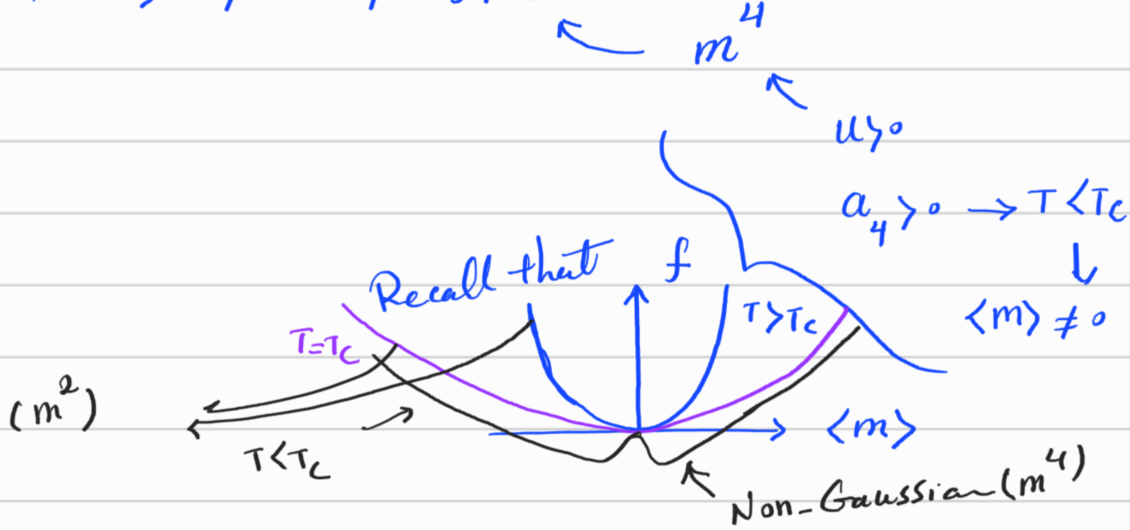
بسم الله الرحمن الرحيم

Diagramatic Representation of PT

① $\mathcal{H} = \mathcal{H}_0 + \mathcal{U} \longrightarrow \beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U}$



$$\underbrace{\beta\mathcal{H} = \int d^dx \left(\frac{t m^2 + K(\nabla m)^2 + L(\nabla^2 m)^2 + \dots}{2} \right)}_{\substack{\beta\mathcal{H}_0 \\ \downarrow \\ \text{Gaussian term}}} + \underbrace{u \int d^dx\, \underbrace{(\vec{m}\cdot\vec{m})^2}_{[m^4]}}_{\substack{\mathcal{U} \\ \downarrow \\ \text{Non-Gaussian term}}}$$

For a typical observable quantity, we have

② $$\langle \mathcal{O} \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle \mathcal{O}\, \mathcal{U}^n \rangle_0^{\text{Connected}}$$

One can compute every observable, perturbatively (without RG)

③ Up to $(p+1)$ of perturbation $\mathcal{O}(u^{p+1})$, we are interested in determining $\langle \mathcal{O} \rangle = \checkmark$

④ Suppose that $\mathcal{O} \equiv \prod_{i=1}^{g} m_{\alpha_i}(q_i)$ $\qquad g \geq 1$

$$\langle \mathcal{O} \rangle = \left\langle \prod_{i=1}^{g} m_{\alpha_i}(q_i) \right\rangle$$

$\alpha_i = 1, \dots n$

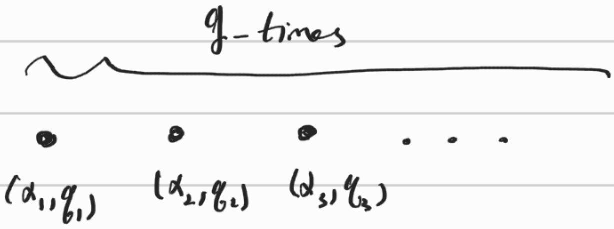
n-vector field

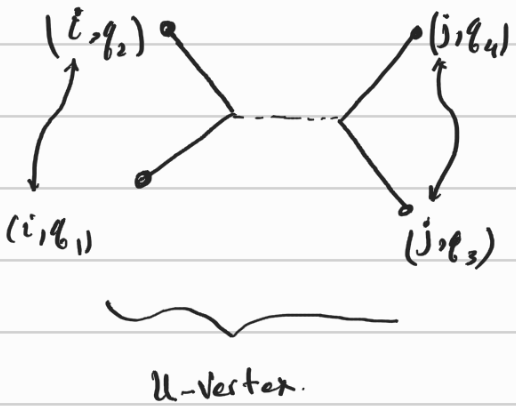
$i = 1, \dots g \leftarrow$ g-Point Correlation function

p, g $\leftarrow$ g-Point Correlation function

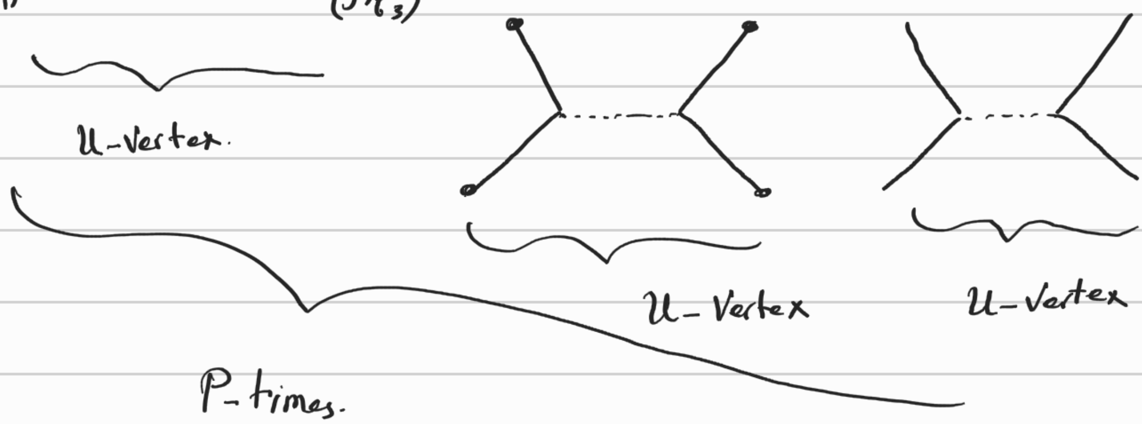
$\mathcal{O}(u^{p+1})$ or including u^p in our perturbative approach.

⑤





$$(\vec{m}\cdot\vec{m})^2 = m_i(q_1)\, m_i(q_2)\, m_j(q_3)\, m_j(q_4)$$

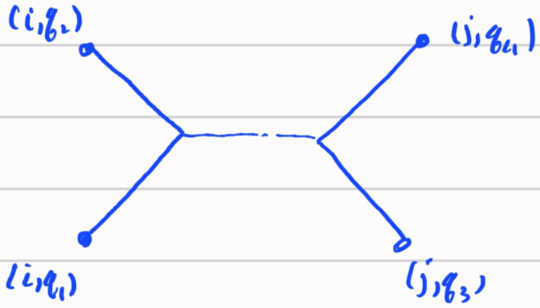


⑥ ☆ $\langle m_\alpha(q)\, m_\beta(q')\rangle_0 \equiv$

☆

$$= \frac{\delta_{\alpha\beta}\,\delta_D(q+q')(2\pi)^d}{t+Kq^2+\cdots}$$

⑦



$$\equiv u(2\pi)^d\,\delta_D(q_1+q_2+q_3+q_4)$$

⑧ We should integrate out on 4P-times internal momentums

⑨ $\delta_{\alpha\beta}$, δ_{ij}, $\delta_{ii}=n$ ← # of vector field.

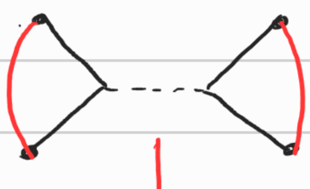
\# of Dof of field at each location.

⑩ Only so-called Fully-Connected Diagrams are non-Zero

⑪ $\frac{(-1)^p}{p!}$ Coefficient should be taken into account

⑫ As an illustration

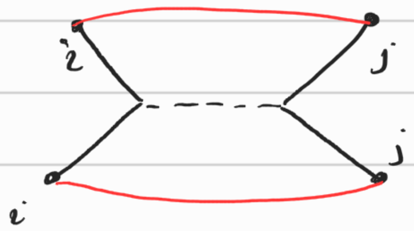
$$\langle m_\alpha(q)\, m_\beta(q')\rangle_0 \langle m_i(q_1)\, m_i(q_2)\, m_j(q_3)\, m_j(q_4)\rangle_0$$



1-term

↑ ↓

$$\langle m_i(q_1)\, m_i(q_2)\rangle_0 \langle m_j(q_3)\, m_j(q_4)\rangle_0$$



2-term

↓

$$\langle m_i(q_1)\, m_j(q_3)\rangle \langle m_i(q_2)\, m_j(q_4)\rangle$$

3-terms

(13) $p=1$, $g=2$

$$\Rightarrow \overbrace{\langle m_\alpha(q)\, m_\beta(q')\rangle}^{\mathcal{O}} = \langle m_\alpha(q)\, m_\beta(q')\rangle_0 + \overbrace{\frac{(-1)^1}{1!}}^{\frac{(-1)^p}{p!}} \langle \mathcal{O}\, \mathcal{U}^1\rangle_0^{\text{Conn}}$$

$$= \langle m_\alpha(q)\, m_\beta(q')\rangle_0 + \frac{(-1)^1}{1!} \langle m_\alpha(q)\, m_\beta(q')\; m_i(q_1)\, m_i(q_2)\, m_j(q_3)\, m_j(q_4)\rangle_0^{\text{Connected}} + \mathcal{O}(u^2)$$

from (6)

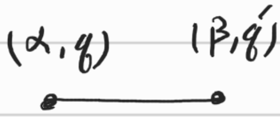
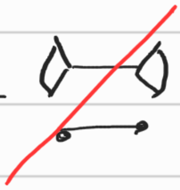
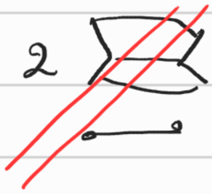
$= $  $+ \frac{(-1)^1}{1!} \Big[\langle m_\alpha(q)\, m_\beta(q')\, m_i(q_1)\, m_i(q_2)\, m_j(q_3)\, m_j(q_4)\rangle_0$

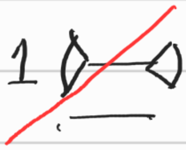
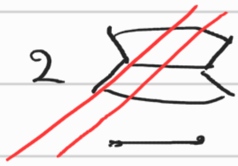
$- \langle m_\alpha(q)\, m_\beta(q')\rangle_0 \langle m_i(q_1)\, m_i(q_2)\, m_j(q_3)\, m_j(q_4)\rangle_0 \Big]$

$+ \mathcal{O}(u^2)$

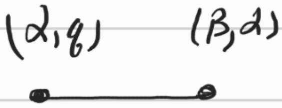
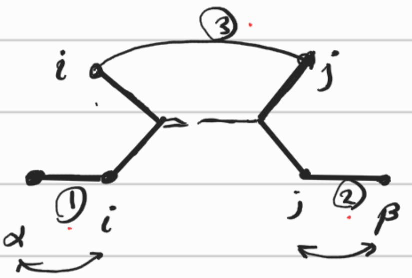
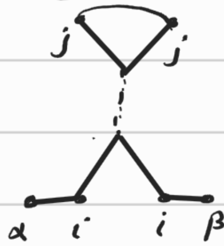
(α, q) (β, q')

$= $  $+ \frac{(-1)^1}{1!} \left[\underbrace{\binom{6}{2}}_{15} - 3 \right]$

$//$ $=$  $+ \frac{(-1)^1}{1!} \Bigg[1$  $+ 2$  $+ \underbrace{12\text{-terms}}_{\text{Fully connected}}$

$\underbrace{-1 \text{ } \text{ } -2}$   $\Bigg]$

3-terms of non-fully connect

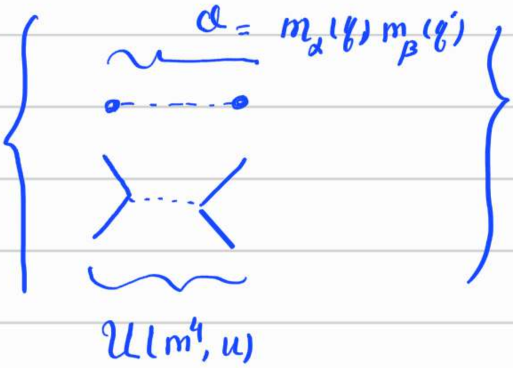
$//$ $=$  $+ \frac{(-1)^1}{1!} \Bigg[\underbrace{8}_{4\times 2 = 8}$  $+4$  $\underset{4\times 1}{} \Bigg]$

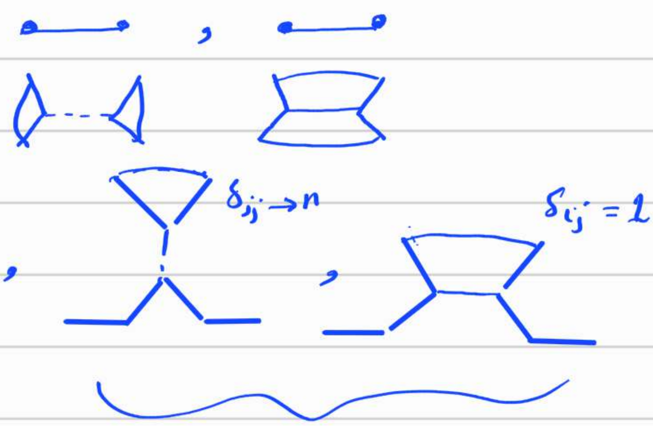
$$\langle m_\alpha(q) m_\beta(q') \rangle = \langle m_\alpha(q) m_\beta(q') \rangle_0 + \frac{(-1)^1}{1!} \Big[8 \; // \; + \; 4 \; // \Big]$$

$$// = \frac{\delta_{\alpha\beta}\, \delta_D(q+q')(2\pi)^d}{t+Kq^2} + \frac{(-1)^1}{1!} \Bigg[\frac{8u\,(2\pi)^d\, \delta_{\alpha i}\, \delta_{\beta j}\, \delta_{ij}}{(t+Kq^2)^2} \int \frac{d^dq'}{(2\pi)^d} \frac{1}{(t+Kq'^2)}$$

$$+ \frac{4u\,(2\pi)^d\, \delta_{\alpha i}\, \delta_{\beta i}\, \overset{n}{\delta_{jj}}}{(t+Kq^2)^2} \int \frac{d^dq'}{(2\pi)^d} \frac{1}{(t+Kq'^2)} \Bigg]$$

$$// = \frac{\delta_{\alpha\beta}\, \delta_D(q+q')(2\pi)^d}{(t+Kq^2)} \Bigg[1 - \underbrace{\frac{(4un + 8u)}{(t+Kq^2)} \int \frac{d^dq'}{(2\pi)^d} \frac{1}{(t+Kq'^2)}}_{\text{Correction-term due to } u'} \Bigg]$$



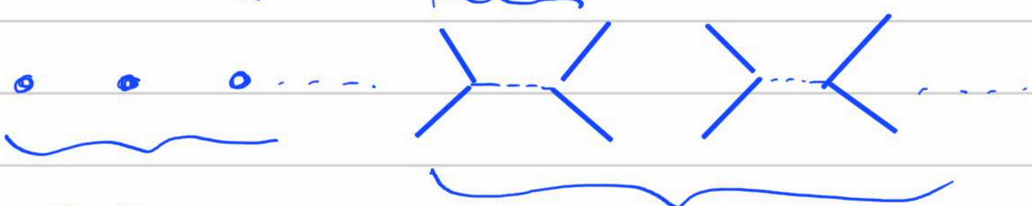


Fully-Connected Diagram

Combination of PT with RG

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U}$$

$\mathcal{U}(m^4, u)$



g-times

P-times

① Coarse graining

$x \geqslant a \longrightarrow x \geqslant la$ Real space

$q \leqslant \Lambda \longrightarrow q \leqslant \frac{\Lambda}{l}$ Fourier Space

marginalization on this range.

$\underbrace{a \leqslant x \leqslant la}_{\text{Real Space}}$

Fourier Space

$\left\{ \begin{array}{l} \frac{\Lambda}{l} \leqslant q \leqslant \Lambda \\ \frac{1}{la} \leqslant q \leqslant \frac{1}{a} \end{array} \right\}$

Singular part

IR-cutoff

UV-cutoff

$$\underbrace{\vec{m}(\vec{x})}_{\text{n-Vector field}} \longrightarrow \widetilde{\vec{m}(\vec{q})} = \begin{cases} \tilde{m}(q) & \text{for } 0 \leq q \leq \frac{\Lambda}{\ell} \\ \tilde{\sigma}(q) & \text{for } \frac{\Lambda}{\ell} < q \leq \Lambda \end{cases}$$

Regular part

Non-Singular part and we integrate out on this range

$$\vec{m}(\vec{x}) = \int \frac{d^d q}{(2\pi)^d} e^{-i\bar{q}\cdot\bar{x}} \vec{m}(\vec{q})$$

$$= \underbrace{\int_0^{\Lambda/\ell} \frac{d^d q}{(2\pi)^d} e^{-i\bar{q}\cdot\bar{x}} \tilde{m}(\bar{q})}_{\boxed{0 \leq q \leq \frac{\Lambda}{\ell}}} + \underbrace{\int_{\Lambda/\ell}^{\Lambda} \frac{d^d q}{(2\pi)^d} e^{-i\bar{q}\cdot\bar{x}} \bar{\sigma}(\bar{q})}_{\boxed{\frac{\Lambda}{\ell} \leq q \leq \Lambda}}$$

Marginalizing on this range

$\overline{\tilde{m}(q)}$

$\sigma(q)$

$$\boxed{\bar{m}(\bar{q}) = \tilde{m}(\bar{q}) + \bar{\sigma}(q)}$$

$0 \leq q \leq \frac{\Lambda}{\ell} \rightarrow \tilde{m}(q) \rightarrow$ n-vector field

$\frac{\Lambda}{\ell} \leq q \leq \Lambda \longrightarrow \bar{\sigma}(q) \rightarrow$ n-vector field

$$\star \quad Z = \int \underbrace{Dm(q)}_{\text{n-vector field}} e^{-\underbrace{\beta \tilde{\mathcal{H}}(m)}_{=\beta\mathcal{H}_0 + \mathcal{U}}}$$

$$= \int D\tilde{m}(q)\, D\sigma(q)\, e^{-\beta\tilde{\mathcal{H}}_0 - \tilde{\mathcal{U}}}$$

$$= \int D\tilde{m}(q)\, D\sigma(q)\, e^{-\int_0^{\Lambda} \frac{d^d q}{(2\pi)^d} \left(\frac{t+Kq^2}{2}\right)\left(\overbrace{|\tilde{m}(q)|^2 + |\sigma(q)|^2}^{|m(q)|^2}\right)}$$

$$\times\, e^{-\mathcal{U}([\tilde{m}(q), \sigma(q), u)}$$

$$Z = \int D\tilde{m}(q)\, e^{-\int_0^{\Lambda/\ell} \frac{d^d q}{(2\pi)^d} \frac{(t+Kq^2)}{2} |\tilde{m}(q)|^2}$$

$$\times \int D\sigma(q)\, e^{-\int_{\Lambda/\ell}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{(t+Kq^2)}{2} |\sigma(q)|^2} \times e^{-\mathcal{U}}$$

(5.29)

Section 5.5: Perturbation RG (first order)