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Perturbative RG $\mathcal{O}(u^3)$

☆ $u \longrightarrow$ Gaussian is no longer valid for $T<T_c$

$$m = \pm\sqrt{\frac{-a_2}{2a_4}} \qquad \underbrace{a_4 m^4}_{u}$$

☆ $u^2 \longrightarrow \epsilon = 4-d$

$$u_\ell = \ell^{\epsilon} u = \ell^{x_u} u$$

Kardar: $a_4 > 0,\ u > 0$

Goldenfeld $\rightarrow \underbrace{u_0 > 0}$

We set this term as a part of Perturbation

$\downarrow$

Irrelevant $x_u < 0$

☐ $x_u < 0$ Just for $d > 4$

☐ $x_u > 0$ for $d < 4 \longrightarrow$ Perturbation is no longer valid

$$\mathcal{O}(u^3)$$

☆ Review on Kardar Notation

① $Z = \int Dm\, e^{-\beta \mathcal{H}} \longrightarrow Z_\ell = Z \longrightarrow [K'] = R_\ell[K]$

$\downarrow$

☆ fixed Point

☆ RG flow

☆ Coupling exponents

(4.41), (4.43), (4.52)

(5.1)

perturbation Part

(5.1): $\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U}$

$\mathcal{O}(m^2) \equiv$ Gaussian term

Real Space.

$$\beta\mathcal{H} = \int d^d r \left[\frac{t m^2}{2} + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \dots \right]$$

$$+ u \int d^d r \, (m \cdot m)^2 \longrightarrow \text{Kardar } n\text{-Vector field}$$

$(m^4) \longrightarrow$ Goldenfeld 1-Vector field

$n = 1$

$\vec{m} = m_\alpha \qquad \alpha = 1 \dots n$

$\delta_{\alpha\beta} \longrightarrow 1$

$\delta_{\alpha\alpha} \longrightarrow n \longrightarrow$ Table at page 90

Fourier space.

$$\boxed{\beta\tilde{\mathcal{H}} = \beta\tilde{\mathcal{H}}_0 + \tilde{\mathcal{U}}}$$

(5.2) & (5.3) $$\beta\tilde{\mathcal{H}} = \int \frac{d^d q}{(2\pi)^d} \frac{(t + Kq^2 + Lq^4 + \dots)}{2} |m(q)|^2$$

$$+ u \int \frac{d^d q_1 \, d^d q_2 \, d^d q_3 \, d^d q_4}{(2\pi)^{4d}} (2\pi)^d \delta_D(q_1 + q_2 + q_3 + q_4)$$

$$\underbrace{\tilde{m}_\alpha(q_1)\, \tilde{m}_\alpha(q_2)\, \tilde{m}_\beta(q_3)\, m_\beta(q_4)}_{(\vec{m} \cdot \vec{m})^2} \qquad \alpha, \beta = 1, \dots n$$

☆ Additional point

$$\int d^d r \, m^2(r) \xrightarrow{F.T.} \int d^d r \int \frac{d^d q_1}{(2\pi)^d} \tilde{m}(q_1) e^{i q_1 \cdot \vec{r}} \times$$

$$\int \frac{d^d q_2}{(2\pi)^d} \tilde{m}(q_2) e^{i q_2 \cdot \vec{r}}$$

$$\text{''} = \int \frac{d^d q_1}{(2\pi)^d} \int \frac{d^d q_2}{(2\pi)^d} \tilde{m}(q_1)\tilde{m}(q_2) \underbrace{\int d^d r \, e^{i r \cdot (q_1 + q_2)}}_{(2\pi)^d \delta_D(q_1+q_2)}$$

$$\boxed{q_1 + q_2 \to 0}$$

$$= \int \frac{d^d q}{(2\pi)^d} \tilde{m}(q) \underbrace{\tilde{m}(-q)}_{\tilde{m}^*(q)} = \int \frac{d^d q}{(2\pi)^d} |\tilde{m}(q)|^2$$

---

② $$\langle \tilde{m}_\alpha(q) \tilde{m}_\beta(q') \rangle_0 = \frac{\delta_D(q+q')(2\pi)^d \delta_{\alpha\beta}}{t + Kq^2 + Lq^4 + \cdots}$$

Green's func

Propagator

$\beta \mathcal{H}_0$ → Gaussian Term → Non-Perturbative part

Page 47

③ RG ☆ Coarse-graining

$$m(q) = \begin{cases} \tilde{m}(q) & 0 < q < \Lambda/\ell \quad (\ell a < r < \infty) \\ \sigma(q) & \frac{\Lambda}{\ell} < q < \Lambda \quad (a < r < \ell a) \end{cases}$$

$$= \{\tilde{m}(q)\} \oplus \{\sigma(q)\}$$

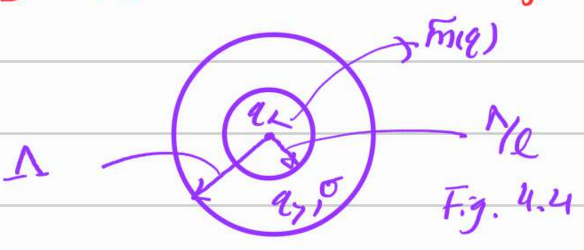


Fig. 4.4

$$\int \mathcal{D}m(q) = \int \underbrace{\mathcal{D}\tilde{m}(q)}_{0<q<\Lambda/\ell} \overbrace{\mathcal{D}\sigma(q)}^{\Lambda/\ell<q<\Lambda}$$

$0<q<\Lambda/\ell$ → It contributes to the Singular part

→ Integrate out part → It contributes to the Regular part

$L=0$

$$Z = \int \mathcal{D}\tilde{m}(q)\,\mathcal{D}\sigma(q)\, e^{-\beta\tilde{\mathcal{H}}}$$

$$= \int \underset{\substack{\updownarrow \\ q_<}}{\mathcal{D}\tilde{m}(q)}\ \underset{\substack{\updownarrow \\ q_>}}{\mathcal{D}\sigma(q)}\ e^{\overbrace{-\int_0^\Lambda \frac{d^dq}{(2\pi)^d}\frac{(t+Kq^2)}{2}\left(|\tilde{m}(q)|^2+|\sigma(q)|^2\right)-\tilde{\mathcal{U}}}^{-\beta\tilde{\mathcal{H}}}}$$

$$= \underbrace{Z_\sigma(t,K)}\int \mathcal{D}\tilde{m}(q)\, e^{-\int_0^{\Lambda/\ell}\frac{d^dq}{(2\pi)^d}\frac{t+Kq^2}{2}|\tilde{m}(q)|^2}\ \underbrace{\left\langle e^{-\tilde{\mathcal{U}}}\right\rangle_\sigma}$$

and $\langle \mathcal{O}\rangle_\sigma \equiv \int \mathcal{D}\sigma(q)\,\dfrac{\mathcal{O}}{Z_\sigma}\exp\left(-\int_{\Lambda/\ell}^{\Lambda}\dfrac{d^dq}{(2\pi)^d}\dfrac{(t+Kq^2)}{2}|\sigma(q)|^2\right)$

→ Gaussian term

after Coarse-graining

$$\text{(5.32)}\quad \beta\tilde{\mathcal{H}} = \beta\tilde{\mathcal{H}}_0 + \int_0^{\Lambda/\ell}\frac{d^dq}{(2\pi)^d}\frac{(t+Kq^2)}{2}|\tilde{m}(q)|^2 - \ln\left\langle e^{-\mathcal{U}}\right\rangle_\sigma$$

$$\text{(5.33)}\quad \ln\left\langle e^{-\mathcal{U}}\right\rangle_\sigma = -\langle \mathcal{U}\rangle_\sigma + \frac{1}{2!}\left[\langle \mathcal{U}^2\rangle_\sigma - \langle \mathcal{U}\rangle_\sigma^2\right] + \mathcal{O}(\mathcal{U}^3)$$

$$\beta\tilde{\mathcal{H}} = \beta\tilde{\mathcal{H}}_\sigma + \beta\tilde{\mathcal{H}}_{\tilde m} + \underbrace{\langle u\rangle_\sigma - \underbrace{\frac{1}{2!}\left[\langle u^2\rangle_\sigma - \langle u\rangle_\sigma^2\right]}_{\mathcal{O}(u^2)} + \underbrace{\mathcal{O}(u^3)}_{\mathcal{O}(u^3)}}$$

(5.28)

(5.29)

(5.30)

(5.31)

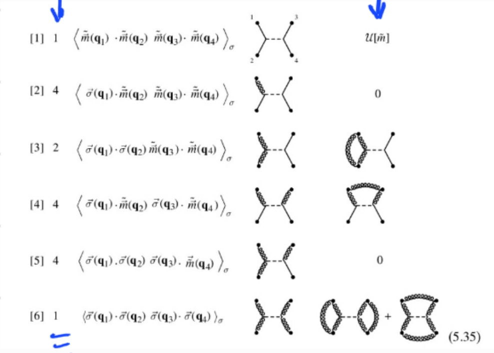
Section 5.5

$$\langle u\rangle_\sigma = u\int \frac{d^dq_1\, d^dq_2\, d^dq_3\, d^dq_4}{(2\pi)^{4d}} (2\pi)^d \delta_D(q_1+q_2+q_3+q_4)$$

$$\left\langle [\tilde m_\alpha(q_1)+\sigma_\alpha(q_1)][\tilde m_\alpha(q_2)+\sigma_\alpha(q_2)][\tilde m_\beta(q_3)+\sigma_\beta(q_3)][\tilde m_\beta(q_4)+\sigma_\beta(q_4)]\right\rangle_\sigma$$

$\tilde m \rightarrow \int_0^{\Lambda/\ell}$, $\sigma \rightarrow \int_{\Lambda/\ell}^{\Lambda}$

term ↓ Result ↓

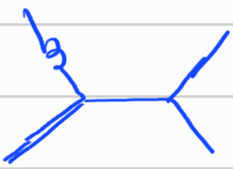


			Result
[1]	1	$\langle \tilde m(\mathbf{q}_1)\cdot\tilde m(\mathbf{q}_2)\ \tilde m(\mathbf{q}_3)\cdot\tilde m(\mathbf{q}_4)\rangle_\sigma$	$\mathcal{U}[\tilde m]$
[2]	4	$\langle \vec\sigma(\mathbf{q}_1)\cdot\tilde m(\mathbf{q}_2)\ \tilde m(\mathbf{q}_3)\cdot\tilde m(\mathbf{q}_4)\rangle_\sigma$	0
[3]	2	$\langle \vec\sigma(\mathbf{q}_1)\cdot\vec\sigma(\mathbf{q}_2)\ \tilde m(\mathbf{q}_3)\cdot\tilde m(\mathbf{q}_4)\rangle_\sigma$	
[4]	4	$\langle \vec\sigma(\mathbf{q}_1)\cdot\tilde m(\mathbf{q}_2)\ \vec\sigma(\mathbf{q}_3)\cdot\tilde m(\mathbf{q}_4)\rangle_\sigma$	
[5]	4	$\langle \vec\sigma(\mathbf{q}_1)\cdot\vec\sigma(\mathbf{q}_2)\ \vec\sigma(\mathbf{q}_3)\cdot\tilde m(\mathbf{q}_4)\rangle_\sigma$	0
[6]	1	$\langle \vec\sigma(\mathbf{q}_1)\cdot\vec\sigma(\mathbf{q}_2)\ \vec\sigma(\mathbf{q}_3)\cdot\vec\sigma(\mathbf{q}_4)\rangle_\sigma$	+ (5.35)

= 16

Result

[1] 1  $\langle \tilde m_\alpha \tilde m_\alpha \tilde m_\beta \tilde m_\beta\rangle_\sigma \longrightarrow u$ ✓

[2] 4  $\langle \sigma m m m\rangle_\sigma \longrightarrow 0$

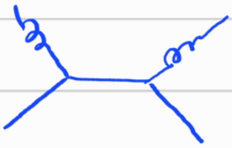
σ ↓ Ga

6-term

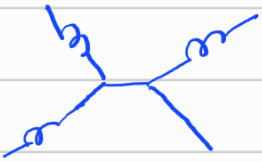
[3] 2  $\langle \sigma\sigma m m\rangle$

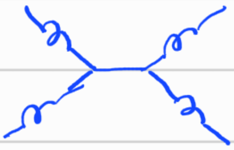
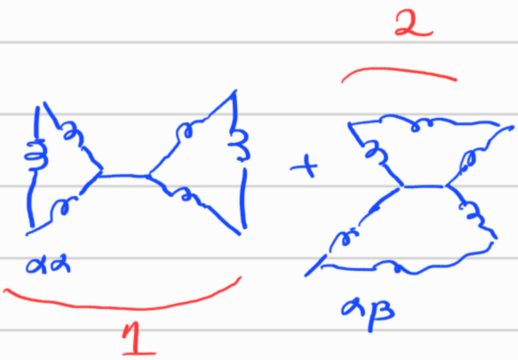
$$\langle \sigma_\alpha\sigma_\alpha m_\beta m_\beta\rangle = \langle\sigma_\alpha\sigma_\alpha\rangle_\sigma m_\beta m_\beta \longrightarrow m^2 \longrightarrow t$$

$\delta_{\alpha\alpha} \rightarrow n$

[4] 4  $\langle \sigma\sigma m m\rangle \to \langle \sigma_\alpha \sigma_\beta \rangle_\sigma m_\alpha m_\beta \to m^2 \to t$

$\delta_{\alpha\beta} \to 1$

[5] 4  $\langle \sigma\sigma\sigma \tilde{m}\rangle_\sigma \to 0$

[6] 1  $\langle \sigma\sigma\sigma\sigma\rangle_\sigma =$ 

$\underbrace{16\text{-term}}$

(5.35)

$\delta f_\ell^{(1)}$ $\qquad Z_\sigma - F_\sigma$

↑ Regular Par

We can determine the form of $\beta\tilde{\mathcal{H}}$ due to

Coarse graining part.

$$\boxed{t \to \tilde{t} = t + 4u(n+2)\int_{\Lambda/\ell}^{\Lambda} \frac{d^d q}{(2\pi)^d}\frac{1}{t+Kq^2}, \quad \tilde{K}=K \quad \& \quad \tilde{u}=u \quad (5.38)\ (5.39)}$$

☆ Rescaling $\quad q' = q\ell \quad q = q'/\ell$

☆ Renormalizing $\quad m' = m/z$

$Z_\ell = Z \to$ Recursive Relation $\to$ Fixed point

$\to$ RG Flow and scaling Expon..

$$\beta \tilde{\mathcal{H}}'[m'] = V\left(\delta f_\ell^0 + u\,\delta f_\ell^1\right) + \int_0^\Lambda \frac{d^d q'}{(2\pi)^d}\, \ell^{-d} z^2 \frac{\left(\tilde{t} + K \ell^{-2} q'^2\right)}{2}$$

(5.40) $\times |m'(q')|^2$

$$+ u z^4 \ell^{-3d} \int_0^\Lambda \frac{d^d q_1'\, d^d q_2'\, d^d q_3'}{(2\pi)^{3d}}\, \underbrace{m'(q_1') \cdot m'(q_2')}_{m.m}\, \overbrace{m'(q_3') \cdot m'(-q_1'-q_2'-q_3')}^{m.m}$$

$$t' = \ell^{d} z^2 \tilde{t}$$

$$K' = \ell^{-d-2} z^2 K \longrightarrow K' = K \rightarrow \ell^{-d-2} z^2 = 1 \rightarrow z = \ell^{1+\frac{d}{2}}$$

$$u' = \ell^{-3d} z^4 u$$

$$\Downarrow$$

$$\begin{cases} t'_\ell = \ell^2\left[t + 4u(n+2)\displaystyle\int_{\Lambda/\ell}^\Lambda \frac{d^d q}{(2\pi)^d} \frac{1}{t + Kq^2}\right] \\ u'_\ell = \ell^{4-d} u = \ell^{\epsilon} u \end{cases}$$ (5.42)

$$\boxed{\epsilon \equiv 4 - d}$$

$\downarrow$

ϵ – Expansion (5.7)

By linearizing $\ell = (1 + \delta\ell) = e^{\delta \ell}$

$$\begin{cases} t'_\ell = t(\ell=1) + \delta\ell \dfrac{dt}{d\ell} + \mathcal{O}(\delta\ell^2) \\ u'_\ell = u(\ell=1) + \delta\ell \dfrac{du}{d\ell} + \mathcal{O}(\delta\ell^2) \end{cases}$$

$$(5.42) \Longrightarrow \begin{cases} t + \delta\ell \dfrac{dt'}{d\ell} = (1 + 2\delta\ell)\left(t + 4u(n+2) \underbrace{\dfrac{S_d}{(2\pi)^d}}_{K_d} \dfrac{\Lambda^d \delta\ell}{t + K\Lambda^2}\right) \\ u + \delta\ell \dfrac{du'}{d\ell} = (1 + (4-d)\delta\ell)\, u \end{cases}$$

$$\begin{cases} \dfrac{dt'}{d\ell} = 2t + \dfrac{4u(n+2) K_d \Lambda^d}{t + K\Lambda^2} \\ \dfrac{du'}{d\ell} = (4-d)u \end{cases}$$

$$\frac{d}{d\ell}\begin{pmatrix} \delta t \\ \delta u \end{pmatrix} = \begin{pmatrix} 2 & \dfrac{4(n+2) K_d \Lambda^{d-2}}{K} \\ 0 & 4-d \end{pmatrix}\begin{pmatrix} \delta t \\ \delta u \end{pmatrix}$$

Gauss $\leftarrow \begin{cases} t^* = 0 \\ u^* = 0 \end{cases}$

Fig (5.3)

$\rightarrow d > 4 \longrightarrow x_u < 0 \longrightarrow$ Irrelevant

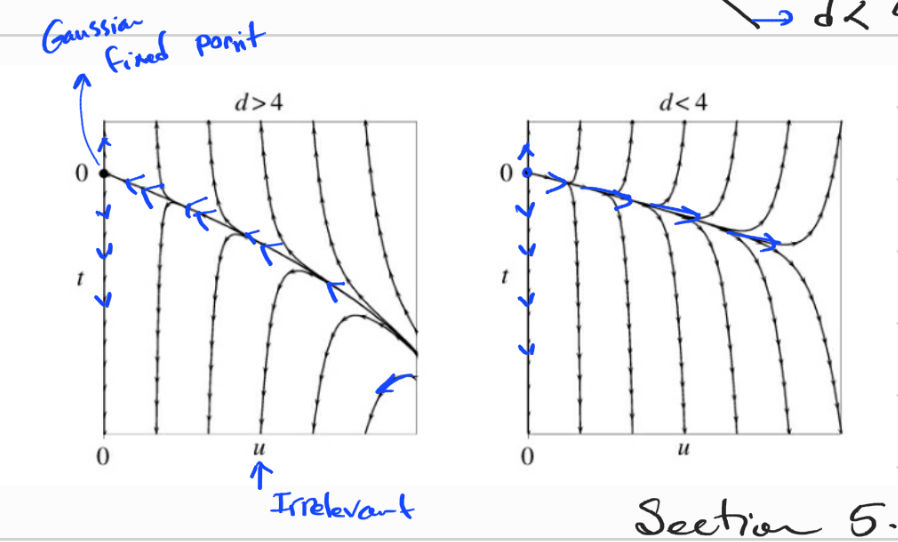
$\rightarrow d < 4 \longrightarrow x_u > 0 \longrightarrow$ Relevant

$\downarrow$

Breaking Perturbation

$\downarrow$

$\mathcal{O}(u^3)$



Section 5.6

$$\beta\tilde{\mathcal{H}} = \overset{V\delta f_\ell^{(0)}}{\widetilde{\beta\mathcal{H}_\sigma}} + \underbrace{\int_0^{\Lambda/\ell} \frac{d^d q}{(2\pi)^d}\left(\frac{t+Kq^2}{2}\right)|\tilde{m}(q)|^2 + \langle U\rangle_\sigma}_{\text{Section 5.5}}$$

$$\underbrace{-\frac{1}{2!}\left[\langle U^2\rangle_\sigma - \langle U\rangle_\sigma^2\right]}_{\text{Section 5.6}} + \mathcal{O}(U^3)$$

$$\langle U\rangle_\sigma \Longrightarrow \left\langle [\tilde{m}+\sigma][\tilde{m}+\sigma][\tilde{m}+\sigma][m+\sigma]\right\rangle_\sigma$$

$$\begin{array}{cccccc} [1], & [2], & [3], & [4], & [5], & [6] \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ u & 0 & t & t & 0 & Z_\sigma \end{array}$$

$$\langle U^2\rangle_\sigma \Rightarrow \left\langle [\tilde{m}+\sigma][\tilde{m}+\sigma][\tilde{m}+\sigma][\tilde{m}+\sigma][\tilde{m}+\sigma][\tilde{m}+\sigma][\tilde{m}+\sigma][\tilde{m}+\sigma]\right\rangle_\sigma$$

$$2^8 = \text{term} = 256$$

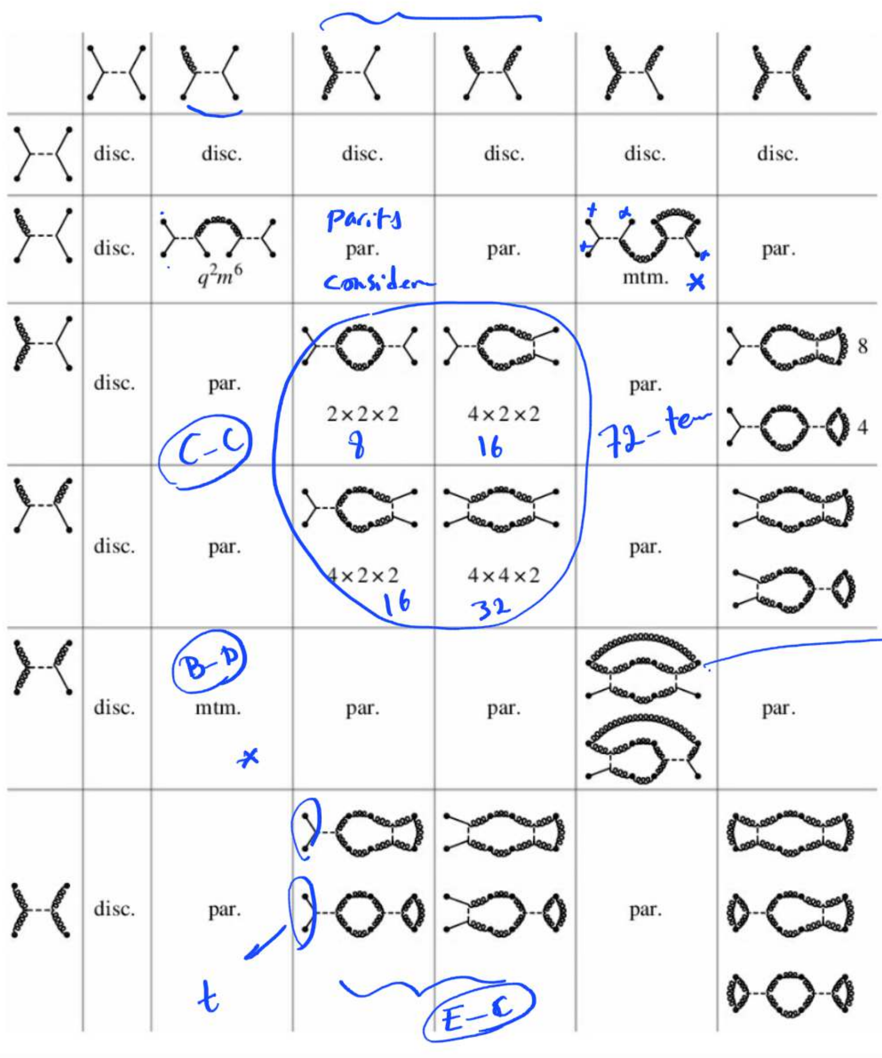
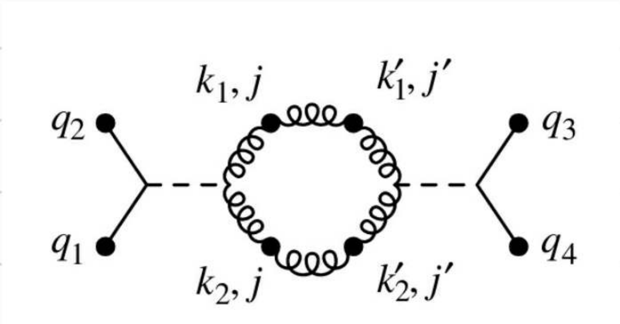


Fig. 5.4 Diagrams appearing in the second-order RG calculation (par. and disc. indicate contributions that are zero due to parity considerations, or being disconnected and mtm. is used to label diagrams that appear at higher order in q^2 due to momentum conservation).

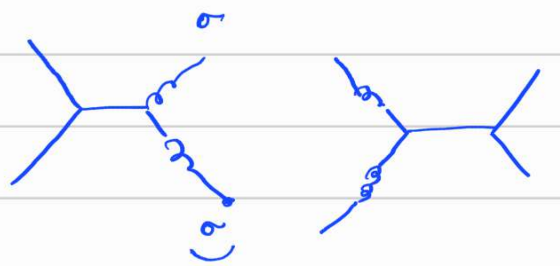
This results in only [3] − [3], [3] − [4], [4] − [4] } 72 terms.

Which contributes to $\mathcal{U}$

[3] − [3]



= ?



$2 \times \binom{2}{1} \times 2$

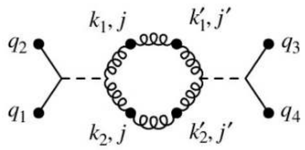
m^4

β, α

انتخاب کدام σ

تعداد حالت های قابل انتخاب

$(m \cdot m)^2$



$$
\begin{aligned}
&\frac{u^2}{2}\times 2\times 2\times 2\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}_1\cdots\mathrm{d}^d\mathbf{q}_4}{(2\pi)^{4d}}\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{k}_1\mathrm{d}^d\mathbf{k}_2\mathrm{d}^d\mathbf{k}_1'\mathrm{d}^d\mathbf{k}_2'}{(2\pi)^{4d}}\\
&\quad\times(2\pi)^{2d}\delta^d(\mathbf{q}_1+\mathbf{q}_2+\mathbf{k}_1+\mathbf{k}_2)\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{q}_3+\mathbf{q}_4)\\
&\quad\times\frac{\delta_{\alpha\alpha'}(2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_1')}{t+Kk_1^{2'}}\frac{\delta_{\alpha\alpha'}(2\pi)^d\delta^d(\mathbf{k}_2+\mathbf{k}_2')}{t+Kk_2^{2'}}\tilde{\tilde{\vec{m}}}(\mathbf{q}_1)\cdot\tilde{\tilde{\vec{m}}}(\mathbf{q}_2)\tilde{\tilde{\vec{m}}}(\mathbf{q}_3)\cdot\tilde{\tilde{\vec{m}}}(\mathbf{q}_4)\\
&=4nu^2\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}_1\cdots\mathrm{d}^d\mathbf{q}_4}{(2\pi)^{4d}}(2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)\tilde{\tilde{\vec{m}}}(\mathbf{q}_1)\cdot\tilde{\tilde{\vec{m}}}(\mathbf{q}_2)\tilde{\tilde{\vec{m}}}(\mathbf{q}_3)\cdot\tilde{\tilde{\vec{m}}}(\mathbf{q}_4)\\
&\quad\times\int\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{(t+Kk^2)\,(t+K(\mathbf{q}_1+\mathbf{q}_2-\mathbf{k})^2)}.
\end{aligned}\tag{5.48}
$$

The contractions from terms $[3]\times[4]$, $[4]\times[3]$, and $[4]\times[4]$ lead to similar expressions with prefactors of 8, 8, and 16 respectively. Apart from the dependence on $\mathbf{q}_1$ and $\mathbf{q}_2$, the final result has the form of $\mathcal{U}[\tilde{\tilde{\vec{m}}}]$. In fact the last integral can be expanded as

Taking into accant $\mathcal{O}(u^2)$. and considering lower order of Momentum and $\epsilon = 4-d$, $\underline{u}$ is modified, therefore

$\mathcal{O}(q^2)$

$$
\begin{aligned}
\beta\tilde{\mathcal{H}} = & V\left(\delta f_b^0 + u\delta f_b^1 + u^2\delta f_b^2\right) + \int_0^{\Lambda/b} \frac{\mathrm{d}^d \mathbf{q}}{(2\pi)^d} |\tilde{m}(\mathbf{q})|^2 \\
& \left[\frac{t+Kq^2}{2} + 2u(n+2)\int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \frac{1}{t+Kk^2} - \frac{u^2}{2} A(t, K, q^2)\right] \\
& + \int_0^{\Lambda/b} \frac{\mathrm{d}^d \mathbf{q}_1 \mathrm{d}^d \mathbf{q}_2 \mathrm{d}^d \mathbf{q}_3}{(2\pi)^{3d}} \tilde{\tilde{m}}(\mathbf{q}_1) \cdot \tilde{\tilde{m}}(\mathbf{q}_2) \tilde{\tilde{m}}(\mathbf{q}_3) \cdot \tilde{\tilde{m}}(\mathbf{q}_4) \times \left[u - \frac{u^2}{2}(8n+64)\right. \\
& \left.\int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \frac{1}{(t+Kk^2)^2} + \mathcal{O}(u^2 q^2)\right] + \mathcal{O}(u^2 \tilde{m}^6 q^2, \cdots) + \mathcal{O}(u^3). \qquad (5.50)
\end{aligned}
$$

and

5.7 The ϵ-expansion

The parameter space (K, t, u) is no longer closed at this order; several new interactions proportional to m^2, m^4, and m^6, all consistent with symmetries of the problem, appear in the coarse-grained Hamiltonian at second order in u. Ignoring these interactions for the time being, the coarse grained parameters are given by

$$\begin{cases} \tilde{K} = K - u^2 A''(0) \\ \tilde{t} = t + 4(n+2)\, u \int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \frac{1}{t+Kk^2} - u^2 A(0) \\ \tilde{u} = u - 4(n+8)u^2 \int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \frac{1}{(t+Kk^2)^2}, \end{cases} \tag{5.51}$$

where $A(0)$ and $A''(0)$ correspond to the first two terms in the expansion of $A(t, K, q^2)$ in Eq. (5.50) in powers of q.

After the *rescaling* $\mathbf{q} = b^{-1}\mathbf{q}'$, and *renormalization* $\tilde{\vec{m}} = z\vec{m}'$, steps of the RG procedure, we obtain

$$K' = b^{-d-2}z^2\tilde{K}, \quad t' = b^{-d}z^2\tilde{t}, \quad u' = b^{-3d}z^4\tilde{u}. \tag{5.52}$$

As before, the renormalization parameter z is chosen such that $K' = K$, leading to $\quad O(u^2) \quad z = \ell^{1+d/2}$

$b=\ell$

$$z^2 = \frac{b^{d+2}}{(1 - u^2A''(0)/K)} = b^{d+2}\left(1 + O(u^2)\right). \tag{5.53}$$

The value of z does depend on the fixed point position u^*. But as u^* is of the order of ϵ, $z = b^{1+\frac{d}{2}+\mathcal{O}(\epsilon^2)}$, it is not changed at the lowest order. Using this value of z, and following the previous steps for constructing differential recursion relations, we obtain

$$\begin{cases} \frac{\mathrm{d}t}{\mathrm{d}\ell} = 2t + \frac{4u(n+2)K_d\Lambda^d}{t+K\Lambda^2} - A(t, K, \Lambda)u^2 \\ \frac{\mathrm{d}u}{\mathrm{d}\ell} = (4-d)u - \frac{4(n+8)K_d\Lambda^d}{(t+K\Lambda^2)^2}u^2. \end{cases} \tag{5.54}$$

$\beta_\ell = 0$

The fixed points are obtained from $\mathrm{d}t/\mathrm{d}\ell = \mathrm{d}u/\mathrm{d}\ell = 0$. In addition to the Gaussian fixed point at $u^* = t^* = 0$, discussed in the previous section, there is now a non-trivial fixed point located at

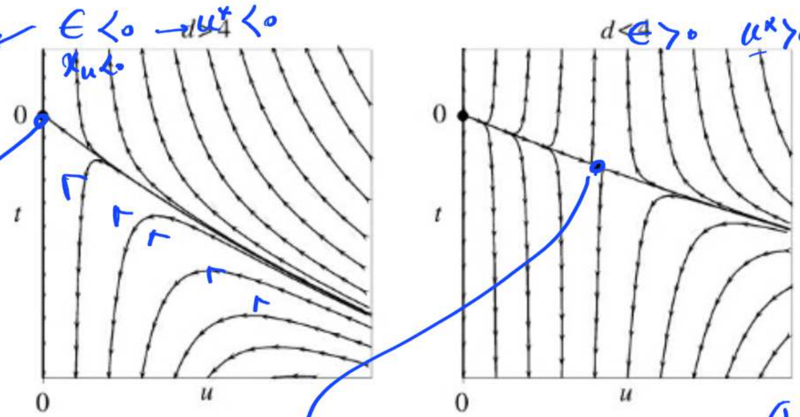
$$\begin{cases} u^* = \frac{(t^*+K\Lambda^2)^2}{4(n+8)K_d\Lambda^d}\epsilon = \frac{K^2}{4(n+8)K_4}\epsilon + \mathcal{O}(\epsilon^2) \\ t^* = -\frac{2u^*(n+2)K_d\Lambda^d}{t^*+K\Lambda^2} = -\frac{(n+2)}{2(n+8)}K\Lambda^2\epsilon + \mathcal{O}(\epsilon^2). \end{cases} \tag{5.55}$$

The above expressions have been further simplified by systematically keeping terms to first order in $\epsilon = 4 - d$.

Fig. 5.5 RG flows obtained perturbatively to second order.

$m = \pm\sqrt{-\frac{a_2}{2a_4}}$ $a_4 < 0$ $a_2 > 0$ ← $\epsilon < 0 \to u^* < 0$

$d > 4$ $\epsilon > 0$ $u^* > 0 \to$ فیزیکی



$(t^*, u^*) = (0,0)$ Gaussian fixed point

Non-trival Fixed point ← فیزیکی

$(t^*, u^*) =$

Fig (12.2) Goldenfe

Linearizing the recursion relations in the vicinity of the fixed point results in

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix} = \begin{pmatrix} 2 - \dfrac{4(n+2)K_d\Lambda^d}{(t^*+K\Lambda^2)^2}u^* - A'u^{*2} & \dfrac{4(n+2)K_d\Lambda^d}{t^*+K\Lambda^2} - 2Au^* \\ \dfrac{8(n+8)K_d\Lambda^d}{(t^*+K\Lambda^2)^3}u^{*2} & \epsilon - \dfrac{8(n+8)K_d\Lambda^d}{(t^*+K\Lambda^2)^2}u^* \end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}. \quad (5.56)$$

At the Gaussian fixed point, $t^* = u^* = 0$, and Eq. (5.45) is reproduced. At the new fixed point of Eqs. (5.55),

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix} = \begin{pmatrix} 2 - \dfrac{4(n+2)K_4\Lambda^4}{K^2\Lambda^4}\dfrac{K^2\epsilon}{4(n+8)K_4} & \cdots\cdots \\ \mathcal{O}(\epsilon^2) & \epsilon - \dfrac{8(n+8)K_4\Lambda^4}{K^2\Lambda^4}\dfrac{K^2\epsilon}{4(n+8)K_4} \end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}. \quad (5.57)$$

We have not explicitly calculated the top element of the second column as it is not necessary for calculating the eigenvalues. This is because the lower element of the first column is zero to order of ϵ. Hence the eigenvalues are determined by the diagonal elements alone. The first eigenvalue is positive, controlling the instability of the fixed point,

$$x_t \to y_t = 2 - \frac{(n+2)}{(n+8)}\epsilon + \mathcal{O}(\epsilon^2). \quad (5.58)$$

$\boxed{\mathcal{O}(\epsilon^2) \quad x_u = y_u = \epsilon}$ $(t^* = 0, u^* = 0)$ $d < 4$ $x_u > 0$

The second eigenvalue,

$(t^* \neq 0, u^* \neq 0)$

$$x_u \to y_u = -\epsilon + \mathcal{O}(\epsilon^2), \quad (5.59)$$

is negative for $d < 4$. The new fixed point thus has co-dimension of one and can describe the phase transition in these dimensions. It is quite satisfying that while various intermediate results, such as the position of the fixed point, depend on such microscopic parameters as K and Λ, the final eigenvalues are pure numbers, only depending on n and $d = 4 - \epsilon$. These eigenvalues characterize the *universality classes* of rotational symmetry breaking in $d < 4$, with short-range interactions. (As discussed in the problem section, long-range interaction may lead to new universality classes.)

The divergence of the correlation length, $\xi \sim (\delta t)^{-\nu}$, is controlled by the exponent

$$\nu = \frac{1}{y_t} = \left\{2\left[1 - \frac{(n+2)}{2(n+8)}\epsilon\right]\right\}^{-1} = \frac{1}{2} + \frac{1}{4}\frac{n+2}{n+8}\epsilon + \mathcal{O}(\epsilon^2). \tag{5.60}$$

The singular part of the free energy scales as $f \sim (\delta t)^{2-\alpha}$, and the heat capacity diverges with the exponent

$$\alpha = 2 - d\nu = 2 - \frac{(4-\epsilon)}{2}\left[1 + \frac{1}{2}\frac{n+2}{n+8}\epsilon\right] = \frac{4-n}{2(n+8)}\epsilon + \mathcal{O}(\epsilon^2). \tag{5.61}$$

To complete the calculation of critical exponents, we need the eigenvalue associated with the (relevant) symmetry breaking field h. This is easily found by adding a term $-\vec{h}\cdot\int \mathrm{d}^d\mathbf{x}\,\vec{m}(\mathbf{x}) = -\vec{h}\cdot\vec{m}(\mathbf{q}=\mathbf{0})$ to the Hamiltonian. This term is not affected by coarse graining or rescaling, and after the renormalization step changes to $-z\vec{h}\cdot\vec{m}'(\mathbf{q}'=\mathbf{0})$, implying

$$h' = zh = b^{1+\frac{d}{2}}h, \quad \Longrightarrow \quad y_h = 1 + \frac{d}{2} + \mathcal{O}(\epsilon^2) = 3 - \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2). \tag{5.62}$$

The vanishing of magnetization as $T \to T_c^-$ is controlled by the exponent

$$\begin{aligned}\beta = \frac{d - y_h}{y_t} &= \left(\frac{4-\epsilon}{2} - 1\right) \times \frac{1}{2}\left(1 + \frac{n+2}{2(n+8)}\epsilon + \mathcal{O}(\epsilon^2)\right) \\ &= \frac{1}{2} - \frac{3}{2(n+8)}\epsilon + \mathcal{O}(\epsilon^2),\end{aligned} \tag{5.63}$$

while the susceptibility diverges as $\chi \sim (\delta t)^{-\gamma}$, with

$$\gamma = \frac{2y_h - d}{y_t} = 2 \times \frac{1}{2}\left(1 + \frac{n+2}{2(n+8)}\epsilon\right) = 1 + \frac{n+2}{2(n+8)}\epsilon + O(\epsilon^2). \tag{5.64}$$

Using the above results, we can estimate various exponents as a function of d and n. For example, for $n = 1$, by setting $\epsilon = 1$ or 2 in Eqs. (5.60) and Eqs. (5.63) we obtain the values $\nu(1) \approx 0.58$, $\nu(2) \approx 0.67$, and $\beta(1) \approx 0.33$, $\beta(2) \approx 0.17$. The best estimates of these exponents in $d = 3$ are $\nu \approx 0.63$, and $\beta \approx 0.32$. In $d = 2$ the exact values are known to be $\nu = 1$ and $\beta = 0.125$. The estimates for β are quite good, while those for ν are less reliable. It is important to note that in all cases these estimates are an improvement over the mean field (saddle point) values. Since the expansion is around four dimensions, the results are more reliable in $d = 3$ than in $d = 2$. In any case, they correctly describe the decrease of β with lowering dimension, and the increase of ν. They also correctly describe the trends with varying n at a fixed d as indicated by the following table of exponents $\alpha(n)$.

Although the sign of α is incorrectly predicted at this order for $n = 2$ and 3, the decrease of α with increasing n is correctly described.

$$Z = \int Ds\, e^{+\mathcal{H}} = \int Ds\, e^{-\beta \mathcal{H}}$$

$$\mathcal{H}_{\text{Goldn}} \equiv -\beta \mathcal{H}_{\text{Kar}}$$

$$\mathcal{H}_{\text{Kar}} = -\mathcal{H}_{\text{Goldn}}$$

$$-\beta \mathcal{H}\Big|_{\text{Kar}} \longrightarrow \quad -\mathcal{H} = \int d^d r \left[\overbrace{\tfrac{1}{2}}^{K=1} (\nabla s)^2 + \tfrac{1}{2} \overbrace{r_0}^{t} s^2 + \tfrac{1}{4} u_0 s^4 - \underbrace{h}_{h=0} s \right]$$