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※ Characteristic Function (Recall)

※ $\langle \tilde{m}_\alpha(q_1)\, \tilde{m}_\beta(q_2) \rangle \longrightarrow \chi \longrightarrow t_c = ?$

$$\mathcal{O} \equiv \tilde{m}_\alpha(q_1)\, \tilde{m}_\beta(q_2) \quad , \quad \beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U}$$

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① Characteristic Function (Generating Function)

$$\mathcal{Z}_m(\vec{\lambda}) \quad (\mathcal{Z}_\phi(\vec{\lambda}))$$

(Analogous to Partition Function)

$$\ln \mathcal{Z}_\phi(\lambda) = \quad \text{Free Energy}$$

Ref. ① Course 3,4 Cosmological field Workshop

② Stochastic 99/2/2

② $\{A_r\} : \{\phi_1, \phi_2 \ldots \phi_n\} = \vec{\phi}(r)$

$\{A\} : \{\vec{\phi}(r_1), \vec{\phi}(r_2) \ldots, \partial\vec{\phi}, \partial^2\phi, \ldots\}$

$\{A\} : \{m_1 \ldots m_n\} = \vec{m}(r)$ n-Vector field

$\{A\} : \{\vec{m}(r_1) \sim \vec{m}(r_p)\}$

Our Goal $\langle f \rangle = \int d\vec{A}\, f(A)\, P(\vec{A})$

$$1 = \int dA \underbrace{P(A)}_{\text{Joint-PDF}} \quad \text{Normalization Condition.}$$

Characteristic Funct is useful to determin $\underbrace{P(\vec{A})}$

ρ_{mic} $\quad \rho_c$ $\quad \rho_{gc}$

$P(\phi)$

$$Z_\phi(\vec{\lambda}) \equiv \left\langle e^{i\vec{\lambda}\cdot\vec{\phi}} \right\rangle = \int D\phi\, e^{i\vec{\lambda}\cdot\vec{\phi}} P(\phi)$$

For $n=1$ $\quad Z_\phi(\lambda) = \langle e^{i\lambda\phi} \rangle = \int d\phi\, e^{i\lambda\phi} P(\phi)$

$$\rightarrow Z_\phi(\lambda) = \sum_{l=0}^{\infty} \frac{(i\lambda)^l}{l!} M_\phi^{(l)} = 1 + i\lambda\langle\phi\rangle - \frac{\lambda^2}{2!}\langle\phi^2\rangle + \dots$$

$$\begin{cases} M_\phi^{(l)} \equiv \langle \phi^l \rangle & : l\text{th moment of } \phi \\ M_\phi^{(l)} = \dfrac{d^l}{d(i\lambda)^l} Z_\phi(\lambda)\Big|_{\lambda=0} \end{cases}$$

Inverse F. T. $\quad P(\vec{\phi}) = \int D\lambda\, e^{-i\vec{\lambda}\cdot\vec{\phi}} Z_\phi(\vec{\lambda})$

$$Z_\phi(\lambda) = \exp\left(\sum_{l=1}^{\infty} \frac{(i\lambda)^l}{l!} K_\phi^{(l)} \right)$$

$$* \quad \ln Z_\phi(\lambda) = \sum_{l=1}^{\infty} \frac{(i\lambda)^l}{l!} K_\phi^{(l)} \quad *$$

$$K_\phi^{1} = M_\phi^{1} = \langle \phi \rangle$$

$$K_\phi^{2} = M_{\phi^2} - M_\phi^{2} = \langle \phi^2 \rangle - \langle \phi \rangle^2$$

Summary on Z in general Form.

$$A_\mu : (A_1, \ldots A_N)$$

$$Z_A(\vec{\lambda}) = \int d^N A \; e^{i\vec{\lambda}\cdot\bar{A}} P(\bar{A})$$

$$\ln Z_A(\bar{\lambda}) = \sum_{n=1}^{\infty} \frac{i^n}{n!} \left(\sum_{\mu_1=1}^{N} \cdots \sum_{\mu_n=1}^{N} K^{(n)}_{\mu_1 \mu_2 \ldots \mu_n} \lambda_{\mu_1} \lambda_{\mu_2} \cdots \lambda_{\mu_n} \right)$$

$$K^{(n)}_{\mu_1 \mu_2 \ldots \mu_n} = \langle A_{\mu_1} A_{\mu_2} \ldots A_{\mu_n} \rangle^{\text{Connected}} \quad : \text{Cumulant}$$

e.g. $n=2$, $\mu_1 = 1$

$$K^{(2)}_{11} = \langle A_1 A_1 \rangle^{\text{Connected}} = \langle A_1 A_1 \rangle - \langle A_1 \rangle \langle A_1 \rangle$$

$$= M^{(2)}_{A_1} - \left(M^{(1)}_{A_1}\right)^2$$

$n=2$, $\mu_1 = 1$ $\mu_2 = 2$

$$K^{(2)}_{12} = \langle A_1 A_2 \rangle^{\text{Connected}} = \langle A_1 A_2 \rangle - \langle A_1 \rangle \langle A_2 \rangle$$

3 $K^{(1)}_\mu = 0 = \underline{\underline{\langle A_\mu \rangle = 0}}$

$n = 1 \Longrightarrow 0$

$$\ln Z_A(\bar{\lambda}) = \sum_{n=2} \frac{i^n}{n!} \Big\{ \underbrace{\sum_{\mu_1=1}^{N} \cdots \sum_{\mu_{n}=1}^{N}}_{n\text{-times}} K^{(n)}_{\underbrace{\mu_1 \mu_2 \cdots \mu_n}_{n\text{-times}}} \underbrace{\lambda_{\mu_1} \lambda_{\mu_2} \cdots \lambda_{\mu_n}}_{n\text{-times}} \Big\}$$

$$☆ \quad Z_A(\bar{\lambda}) = \exp\Big[\overbrace{-\frac{1}{2} \lambda^T \cdot K^{(2)} \cdot \lambda}^{n=2} \Big]$$

$$\times \exp\Big[\underbrace{\sum_{n=3}^{\infty} \frac{i^n}{n!} \Big\{ \underbrace{\sum_{\mu_1=1}^{N} \cdots \sum_{\mu_n=1}^{N}}_{n\text{-times}} K^{(n)}_{\mu_1 \cdots \mu_n} \lambda_{\mu_1} \cdots \lambda_{\mu_n}}_{n \geqslant 3} \Big]$$

$$\lambda^T = [\lambda_1 \ \lambda_2 \ \cdots \ \lambda_N]_{1\times N}$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix}_{N\times 1}$$

$$K^{(2)} = \begin{bmatrix} \underset{\sigma_{11}^2}{\langle A_1 A_1 \rangle_c} & \langle A_1 A_2 \rangle_c & \cdots & \langle A_1 A_N \rangle_c \\ \langle A_2 A_1 \rangle_c & \underset{\sigma_{22}^2}{\langle A_2 A_2 \rangle_c} & & \langle A_2 A_N \rangle_c \\ \vdots & & & \\ \langle A_N A_1 \rangle_c & & & \underset{\sigma_{NN}^2}{\langle A_N A_N \rangle_c} \end{bmatrix}_{N\times N}$$

e.g. $N=2$

$$Z_A(\vec{\lambda}) = e^{-\frac{1}{2}(\lambda_1 \ \lambda_2) \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}} \times e^{\sum_{n=3}^{\infty} \cdots}$$

For Gaussian Case $\quad K^{(n)}_{\mu_1 \cdots \mu_n} = 0 \quad n \geqslant 3 \qquad$ for $K'_\mu = 0$

$$Z_A(\vec{\lambda}) = e^{-\frac{1}{2}\left[\lambda_1^2 \sigma_{11}^2 + \lambda_2^2 \sigma_{22}^2 + 2\lambda_2 \lambda_1 \sigma_{12}^2\right]}$$

for $K^{(2)} = \begin{bmatrix} \sigma_{11}^2 & 0 \\ 0 & \sigma_{22}^2 \end{bmatrix}$

$$Z_A(\bar{\lambda}) = e^{-\frac{1}{2}(\lambda_1^2 \sigma_{11}^2)} \; e^{-\frac{1}{2}(\lambda_2^2 \sigma_{22}^2)}$$

$\downarrow$

$$P(A) = \frac{1}{(2\pi)^2} \int d\lambda_1 \, d\lambda_2 \; e^{-i\lambda_1 A_1 - i\lambda_2 A_2} \; Z_A(\lambda_1, \lambda_2)$$

$$P(A_1, A_2) = \frac{e^{-\frac{A_1^2}{2\sigma_{11}^2}} \; e^{-\frac{A_2^2}{2\sigma_{22}^2}}}{\sqrt{(2\pi)^2 \underbrace{\mathrm{Det}(K^{(2)})}_{\sigma_{11}^2 \sigma_{22}^2}}} = P_G(A_1) \times P_G(A_2)$$

4 $\langle F \rangle = \int d\vec{A} \; F(A) \, P(A)$ $\qquad \langle \vec{A} \rangle = 0$

$$Z_A(\bar{\lambda}) = e^{-\frac{1}{2}\lambda^T \cdot K^{(2)} \cdot \lambda} \; e^{\sum_{n=3}^{\infty} \frac{i^n}{n!} \left\{ \sum_{\mu_1=1}^{N} \cdots \sum_{\mu_n=1}^{N} K^{(n)}_{\mu_1 \cdots \mu_n} \lambda_{\mu_1} \cdots \lambda_{\mu_n} \right\}}$$

$$P(A) = \int D\lambda \; e^{-i\bar{\lambda} \cdot \bar{A}} \; Z_A(\bar{\lambda})$$

$$* \quad P(\vec{A}) = \hat{\mathcal{L}} P_G(\vec{A}) \qquad \lambda = i\frac{\partial}{\partial A_\mu}$$

$$P_G(A) = \frac{1}{(2\pi)^N} \int D\lambda \, e^{-i\vec{\lambda}\cdot\vec{A} - \frac{\lambda^T \cdot K^{(2)} \cdot \lambda}{2}}$$

$$\hat{\mathcal{L}} \equiv e^{\left[\sum_{n=3}^{\infty} \frac{(-1)^n}{n!} \left\{ \sum_{\mu_1=1}^{N} \cdots \sum_{\mu_n=1}^{N} K^{(n)}_{\mu_1 \cdots \mu_n} \frac{\partial^n}{\partial A_{\mu_1} \cdots \partial A_{\mu_n}} \right\} \right]}$$

$$\langle f \rangle = \int DA \, f(A) \, P(A)$$

$$= \int DA \, f(A) \, \hat{\mathcal{L}} P_G(A)$$

By Part

$$= \int DA \left[\hat{\mathcal{L}}^\dagger f(A) \right] P_G(A)$$

$$\boxed{\langle f \rangle = \left\langle \hat{\mathcal{L}}^\dagger f(A) \right\rangle_G}$$

$$\hat{\mathcal{L}}^\dagger \equiv e^{\left[\sum_{n=3}^{\infty} \frac{(+1)^n}{n!} \left\{ \sum_{\mu_1=1}^{N} \cdots \sum_{\mu_n=1}^{N} K^{(n)}_{\mu_1 \cdots \mu_n} \frac{\partial^n}{\partial A_{\mu_1} \cdots \partial A_{\mu_n}} \right\} \right]}$$

For Gaussian Case $\quad \hat{\mathcal{L}} = \hat{\mathcal{L}}^\dagger = 1$

⑤ $$\langle F\rangle = \langle F\rangle_G + \left\langle \sum_{n=3}^{\infty} \frac{1}{n!} \left(\sum_{\mu_1=1}^{N} \cdots \sum_{\mu_n=1}^{N} K^{(n)}_{\mu_1 \mu_2 \cdots \mu_n} \left(\frac{\partial^n}{\partial A_{\mu_1} \cdots \partial A_{\mu_n}} \right) \right) F \right\rangle_G + O(\sigma^2)$$

⑥ $$Z = \int D^n\phi \, e^{-\beta \mathcal{H}}$$

$$Z = \int D^n\phi \, e^{-\beta \mathcal{H}_0} \, e^{-\mathcal{U}}$$

$\mathcal{U}=0$ $$Z_0 = \int D^n\phi \, e^{-\beta \mathcal{H}_0} = \frac{\sqrt{(2\pi)^n}}{\sqrt{\beta \, \mathrm{Det}(\mathrm{Cov})}}$$

$\mathcal{H}_0 =$ Gaussian

for $h \neq 0$

$$Z_0(\bar{\lambda}) = \int D^n\phi \, e^{-\beta \mathcal{H}_0 + \int d^dx \, \bar{\lambda}(x) \cdot \phi(x)}$$

$$= \frac{\sqrt{(2\pi)^N}}{\sqrt{\beta \, \mathrm{Det}(\mathrm{Cov})}} \, e^{-\frac{1}{2} \lambda^T \cdot \mathrm{Cov}^{-1} \cdot \lambda}$$

⑦ $$\underbrace{\langle \phi_i \rangle_{\text{Connect}}}= \left. \frac{\partial \ln Z_0(\bar{\lambda})}{\partial \lambda_i} \right|_{\bar{\lambda}=0}$$

$$\underbrace{\langle \phi_i \, \phi_j \rangle_{\text{Connecte}}}_{\text{رده کوسی}} = \frac{\partial^2 \ln Z_0(\vec{\lambda})}{\partial \lambda_i \, \partial \lambda_j} \Bigg|_{\vec{\lambda}=0}$$

(8) $$\langle O \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle O \, \mathcal{U}^n \rangle_0^{\text{Connected}}$$

$$= \langle O \rangle_0^{\text{Connected}} - 1 \, \langle O \, \mathcal{U} \rangle_0^{\text{Connected}}$$

(9) Ex1: $\quad O \equiv m_i(q_1)\, m_j(q_2)$

$$\langle m_i(q_1)\, m_j(q_2) \rangle = \langle m_i(q_1)\, m_j(q_2) \rangle_0^{\text{Connected}}$$

$$- \langle m_i(q_1)\, m_j(q_2)\, \mathcal{U} \rangle_0^{\text{Connected}}$$

Recall $\quad \beta \mathcal{H} = \beta \mathcal{H}_0 + \mathcal{U}$

$$= \overbrace{\int d^d x \left[\frac{t}{2} \, \vec{m}\cdot\vec{m} + K (\nabla m)^2 + L (\nabla^2 m)^2 + \dots \right] - \vec{h}\cdot\vec{m}}^{\beta \mathcal{H}_0}$$

$$+ \underbrace{u \int d^d x \, (\vec{m}\cdot\vec{m})^2}_{\mathcal{U}}$$