بسم الله الرحمن الرحيم

Perturbative RG

Chapter 5 Kardar's Book

Chapter 12 Goldenfeld's Book

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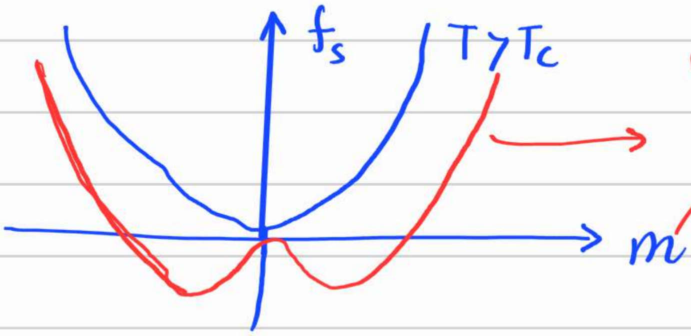
① We are looking for critical behaviour for

$$\underbrace{\mathcal{H}}_{} = \underset{\uparrow\ \text{Gaussian}}{\mathcal{H}_0} + \mathcal{U} \leftarrow \underbrace{\text{Perturbation}}_{?}$$

☆ the importance role of m^4 in $\mathcal{H}$



$\cancel{m^2} \longrightarrow (m^2, m^4)$

$\langle m \rangle (t > \cdot\ (T > T_c), h = 0) = 0 \quad \longrightarrow \quad \mathcal{H}(m^2) = \checkmark$

$\langle m \rangle (t < 0\ (T < T_c), h = 0) \neq 0 = \pm\sqrt{\dfrac{-t}{a_4}} \qquad \boxed{a_4 \neq 0}$

$$\mathcal{L} = a_0 + a_2 m^2 + a_4 m^4 - hm$$

$$= \underbrace{\mathcal{L}_0}_{} + \mathcal{L}_4 \longrightarrow \text{Higher term}$$

Gaussian part

☆ Therefore we necessarily should take into account a_4 in $\mathcal{H}$ to examine $T < T_c$ properly.

Also we will see that for $d < 4$, a_4 has relevant contribution. and one can not consider its impact perturbatively, in this case we need to consider $\mathcal{O}(\mathcal{U}^2)$ to regularize $\mathcal{U}$ in $\mathcal{H}$. (RG & Perturbation)

☆ $$Z = \mathrm{Tr}\left(e^{-\beta \mathcal{H}}\right) = \mathrm{Tr}\left(e^{-\beta(\mathcal{H}_0 + \mathcal{U})}\right)$$

$$F = -K_B T \ln Z$$

$$f_s = \frac{F}{N(\Omega)} = \frac{F}{V(\Omega)}$$

Using RG $\longrightarrow$ $[K'] = R_\ell [K]$

$B_\ell \implies$ RG_flow

② $\mathcal{H} = \mathcal{H}_0 + \mathcal{U}/\beta$

$h = 0$

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U}$$

$$= \int d^d x \left[\frac{t}{2} \vec{m}.\vec{m} + K(\vec{\nabla} m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots \right]$$

$$+ u \int d^d x \, (\vec{m}.\vec{m})^2$$

For $u > 0 \longrightarrow t(u=0) - t(u \neq 0) > 0$

③ Our New Coupling Constant $[K] = [\underbrace{t, k, L}_{\text{Gaussian Part}}, \overset{\text{Perturbation}}{u}]$

$$\left.\begin{aligned} t_\ell &= \ell^{x_t}\, t \\ k_\ell &= \ell^{x_k}\, k \\ L_\ell &= \ell^{x_L}\, L \\ u_\ell &= \ell^{x_u}\, u \end{aligned}\right\} \quad \left.\begin{aligned} x > 0 &\longrightarrow \text{Relevant} \\ x < 0 &\longrightarrow \text{Irrelevant} \\ x = 0 &\longrightarrow \text{Marginal.} \end{aligned}\right\} \quad x = ? \quad [K_c] = ?$$

$$m_\alpha(\vec{x}) = \int \frac{d^d q}{(2\pi)^d} \, \tilde{m}_\alpha(q)\, e^{i\vec{x}\cdot\vec{q}}$$

④ In Fourier Space :

$$\beta\mathcal{H}_0 = \int \frac{d^d q}{(2\pi)^d} \, \frac{t + Kq^2 + Lq^4 + \cdots}{2} \, |\tilde{m}(q)|^2$$

$$\mathcal{U} = u \int d^d x \int \frac{d^d q_1}{(2\pi)^d} \int \frac{d^d q_2}{(2\pi)^d} \int \frac{d^d q_3}{(2\pi)^d} \int \frac{d^d q_4}{(2\pi)^d} \, e^{-i x.(\bar{q}_1 + \bar{q}_2 + \bar{q}_3 + \bar{q}_4)}$$

$(2\pi)^d \, \delta_D(\bar{q}_1 + \bar{q}_2 + \bar{q}_3 + \bar{q}_4)$

$$\tilde{m}_\alpha(q_1)\,\tilde{m}_\alpha(q_2)\,\tilde{m}_\beta(q_3)\,\tilde{m}_\beta(q_4)$$

$$\mathcal{U} = u\int \frac{d^dq_1\, d^dq_2\, d^dq_3}{(2\pi)^{3d}}\, m_\alpha(q_1)\, m_\alpha(q_2)\, m_\beta(q_3)\, m_\beta(-q_1-q_2-q_3)$$

⑤ $\{m\} : \{m_1, m_2 \dots m_n\} = \vec{m}$ n- Vector field

$\alpha = 1, \dots n$

$$Z = \int D^n m\, e^{-\beta\mathcal{H}} = \int D^n m\, e^{-\beta\mathcal{H}_0 - \mathcal{U}}$$

$$= \int D^n m\, e^{-\beta\mathcal{H}_0}\, e^{-\mathcal{U}} \times \frac{\int D^n m\, e^{-\beta\mathcal{H}_0}}{\int D^n m\, e^{-\beta\mathcal{H}_0}}$$

$$Z = \left(\int D^n m \underbrace{\frac{e^{-\beta\mathcal{H}_0}}{\int D^n m\, e^{-\beta\mathcal{H}_0}}}_{\mathcal{S}_0}\, e^{-\mathcal{U}}\right) \underbrace{\int D^n m\, e^{-\beta\mathcal{H}_0}}_{Z_0}$$

Gaussian Part

$$Z = Z_0 \langle e^{-\mathcal{U}} \rangle_0$$

Gaussian Part

$$= Z_0\left[1 - \langle \mathcal{U}\rangle_0 + \frac{1}{2!}\langle \mathcal{U}^2\rangle_0 - \frac{1}{3!}\langle \mathcal{U}^3\rangle_0 + \dots\right]$$

⑥ therefore we need to Compute $\langle \mathcal{U}^p \rangle_0 =$

To this end: $\langle \hat{O} \rangle = ? = \dfrac{\int Dm\, \hat{O}\, e^{-\beta\mathcal{H}}}{\int Dm\, e^{-\beta\mathcal{H}}}$

$$= \frac{\int Dm\, \mathcal{O}\, e^{-\beta \mathcal{H}_0} e^{-\mathcal{U}}}{\int Dm\, e^{-\beta \mathcal{H}_0} e^{-\mathcal{U}}}$$

$$= \frac{Z_0 \int Dm\, \mathcal{O}\, e^{-\beta \mathcal{H}_0} \left[1 - \mathcal{U} + \frac{\mathcal{U}^2}{2!} \cdots\right]}{Z_0 \int Dm\, e^{-\beta \mathcal{H}_0} \left[1 - \mathcal{U} + \frac{\mathcal{U}^2}{2!} + \cdots\right]}$$

$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} \rangle_0 - \langle \mathcal{O}\mathcal{U} \rangle_0 + \frac{1}{2!}\langle \mathcal{O}\mathcal{U}^2 \rangle_0}{1 - \langle \mathcal{U} \rangle_0 + \frac{\langle \mathcal{U} \rangle_0}{2!} + \cdots}$$

$$\simeq \left[\langle \mathcal{O} \rangle_0 - \langle \mathcal{O}\mathcal{U} \rangle_0 + \frac{1}{2!}\langle \mathcal{O}\mathcal{U}^2 \rangle_0 \cdots\right]\left[1 + \langle \mathcal{U} \rangle_0 - \frac{\langle \mathcal{U}^2 \rangle_0}{2!} \cdots\right]$$

$$\simeq \langle \mathcal{O} \rangle_0 - \left[\langle \mathcal{O}\mathcal{U} \rangle_0 - \langle \mathcal{O} \rangle_0 \langle \mathcal{U} \rangle_0\right]$$

$$+ \frac{1}{2!}\left[\langle \mathcal{O}\mathcal{U}^2 \rangle_0 - 2\langle \mathcal{O}\mathcal{U} \rangle_0 + 2\langle \mathcal{O} \rangle_0 \langle \mathcal{U} \rangle_0^2 - \langle \mathcal{O} \rangle_0 \langle \mathcal{U}^2 \rangle_0\right]$$

$$\boxed{\langle \mathcal{O} \rangle = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \underbrace{\langle \mathcal{O}\mathcal{U}^p \rangle_0^{\text{Connected}}}_{\text{Cumulent}}}$$

$$\begin{cases} \langle x \rangle_c = \langle x \rangle \\ \langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2 \end{cases}$$

e.g. $\underline{\underline{Q}} = m_\alpha(x)\, m_\beta(x')$

$$\langle Q \rangle = \langle m_\alpha(x)\, m_\beta(x') \rangle = G_{\alpha\beta}(x, x')$$

Characteristic Function