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Outline

- ✓ * Gaussian Model (Renormalization Group)
- * Eample : $\underbrace{\text{Real space}}_{(\vec{r})\text{ or }(\vec{X})}$ with respect to $\underbrace{\text{Fourier Space}}_{(\vec{q})}$ $[q] \sim L^{-1}$
- * Finite Size Effect (9.11 : Goldenfeld)

4.7 : Gaussian Model (RG)

(1) $\mathcal{H} = \mathcal{H}_0 + \mathcal{U} = \mathcal{H}_{\text{Gaussian}} + \mathcal{U}$ ← Perturbation

Gaussian ↓ $(\vec{m})^2$

Perturbation ↑ Beyond Gaussian model.

Gaussian Regime

$\mathcal{H} = \mathcal{H}_0 \longrightarrow [K] \text{ or } [x_K] \longrightarrow [K]_0 \text{ or } [x]_0$

Coupling Constant for Gaussian Model.

$\mathcal{H} = \mathcal{H}_0 \longrightarrow \underbrace{\mathcal{H} = \mathcal{H}_0 + \mathcal{U}} \longrightarrow [K]_0 \longrightarrow [K]_0 + [\Delta K]$

Due to assuming $\mathcal{U}$ in $\mathcal{H}$

Chapter 5: Perturbative RG

② Gaussian Model (RG)

$$Z = \int D\vec{m}(x)\, e^{-\int d^d x \left[\frac{t m^2}{2} + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots - \vec{h}\cdot\vec{m}(x) \right]}$$

In Real Space.

$$\langle \eta(t)\eta(t') \rangle = 2\sigma^2 \delta_D(t-t')$$

our assumption = Cts

In Fourier Space.

$$Z \sim \int D\vec{m}(q)\, e^{-\int_0^{\Lambda} \frac{d^d q}{(2\pi)^d} \left(\frac{t + Kq^2 + Lq^4 + \cdots}{2} \right) |m(q)|^2 + \vec{h}\cdot\vec{m}(q)}$$

③ Now we are applying RG on Z $\left[Z \xrightarrow{RG} Z' = ? \right.$

$[K'] = R_\ell [K]$

ROROI

Integration out

☆ Integration out : Coarse grain

$a \leqslant X < L \Rightarrow \int_a^{\ell a} d\vec{x} \longrightarrow \int_a^L = \int_a^{\ell a} + \int_{\ell a}^{L}$

In Real Space

$L \to \infty$

Thermodynamical limit

It Contains Singular Part

Integration out

In Fourier Space

$\Lambda = 1/a$

$0 = \frac{1}{L} \langle q \leqslant \Lambda$

UV-Cut off

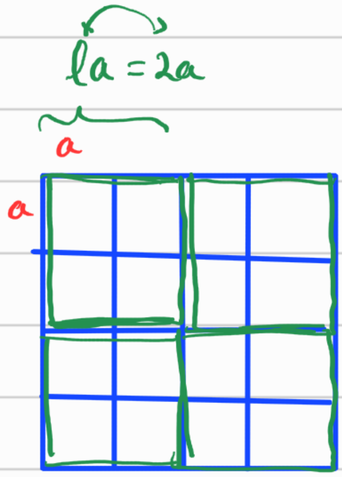
$$\int_0^{\Lambda} d\vec{q} \longrightarrow \int_0^{\Lambda/\ell} + \int_{\Lambda/\ell}^{\Lambda}$$

$$\int_a^L d\vec{x} \longrightarrow \int_a^{\ell a} + \int_{\ell a}^{L}$$

$$0 < q \leq \Lambda \longrightarrow 0 < q \leq \frac{\Lambda}{l}$$

we have $\frac{\Lambda}{l} < q < \Lambda$

Integrated out on

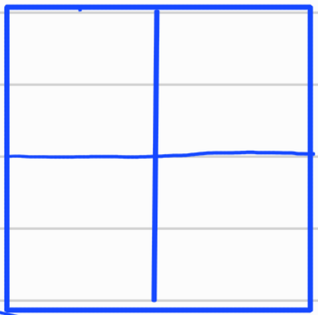


Block - Spin Representation

$$\boxed{\begin{aligned} q &= \frac{1}{x} = \frac{1}{la} = \frac{\Lambda}{l} \\ \Lambda &= 1/a \end{aligned}}$$

Fourier $\quad$ Real

$$q = \frac{\Lambda}{l} \leftrightarrow x = la$$



$$\left\{ \begin{aligned} & a \leq x \leq la \\ & \frac{\Lambda}{l} < q < \Lambda \end{aligned} \right\}$$ Eliminating fluctuations at scal

Now breaking up $\vec{m}(q)$ into two Subspace.

$$\{\vec{m}(\vec{q})\} = \underbrace{\{\vec{\sigma}(q_>)\}}_{\frac{\Lambda}{l} \leq q \leq \Lambda} \oplus \underbrace{\{\vec{m}(q_<)\}}_{0 < q \leq \Lambda/l}$$

Short modes $\quad$ Long modes

{ They are Related to Singular behaviour }

$$Z = \int \mathcal{D}\vec{\tilde{m}}(q_<) \int \mathcal{D}\vec{\sigma}(q_>)\, e^{-\beta \mathcal{H}[\tilde{m}, \sigma \ldots]}$$

Recall $Z = \sum_{\{s\}} e^{-\beta \mathcal{H}}$

$= \underline{\sum_{\{s\} = \{s'\} + \{s''\}} e^{-\beta H}}$

$$= Z_< \times Z_>$$

Singular part → $Z_<$

Regular Part → $Z_>$

$$\boxed{f_{sing} = \frac{-\ln Z_<}{V}}$$

So

☆ $$Z = Z_> Z_< \sim \overbrace{e^{-\frac{n\bar{V}}{2} \int_{\Lambda/\ell}^{\Lambda} \frac{d^d q}{(2\pi)^d} \ln(t + Kq^2 + Lq^4 + \ldots)}}^{Z_>} \times$$

$$\text{(A)} \quad \underbrace{\int \mathcal{D}\tilde{m}(q_<)\, e^{-\left[\int_0^{\Lambda/\ell} \frac{d^d q}{(2\pi)^d} \left(\frac{t + Kq^2 + Lq^4 + \ldots}{2}\right) |\tilde{m}(q_<)|^2 + \bar{h} \cdot \tilde{m}(q=0)\right]}}_{Z_<}$$

Rescale $\Longrightarrow$ $\begin{cases} x \longrightarrow x' = x/\ell & \text{in Real space.} \\ q \longrightarrow q' = q\ell & \text{in Fourier Space.} \end{cases}$

$Z \to Z'$

$$Z' = Z'_{>} Z'_{<} = e^{-V f_b} \int \mathcal{D}\tilde{m}(q'_{<}) \times e^{-\left[\int_0^{\Lambda} \frac{d^d q'}{(2\pi)^d \ell^d} \left(\frac{t + K\ell^{-2} q'^2 + L \ell^{-4} q'^4 + \cdots}{2} \right) |\tilde{m}(q'_{<})|^2 + \bar{h}\cdot\tilde{m}(q \cdots \right.}$$

Renormalization $\Longrightarrow$

$$\begin{cases} \vec{m}(x') \longrightarrow m'(x') = \dfrac{m(x')}{\upsilon} = \dfrac{m(x')}{\zeta} = \ell^{\Delta} m(x') \qquad \boxed{\upsilon \equiv \ell^{-\Delta}} \\[2ex] \tilde{\vec{m}}(q') \longrightarrow \tilde{m}'(q') = \dfrac{\tilde{m}(q')}{z} \end{cases}$$

$$Z' = Z'_{>} Z'_{<} \sim Z'_{>} \int \mathcal{D}\tilde{m}'(q')\, e^{-\left[\int_0^{\Lambda} \frac{d^d q'}{(2\pi)^d} \ell^{-d} \left(\frac{t + K\ell^{-2} q'^2 + L\ell^{-4} q'^4 + \cdots}{2} \right) z^2 |\tilde{m}'(q'_{<})|^2 + \bar{h}\cdot\tilde{m}'(q'=0)\, z \right]}$$

(B)

Now we turn to our coupling constants (By comparing (A) and (B)):

$$\begin{bmatrix} t \longrightarrow t' = \ell^{-d} z^2 t \\ K \longrightarrow K' = \ell^{-d} \ell^{-2} z^2 K = z^2 \ell^{-d-2} K \\ h \longrightarrow h' = z h \\ L \longrightarrow L' = \ell^{-d} \ell^{-4} z^2 L = z^2 \ell^{-d-4} L \end{bmatrix}$$

$$[K'] = R_\ell [K]$$

$$[K^*] = R_\ell [K^*]$$

$$\begin{bmatrix} t_c = t^* = 0 \\ h_c = h^* = 0 \end{bmatrix} \longrightarrow \underbrace{Z_< = Z'_<}_{\text{Demand}} \Longrightarrow K = K'$$

$$z^2 \ell^{-d-2} = 1 \longrightarrow \boxed{z = \ell^{1+\frac{d}{2}}}$$

$$L' = z^2 \ell^{-d-4} L = \ell^{2+\cancel{d}-\cancel{d}-4} L = \ell^{-2} L$$

$x_L = -2 < 0 \rightarrow$ Irrelevant

$$t' = z^2 \ell^{-d} t \longrightarrow t' = \ell^{2+d-d} t = \ell^2 t \rightarrow \boxed{x_t = 2}$$

$$h' = zh \longrightarrow h' = \ell^{1+\frac{d}{2}} h \longrightarrow \boxed{x_h = 1 + \frac{d}{2}}$$

Recall that

$$\left.\begin{aligned} \nu &= \frac{1}{x_t} \\ \alpha &= 2 - \frac{d}{x_t} \\ \beta &= \frac{d - x_h}{x_t} \end{aligned}\right\}$$

Linearization approach

$$\beta_\ell = ? \qquad \begin{cases} t' = \ell^2 t \\ h' = \ell^{1+\frac{d}{2}} h \end{cases} \xRightarrow{\ell = (1+\delta\ell)} \begin{cases} t' = (1 + 2\delta\ell) t \\ h' = (1 + (1+\frac{d}{2})\delta\ell) h \end{cases}$$

$$\beta_\ell = \left\{ \begin{aligned} \frac{\delta t'}{\delta \ell} &= 2t \\ \frac{\delta h'}{\delta \ell} &= (1+\frac{d}{2}) h \end{aligned} \right\}$$

$$\beta_\ell = 0 \Longrightarrow \left\{ \begin{aligned} t_c &= t^* = 0 \\ h_c &= h^* = 0 \end{aligned} \right\}$$

$$\boxed{\left.\frac{\delta \beta_\ell}{\delta h}\right|_{\substack{h=0 \\ t=0}} = x_h = (1+\frac{d}{2})}$$

$$\boxed{\left.\frac{\delta \beta_\ell}{\delta t}\right|_{\substack{h=0 \\ t=0}} = x_t = 2}$$

$$\beta_l = \begin{bmatrix} 2 & 0 \\ 0 & (1+\frac{d}{2}) \end{bmatrix} \rightarrow \text{Eigen values represent } [x_k]$$

☆ Example (Real and Fourm)

☆ Finite Size