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✓ # RG in Real space & Fourier space

Finite Size effect on phase transition.

Real Space

① $\mathcal{H} = \int d^d x \left[t m^2 + K (\nabla m)^2 - h m \right]$

② RG Processes

Ⓐ → Coarse - graining

$m(x) = \frac{1}{\ell^d} \int_{x' \in X} d^d x' \, m(x')$

$= \frac{1}{\ell^d} \int d^d x' \underbrace{\underbrace{W(x-x')}_{\text{Window function}} m(x')}_{\text{Convolution method}}$

$\boxed{x \geq a \longrightarrow x \geq \ell a}$

Ⓑ → Rescaling

$x_{New} = \frac{x_{old}}{\ell}$

$x' = \frac{x}{\ell}$

Ⓒ → Renormalizing

$m'(x') = \frac{m(x')}{\zeta} = \ell^{\Delta} m(x') = \ell^{-x_m} m$

③ $\mathcal{H} \xrightarrow{RG} \mathcal{H}' = \mathcal{H} \longrightarrow [K'] = R_\ell [K]$

$$\mathcal{H} = \int \underbrace{d^d x}_{} \left[t m^2 + K (\nabla m)^2 - h m \right]$$

RG $\downarrow$ Rescaling, Renormalization

$$\mathcal{H}' = \int d^d x' \, \ell^d \left[t \vartheta^2 m'^2 + K \ell^{-2} \vartheta^2 (\nabla' m')^2 - h \vartheta m' \right]$$

$\vartheta = \ell^{x_m}$

$$t' = \ell^d \vartheta^2 t \equiv \ell^{x_t} t \Longrightarrow \boxed{x_t = d + 2x_m} \;\star$$

$$K' = \ell^{d-2} \vartheta^2 K \equiv \ell^{x_K} K \Longrightarrow \boxed{x_K = d - 2 + 2x_m} \;\star$$

$$h' = \ell^d \vartheta h \equiv \ell^{x_h} h \Longrightarrow \boxed{x_h = d + x_m} \;\star$$

④ A general point

$$\int d^d x \, F \phi \xrightarrow{RG} \left\{ \begin{array}{l} x' = \dfrac{x}{\ell} \\ \phi' = \dfrac{\phi}{\vartheta} = \ell^{-x_\phi} \phi \end{array} \right\}$$

F: Coupling, ϕ: field

$$\underbrace{}_{} \Big\downarrow RG$$

$$\int d^d x' \, \ell^d \, F \vartheta \phi = \int d^d x' \, \ell^d F \ell^{x_\phi} \phi'$$

$$= \int d^d x' \, F' \phi'$$

$$F' \equiv \ell^{d + x_\phi} F = \ell^{x_F} F \qquad \boxed{x_F = d + x_\phi}$$

$$\boxed{x_\phi = x_F - d}$$

Fourier Space

$$l' = \ell^{x_l} l \qquad x_l < .$$

① $\mathcal{H} = \int d^dx \left[t m^2 + K(\nabla m)^2 - h m \right]$

$\downarrow$

$x \longrightarrow x' = x/\ell \qquad m \longrightarrow m' = \frac{m}{\vartheta} = \ell^{-x_m} m$

$q \longrightarrow q' = \ell q \qquad \tilde{m} \longrightarrow \tilde{m}' = \frac{\tilde{m}}{z} = \ell^{-x_m'} \tilde{m}$

$\nabla \to iq \qquad z = \ell^{+x_m'}$

② $\mathcal{H} \xrightarrow{F.S.} \tilde{\mathcal{H}} = \int \frac{d^dq}{(2\pi)^d} \left[t\, \tilde{m}^2(q) + K q^2 \tilde{m}^2(q) \right] - h \tilde{m}(q=0)$

$\downarrow$ cts

Extra point

$$m(x) = \int \frac{d^dq_1}{(2\pi)^d} e^{i q_1 \cdot x} \tilde{m}(q_1)$$

$\boxed{q_1 = -q_2}$, $\boxed{\nabla \to iq}$

$$m(x) \cdot m(x) = \int \frac{d^dq_1}{(2\pi)^d} \int \frac{d^dq_2}{(2\pi)^d} e^{i(q_1+q_2)\cdot x} \tilde{m}(q_1) \tilde{m}(q_2)$$

$$\int d^dx\, t\, m(x) \cdot m(x) = \frac{1}{(2\pi)^{2d}} \int d^dq_1 \int d^dq_2 \underbrace{\int d^dx\, e^{i(q_1+q_2)\cdot x}}_{(2\pi)^d \delta_D(q_1+q_2)} t\, \tilde{m}(q_1) \cdot \tilde{m}(q_2)$$

$$= \frac{1}{(2\pi)^d} \int d^dq\, t\, \underbrace{\tilde{m}(q) \cdot \tilde{m}(-q)}$$

✓

③ $\tilde{\mathcal{H}} \xrightarrow{RG} \tilde{\mathcal{H}}' = \int d^dq'\, \ell^{-d} \left[t z^2 |\tilde{m}'(q')|^2 + K \ell^{-2} q'^2 z^2 |\tilde{m}'(q')|^2 \right] - h z \tilde{m}'(q'=0)$

$$\tilde{\mathcal{H}} = \int d^dq \left[t |\tilde{m}(q)|^2 + K q^2 |\tilde{m}(q)|^2 \right] - h \tilde{m}(q=0)$$

$$t' = \ell^{-d} z^2 t = \ell^{x_t} t \implies \boxed{x_t = -d + 2x_m'}$$

$$K' = \ell^{-d-2} z^2 K = \ell^{x_K} K \implies \boxed{x_K = -d - 2 + 2x_m'}$$

$$h' = z h = \ell^{x_h} h \implies \boxed{x_h = x_m'}$$

Real Space | Fourier Space

$$x_t = d + 2x_m \longleftrightarrow x_t = -d + 2x_m'$$

$$x_K = d - 2 + 2x_m \longleftrightarrow x_K = -d - 2 + 2x_m'$$

$$x_h = d + x_m \longleftrightarrow x_h = x_m'$$

Fixed point $t = 0$, $h = 0$ $\rightarrow$ $\boxed{K = K'}$

Real space

$$d - 2 + 2x_m = 0 \implies \boxed{x_m = \frac{2-d}{2}}$$

Fourier space

$$-d - 2 + 2x_m' = 0$$

$$\Downarrow$$

$$\boxed{x_m' = \frac{d+2}{2}}$$

$x_t = d + 2\left(\frac{2-d}{2}\right) = d + 2 - d = 2$ in Real Space

$x_t = -d + 2\left(\frac{d+2}{2}\right) = -d + d + 2 = 2$ in Fourier span

$x_h = d + 1 - \frac{d}{2} = 1 + \frac{d}{2}$ in Real space ✓

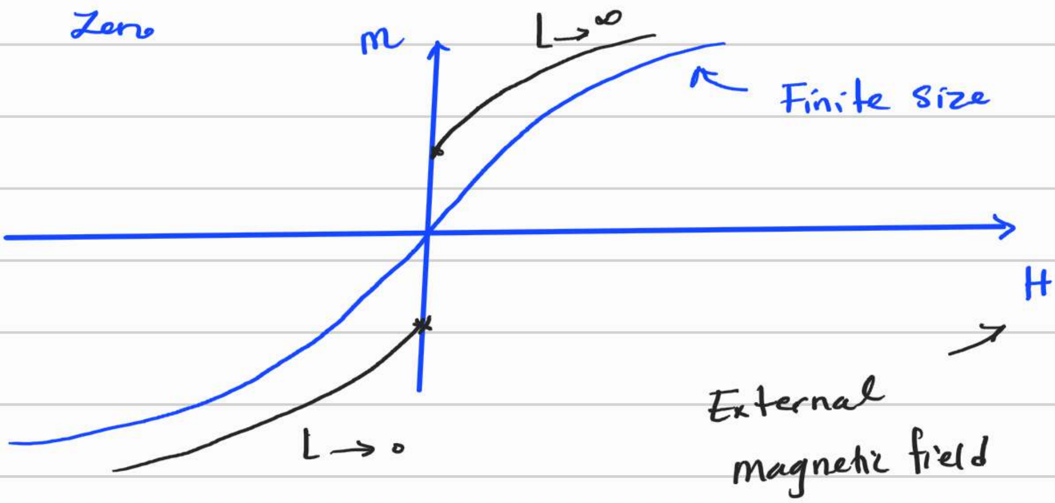
$x_h = x'_m = \frac{d+2}{2} = 1 + \frac{d}{2}$ in Fourier space ✓

Q. E. D.

* Finite Size Effect

① Thermodynamical limit, Spontaneous Symmetry breaking.

$$\underbrace{\lim_{N\to\infty} \lim_{H\to 0}}_{\text{It could be zero}} \neq \underbrace{\lim_{H\to 0} \lim_{N\to\infty}}_{\text{It has non-zero value}}$$



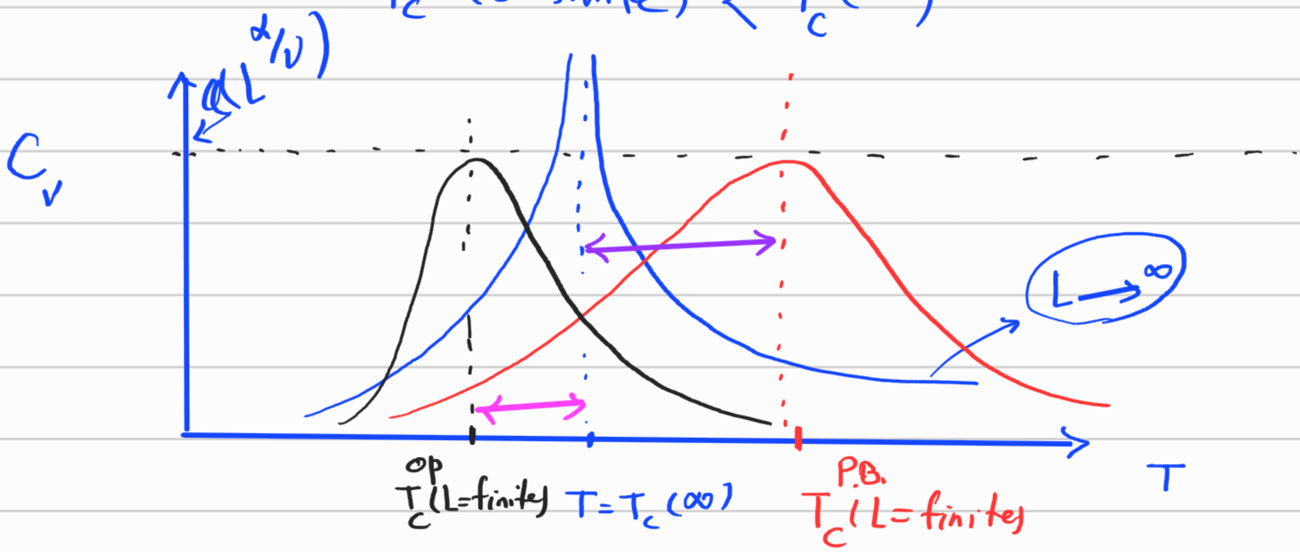
② We are interested in impact of finite size ($L \cancel{\to \infty}$) on phase transition!

③ For Periodic Boundary condition we expect to

have $\underbrace{\text{more order feature}}$ $\longrightarrow$ $T_c^{\text{Periodic}}(L=\text{finite}) > T_c(\infty)$

On the other hand for open Boundary condition

$$T_c^{op}(L=\text{finite}) < T_c(\infty)$$



$T_c^{op}(L=\text{finite}) - T_c(\infty) < 0 \quad \propto L^{-\frac{1}{\nu}}$

$T_c^{P.B}(L=\text{finite}) - T_c(\infty) > 0 \quad \propto L^{-1/\nu}$

$\xi \sim |t|^{-\nu}$

$\downarrow$

$L \sim |t|^{-\nu}$

$L^{-1/\nu} \sim |t|$

$\downarrow$

$$\boxed{t = \frac{T - T_c(\infty)}{T_c(\infty)}}$$

According to physical evidence

physical properties

④

☆ Point 1 $\rightarrow$ No-Singularity in C_v

☆ Point 2 $\longrightarrow$ $|t| \propto L^{-1/\nu}$

☆ Point 3 $\rightarrow$ $\xi \longrightarrow L \neq \infty$

☆ Point 4 $L \rightarrow$ Coupling Constant Relevant plays role in our Analysis. $x_L > 0$

$$f_s([K]) = \ell^{-d} f_s([K']) \qquad L \longrightarrow \infty$$

for $\underbrace{\text{Finite } L}$ $\qquad f_s = \dfrac{F}{V^{(d)}} \sim f_s([K], L^{-1})$

$$f_s([K], L^{-1}) = \ell^{-d} f_s([K'], \ell L^{-1})$$

$$= \ell^{-d} f_s(\ell^{x_t} t, \ell^{x_h} h, \underbrace{\ell L^{-1}}_{\text{New}}, \ldots)$$

$L^{-1} \equiv K_{New} \quad \rightarrow \quad x_L = +1 > 0$ It is Relevant Coupling Constant for finite size.

$$\boxed{L \rightarrow \frac{L}{\ell} \quad , \quad L^{-1} \rightarrow \frac{\ell}{L} = \ell L^{-1}}$$

☆ for Infinite size $[t, h]$ are relevant while for finite size $[t, h, L^{-1}]$ are relevant

⑤ The imprint of finite size on $t, C_v, \xi = ?$

⑥ possibility of ignoreing finite size Effect ?

$$\nu = \frac{1}{x_t}$$

$$f_s(t, L^{-1}) = \ell^{-d} f_s(\ell^{x_t} t, \ell L^{-1})$$

$$= |t|^{d/x_t} f_s(1, |t|^{-\nu} L^{-1})$$

$$= |t|^{d\nu} F(X) \qquad X \equiv t^{-\nu} L^{-1}$$

☆ $\underbrace{F(X)}$ = ?

$$\left\{ X = \frac{|t|^{-\nu}}{L} = 0 \right.$$

$L \to \infty$

"Far from critical point"

But $\underline{\underline{L}}$ is finite $\longrightarrow$ $\cancel{X \to 0}$

$$X \overset{?}{\ll} 1$$

$$\frac{|t|^{-\nu}}{L} \overset{?}{\ll} 1 \longrightarrow \boxed{|t|^{-\nu} \ll L} \quad ①$$

To ignore size effect we essentialy should

take $\underbrace{|t|^{-\nu} \ll L}$, $\underbrace{\text{what is its consequence?}}$

Recall $\boxed{\xi \sim |t|^{-\nu}}$ ②

① & ② $\Longrightarrow$ $\boxed{\xi \ll L}$ in such case we can ignore size effect

$G(r) \sim e^{-r/\xi}$

$\xi \ll L$

$\xi(t) \sim |t|^{-\nu}$

☆ To approach critical point ξ increases

therefore $\xi \to L \Rightarrow \cancel{\times}\ L \to$ we have size effect. For far enough from critical point one can ignore size effect

⑦ The imprint of finite size on $C_v = \dfrac{\partial^2 f_s}{\partial t^2} = ?$

$C_v(t)$: for $L \to \infty \Rightarrow C_v(t, L^{-1})$ For finite L

$$\begin{cases} C_v(t, L^{-1}) = \dfrac{\partial^2 f_s(t, L^{-1})}{\partial t^2} \sim |t|^{-\alpha} \times \text{Correction} \\ f_s(t, L^{-1}) = |t|^{2-\alpha} f_s(1, t^{-\nu} L^{-1}) \end{cases}$$

$$C_v(t, L^{-1}) = |t|^{-\alpha} F(\underbrace{|t|^{-\nu} L^{-1}}_{x})$$

$$\boxed{\lim_{t \to 0} C_v \to \infty}$$ our expected behaviour

$$\lim_{t_L \to 0} F = F(x_0) \equiv \mathrm{Max}\{F\} \equiv \text{finite}$$

$$\xi \sim |t|^{-\nu} \Rightarrow \lim_{t_L \to 0} \xi \simeq L \Rightarrow$$

$$\lim_{t_L \to 0} t_L \sim \xi^{-1/\nu} \sim L^{-1/\nu}$$

$$\lim_{t \to 0} C_v \approx L^{\alpha/\nu} F(x_0) \neq \infty \ll \infty$$

finite

L

$$\lim_{t \to 0} x = x_0$$

$$\lim_{t_L \to 0} C_v \sim \mathcal{O}(L^{\alpha/\nu}) \equiv \text{finite} \ll \infty$$

$$\boxed{\lim_{\substack{t\to 0\\ L\to\infty}} C_v \longrightarrow \infty}$$

(8) The imprint of size effect on ξ ?

$\xi(t)$ for $L\to\infty \Longrightarrow \xi(t, L^{-1})$ for finite L

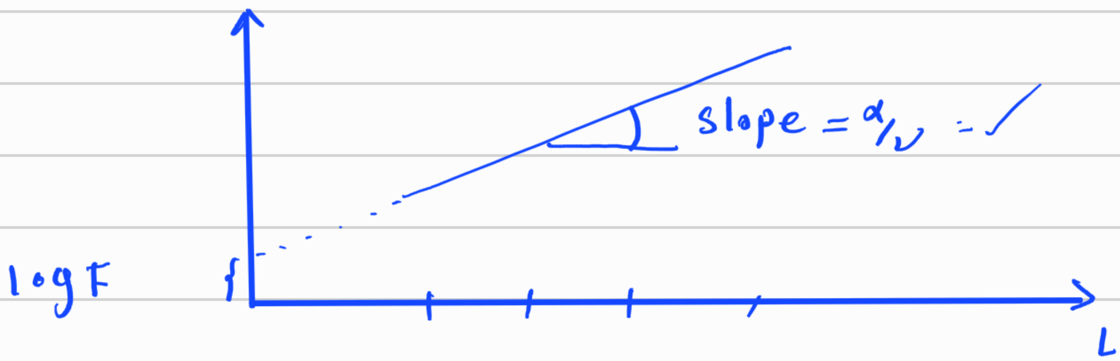
$$\xi(t, L^{-1}) = \ell\, \xi(\ell^{y_t} t, \ell L^{-1})$$

$$\xi(t, L^{-1}) = t^{-\nu}\, \underbrace{G(L^{-1} t^{-\nu})}$$

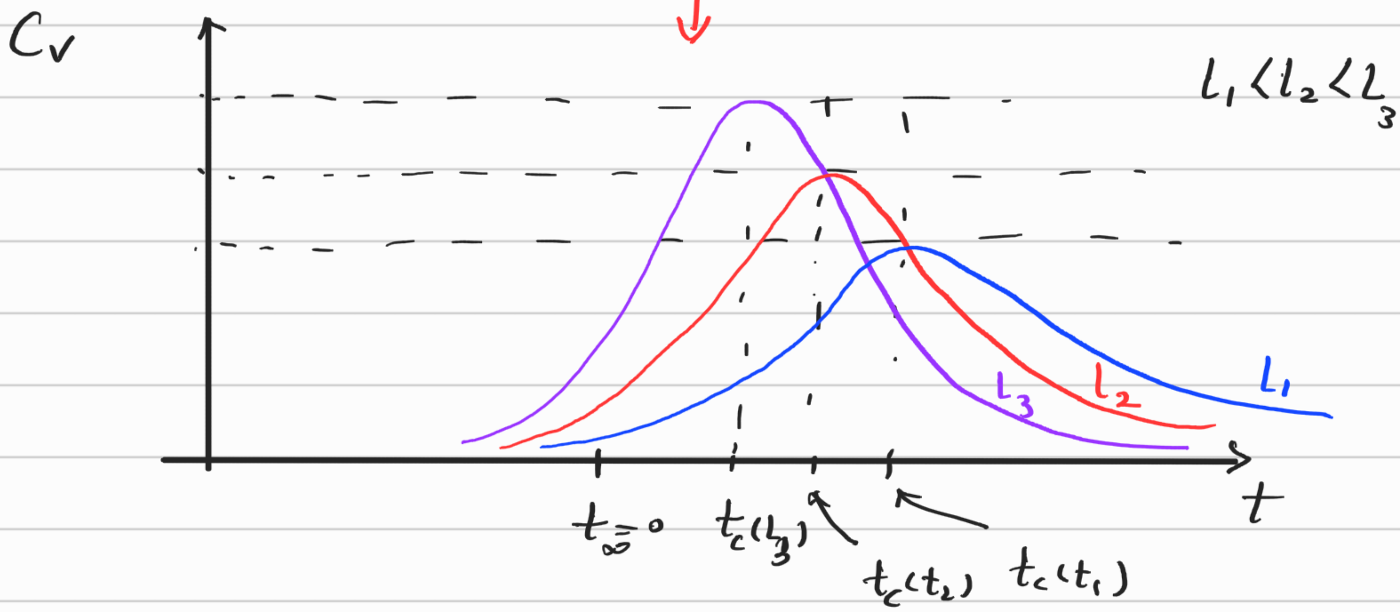
$$\lim_{t\to 0} \xi(t, L^{-1}) \sim a(t^{-\nu}) \sim a(L) \sim L$$

A point: $\quad C_v \sim L^{\alpha/\nu} F$

$$\log C_v = \alpha/\nu \log L + \log F$$



$$C_V \longrightarrow \tilde{C}_V = C_V L^{-\alpha/\nu}$$



$$C_V \longrightarrow \tilde{C}_V = C_V L^{-\alpha/\nu}$$

$$t_L \longrightarrow \tilde{t}_L = t L^{+1/\nu}$$

All diagram will collapse to each other

$$|t|^{-\nu} \simeq L$$

$$t \sim L^{-1/\nu}$$

$$t L^{+1/\nu}$$