4-d}$$

again we introduce following quantities :

$$\varphi \equiv \frac{\phi}{L^{1-d/2}} \qquad x \equiv \frac{r}{L} \qquad \bar{u}_0 \equiv \frac{u_0}{L^{d-4}}$$

$$L \equiv r_0^{-1/2}$$

$$Z = \int D\varphi \, e^{-\beta L} = \int D\varphi \, e^{-\beta\mathcal{H}_0 - \mathcal{U}}$$

$$\beta\mathcal{H}_0 \equiv \int d^d x \left\{ \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}\varphi^2 \right\}$$

$$\mathcal{U} \equiv \int d^d x \left(\overbrace{\frac{1}{4}\bar{u}_0}^{a_4} \varphi^4 \right) \longleftarrow \lambda\phi^4\text{-model.}$$

If $\mathcal{U}=0 \longrightarrow$ Gaussian model. $\left\{(\nabla\varphi)^2, \varphi^2\right\}$

If $\mathcal{U}\neq 0 \longrightarrow$ Non-Gaussian for $t<0$ $(T<T_c)$ = ✓

therefore

$$Z = \int D\varphi \, e^{-\beta\mathcal{H}_0} \underbrace{\left(1 - \mathcal{U} + \frac{1}{2!}\mathcal{U}^2 + \dots\right)}$$

$\left\{ \mathcal{U} < 1 \text{ to have Perturbation} \right\}$

$\bar{u}_0$

$r_0 \equiv \bar{a} t$

$r_0 \equiv l^{-2}$

$$\bar{u}_0 = l^{4-d} u_0 = r_0^{\frac{d-4}{2}} u_0 = \bar{a}^{\frac{(d-4)}{2}} t^{\frac{d-4}{2}} u_0$$

$$\boxed{\bar{u}_0 \sim t^{\frac{d-4}{2}} u_0}$$

$\epsilon = 4 - d$

@ Gaussian fixed Point $t^* = 0$ $T = T_c$

If $d > 4$ $\quad \bar{u}_0 \simeq t^{-\frac{\epsilon}{2}}\Big|_{t = t^* = 0} \longrightarrow 0$ $\quad \underbrace{\text{Convergency}}_{\text{همگرا}}$

$\epsilon < 0$

$-\epsilon > 0$

If $d < 4$ $\quad \bar{u}_0 \simeq t^{-\frac{\epsilon}{2}}\Big|_{t = t^* = 0} \longrightarrow \infty$ $\quad \underbrace{\text{Divergency}}_{\text{واگرا}}$

$\epsilon > 0$

$-\epsilon < 0$

To solve mentioned problem $\longrightarrow$ ϵ- Expansion

$> \mathcal{O}(u^3)$

In this case $[t^*, u^*] = [0, 0] \leftarrow$ Gaussian fixed

In addition above fixed Point we find

another fixed point (so-called [Wilson fisher fixed Point]

$\boxed{u^* = \frac{\epsilon}{A}}$ $\quad A > 0$ $\quad$ for $d < 4$ $(\epsilon > 0)$

$\Rightarrow u^* > 0$ is physically acceptable !

$(m \text{ is Real}) \equiv m\sqrt{-\frac{a_2}{a_4}}$ $\quad$ For $t < 0$

Therefore instead of Gaussian fixed Point

We should consider W.F. fixed Point

$x_u\Big|_{u^*=u_{W.F.}} < 0 \longrightarrow$ u can be considered as irrelevant coupling coefficient

$\underbrace{}$

✓ { Perturbation remaines Valid } ✓

$$\beta_\ell^{(u)} = \epsilon u_0 - A u_0^2 \quad \leftarrow \mathcal{O}(u^3)$$

$$0 = \beta_\ell^{(u)} \Longrightarrow \begin{cases} u_0^* = 0 \\ u_0^* = \epsilon/A \end{cases} \leftarrow \text{W.F. Fixed Point}$$

(8) 12.2.5 Dangerous Irrelevant Variable in Landau Theory.

To show above, suppose the RG for Gaussian Model. $t>0$

$[t>0]$, $h=0$

$$\beta \mathcal{H} = \int d^dx \left[\frac{1}{2}(\nabla S)^2 + \frac{1}{2} r_0 S^2 \right]$$

F.T. $\downarrow$

$$\beta \tilde{\mathcal{H}} = \int \frac{d^d q}{(2\pi)^d} \left[\frac{1}{2} (q^2 + r_0) |\tilde{S}(q)|^2 \right.$$

RG: RoRoI

$$S(x) = \int \frac{d^d q}{(2\pi)^d} S(q) e^{iq\cdot x}$$

$$= \underbrace{\int_0^{\Lambda/\ell} \frac{d^d q}{(2\pi)^d} S(q) e^{iq\cdot x}}_{\text{Singular Part}} + \underbrace{\int_{\Lambda/\ell}^{\Lambda} \frac{d^d q}{(2\pi)^d} \sigma(q) e^{iq\cdot x}}_{\text{Regular Part}}$$

$$q \to q' = \ell q$$

$$S(q) \to S'(q') = \frac{S(q)}{Z} , \quad \boxed{Z = \ell^{1+d/2}}$$

$$r_0 \to r_0(\ell) = \ell^{x_{r_0}} r_0$$

$$t_\ell = \ell^{x_t} t$$

$$x_s = 1 - d/2$$

$$x_t = x_{r_0} = -d + 2x_s = 2$$

$$\boxed{x_h = 1 + d/2}$$

RG

Chapter 5: Widom Hypothesis

$$M \sim |t|^{\beta} \quad t<0$$

$$\chi \sim |t|^{\gamma} \quad t<0$$

$$c \sim |t|^{-\alpha} \quad t<0$$

$$M \sim h^{1/\delta}$$

$$\xi \sim t^{-\nu}$$

$$\left\{\begin{aligned} \nu d &= 2-\alpha \\ \beta &= \frac{d-2}{4} \\ \gamma &= 1 \\ \delta &= \frac{d+2}{d-2} \end{aligned}\right\} \quad \text{Widom}$$

$$\text{Widom} \;\&\; \text{RG} \Longrightarrow \left\{\begin{aligned} \beta &= \underbrace{\frac{1-b}{a}}_{\text{Widom}} = \underbrace{\frac{1-x_h}{x_t}}_{\text{RG}} \\ \delta &= \frac{x_h}{1-x_h} \end{aligned}\right\}$$

From RG

$$\beta_{RG} = \frac{1-x_h}{x_t} = \frac{1-(1+\frac{d}{2})}{2} = -\frac{d}{4}$$

From widom $\beta_w = \dfrac{d-2}{4}$

$$\Longrightarrow \boxed{\beta_{RG}^{t>0} \neq \beta_{w}^{t<0}}$$

Gaussian $(t>0)$

Inconsistent Results why ?

From RG $\delta_{RG} = \dfrac{x_h}{1-x_h} = \dfrac{1+d/2}{1-(1+d/2)} = -\dfrac{d+2}{d}$

$$\delta_w = \frac{d+2}{d-2} \longrightarrow \boxed{\delta_{RG}^{t>0} \neq \delta_{w}^{t<0}}$$

why ? RG is given for Gaussian model that is valid for $t>0$, while for Widom case we have $t<0$!

To clarify above challenge, consider the scaling behaviour of $f(t, h, u_0)$

$$f(t, h, u_0) = \ell^{-d} f(\ell^{x_t} t, \ell^{x_h} h, \ell^{x_{u_0}} u_0)$$

$$M = \frac{-1}{k_B T} \frac{\partial f}{\partial h}\bigg|_{h=0} = \ell^{x_h - d} M(\ell^{x_t} t, \ell^{x_h} h, \ell^{x_{u_0}} u_0)\bigg|_{h=0}$$

$$\boxed{M(t, 0, u_0) = \ell^{x_h - d} \underbrace{M(\ell^{x_t} t, 0, \ell^{x_{u_0}} u_0)}} \longrightarrow |t|^{\beta} \longrightarrow \beta = ?$$

$$\ell^{x_t} t = 1 \longrightarrow \ell = t^{-1/x_t}$$

$$M(t, 0, u_0) = t^{\frac{d - x_h}{x_t}} M(1, 0, \underbrace{t^{-\frac{x_{u_0}}{x_t}} u_0})$$

$$= t^{\frac{d - x_h}{x_t}} M(1, 0, 0) \sim |t|^{\beta} \quad t<$$

$$\Rightarrow \boxed{\beta = \frac{d - x_h}{x_t}} \quad \dot{\varepsilon}$$

For $t < 0 \longrightarrow m = \sqrt{\frac{-a_2}{a_4}} = \sqrt{\frac{-a_2}{u_0}}$

$u_0 \neq 0$

For $u_0 = 0 \longrightarrow m \longrightarrow \infty$

To mitigate above problem for $t<0$:

$$\star \quad M(t,0,u_0) \equiv t^{-\frac{(x_h-d)}{x_t}} \underbrace{M(1,0,u_0 t^{-\frac{x_{u_0}}{x_t}})}$$

$$\star \quad M \propto u_0^{-1/2} \leftarrow M = \sqrt{\frac{a_2}{a_4}} \sim a_4^{-1/2} \sim u_0^{-1/2}$$

$$M(t,0,u_0) \simeq t^{-\frac{(x_h-d)}{x_t}} \underbrace{F(X)}_{X^{-1/2}} \quad = \left(u_0 t^{-\frac{x_{u_0}}{x_t}}\right)^{-1/2}$$

$$\simeq t^{-\frac{(x_h-d)}{x_t}} X^{-1/2} (cts)$$

$$\simeq t^{-\frac{(x_h-d)}{x_t}} \left(t^{-\frac{x_{u_0}}{x_t}}\right)^{-1/2} \underbrace{(cts)}_{\text{without imprint of } t}$$

$$M(t) \sim t^{-\frac{(x_h-d)}{x_t} + \frac{x_{u_0}}{2x_t}} \sim t^{\beta}$$

$$\beta = -\frac{(x_h-d)}{x_t} + \frac{x_{u_0}}{2x_t} = -\frac{(1-d/2)}{2} + \frac{(4-d)}{4} = \boxed{1/2}$$

For δ

$$M(0,h,u_0) \sim h^{-\frac{(x_h-d)}{x_h}} \left(h^{-\frac{x_{u_0}}{x_h}}\right)^{-1/3} \sim M(u_0) \sim u_0^{-1/3}$$

$$\frac{1}{\delta} = -\frac{(x_h-d)}{x_h} + \frac{1}{3}\frac{x_{u_0}}{x_h} = \boxed{1/3}$$

☆ For $d>4$ u_0 is irrelevant

☆ The RG is calculated for Gaussian Case $(u_0=0)$ $\longrightarrow$ x_t ✓
x_h ✓

☆ To obtain critical exponent we should consider the regime for that $T<T_c$ $(t<0)$, in this regime $u_0 \neq 0$

However u_0 is irrelevant variable at Gaussian fixed point.

But It is a dangerous variable for $T<T_c$