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جلسه ۴ر۹ر۱۴۰۳

# * Chapter 4: M. Kardar

## The scaling Hypothesis

فرض مقیاسی

① Widom Hypothesis

$$\phi(x) \longrightarrow \underbrace{\phi' = \phi(ax) = a^{\alpha}\phi(x)}_{\text{scaling Behavior}}$$

$$x \rightarrow ax$$

② A demanding: An invariant behavior of underlying system <u>at</u> (and <u>very close to</u>) critical point

$$\underbrace{\left\{ \begin{array}{c} \text{Invariant behavior with respect to} \\ \text{Length transformation} \end{array} \right\}}$$

Our System becomes <u>Self-Similar</u>

③ the mathematical consequences of ②

$$f = \frac{F}{V^{(d)}} = \frac{F}{N^{(d)}} = f([K]) \xrightarrow{R_\ell} f_\ell([K]_\ell) \overset{?}{=} \underbrace{f([K])}$$

$$\boxed{f_\ell = \ell^{d} f}$$

$$\left\{\begin{array}{l} t \xrightarrow{R_\ell} t_\ell = \ell^{x_t} t \\ h \xrightarrow{R_\ell} h_\ell = \ell^{x_h} t \end{array}\right\} \quad \left.\begin{array}{l} x_t = ? \\ x_h = ? \end{array}\right\}$$

$\boxed{\text{All scaling Exponent can be determined}}$

④ Renormalization Group

RG: $R \circ R \circ I$

- Integration (Coars-grain) ☆
- Rescaling ☆
- Renormalization ☆

Mathematical description of RG (Block-spin)

I: $$S_I = \frac{1}{\ell^d} \sum_{i \in I} S_i \; ; \qquad m_i(x) = \frac{1}{\ell^d} \int_{x' \in x} d^d x' \, m_i(x')$$

R: $$x \longrightarrow x' = x/\ell$$

R: $$S_I(x) \longrightarrow \underbrace{S'_I(x') = \ell^{\Delta} S_I(x')}$$ $$\Delta \equiv \frac{x_h}{x_t}$$

$\hookrightarrow \quad m'(x') = \frac{1}{\underset{\approx}{\zeta}\, \ell^d} \int d^d x \; m(x)$

$$\boxed{\zeta = \ell^{-\Delta}} \qquad (4.42)$$

⑤ According to ROROI $\Longrightarrow$

$N_\ell = \ell^{-d} N$

$\xi_\ell = \ell^{-1} \xi$

$[K] \rightarrow [K_\ell] \longleftarrow \begin{cases} t_\ell = \ell^{x_t} t \\ h_\ell = \ell^{x_h} h \end{cases}$

$f_\ell(t_\ell, h_\ell, \ldots) = \ell^d f(t, h, \ldots)$

⑥ $\mathcal{H} = \mathcal{H}_0 + \mathcal{U}$

Perturbative part

Non_Perturbative

$\mathcal{H} \xrightarrow{R_\ell} \mathcal{H}_\ell[K_\ell] = \mathcal{H}[K]$ An assumption

$Z \xrightarrow{R_\ell} \underbrace{Z_\ell[K_\ell] = Z[K]}$

Our system becomes Self-Similar

$$\boxed{F_\ell = F}$$

$x_t , x_h$

Relevant Coupling Constant

⑦ $\xi \sim t^{-\nu} \longrightarrow \boxed{\nu = \frac{1}{x_t}}$

$M \sim t^{\beta} \longrightarrow \boxed{\beta = \nu \frac{x_h}{x_t} = \nu\Delta}$

$M \sim h^{1/\delta} \longrightarrow \boxed{\delta = \frac{d-\Delta}{\Delta}}$

$\chi \sim t^{\gamma} \longrightarrow \boxed{\gamma = \nu(2\Delta - d)}$

$G(r) \sim r^{-(d-2+\eta)} e^{-r/\xi} \longrightarrow \boxed{2\Delta = d-2+\eta}$

⑧ Recursive Relation $\underbrace{[K]_\ell = R_\ell[K]}$

$[K^*]_\ell = R_\ell[K^*] \leftarrow$ Finding fixed Point ☆ Ⓐ

$$\boxed{[K^*] - R_\ell[K^*] = 0}$$

Relevant

Marginal $\leftarrow \{x_\Delta\}$ Finding scaling Exponents ☆ Ⓑ

Irrelevant

$\underbrace{\text{( Critical Hyper Surface ) RG-flow}}$ ☆ Ⓒ

☆ Universality Class ☆

☆ RG Linearization ☆

$[K]_\ell = R_\ell[K]$

$\vec{K}' = R_\ell[\vec{K}]$

$\vec{K}' = R_\ell[\vec{K}^*] + (\vec{K} - \vec{K}^*) \frac{\partial R_\ell[\vec{K}]}{\partial \vec{K}}\Big|_{\vec{K}=\vec{K}^*} + \mathcal{O}(\Delta K^2)$

$$\underbrace{\vec{K}' - \vec{K}^*}_{} = \underbrace{(\vec{K} - \vec{K}^*)}_{} \frac{\partial R_\ell[\vec{K}]}{\partial \vec{K}}\Big|_{\vec{K}=\vec{K}^*} + \sim$$

$$\vec{k}' \simeq \vec{k} \frac{\partial \vec{K}'}{\partial \vec{K}}\Big|_{\vec{K}=\vec{K}^*} \Longrightarrow \underbrace{\vec{k}'}_{(n\times 1)} = \underbrace{T^*_\ell}_{(n\times n)} \underbrace{\vec{k}}_{(n\times 1)}$$

$$T \phi = \lambda \phi$$

$$\left\{ \begin{array}{l} \vec{\Delta K} = \vec{k} = \sum u_i \vec{\phi}_i \\ \vec{\Delta K'} = \vec{k}' = \sum u_i' \vec{\phi}_i \end{array} \right\} \quad \boxed{u' = \lambda u}$$

$$\lambda_i = \ell^{x_i} \qquad x_i = \frac{d \ln \lambda_i}{d \ln \ell}$$

$$\left\{ \begin{array}{ll} \lambda_i > 1 \ (x_i > 0) & \rightarrow \text{Relevant} \\ \lambda_i < 1 \ (x_i < 0) & \rightarrow \text{Irrelevant} \\ \lambda_i = 1 \ (x_i = 0) & \rightarrow \text{Marginal.} \end{array} \right.$$

Stream flow $\longleftrightarrow$ Dynamical system

⑨ $\beta$_ function

$$\vec{K}' = \vec{K}^* + \vec{k} \frac{\partial R_\ell}{\partial K}\Big|_{\vec{K}=\vec{K}^*}$$

$$\vec{K}' = \vec{K} + \frac{\partial \vec{K}'}{\partial \ell}\Big|_{K'=K} \delta\ell + O(\delta\ell^2)$$

$$\downarrow \quad \vec{K}' = \vec{K} - \vec{\beta}_\ell \, \delta\ell + \mathcal{O}(\delta\ell^2)$$

$$\vec{\beta}_\ell \equiv - \left. \frac{\partial \vec{K}'}{\partial \ell} \right|_{\vec{K}' = \vec{K}} , \quad \boxed{\vec{\beta}_\ell (K = K^*) = 0}$$

$$\left( \mathbb{1} - \frac{\partial \vec{\beta}_\ell}{\partial k} \delta\ell \right) \phi_i = \lambda_i \phi_i$$

$$\downarrow$$

$$\ell^{x_i}$$

$$\boxed{\ell = 1 + \delta\ell} \quad \hookrightarrow \quad (1+\delta\ell)^{x_i} \simeq 1 + x_i \delta\ell$$

$$- \left. \frac{\partial \vec{\beta}_\ell}{\partial k} \right|_{K=K^*} \phi_i = x_i \cdot \phi_i$$

Recall that Migdal – Kadanoff

$$K_p' = \ell^{d-p} R_\ell (\ell^{p-1} K_p)$$

$$R_\ell = \tanh^{-1} \left[ (\tanh K)^\ell \right]$$

$d = 2$

$$K_1' = \ell R_\ell (K_1)$$

$$K_2' = R_\ell (K_2)$$

$$\ell = 1 + \delta\ell = e^{\delta\ell}$$

$$K'_p = e^{(d-p)\delta\ell} \underset{(1+\delta\ell)}{R} \left( e^{\delta\ell(p-1)} K_p \right)$$

$$= e^{(d-p)\delta\ell} \tanh^{-1}\left[ \left( \tanh\left(e^{\delta\ell(p-1)} K_p\right)\right)^{1+\delta\ell} \right]$$

$$\underbrace{K' = K'_p - K_p}_{} = (d-1) K_p \delta\ell + \delta\ell [\quad] - -$$

$$\beta_\ell^{(p)} = \frac{K'_p}{\delta\ell} \checkmark \longrightarrow \beta_\ell = 0 \longrightarrow \text{fixed Point}$$

$$\left.\frac{\delta\beta_\ell}{\delta\ell}\right|_{\text{fixed point}} = \text{Exponent}$$

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Sec. 4.6: Gaussian model (Direct Solution)

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{U}$$

$$Z = \int Dm \, e^{-\beta\mathcal{H}}$$

$\cancel{m^4}$

For Gaussian model without RG

$$Z = \int D\vec{m} \; e^{-\int d^dx \left[ \frac{t}{2} m^2 + \frac{K}{2} (\nabla m)^2 + \frac{L}{2} (\nabla^2 m)^2 + \dots - \vec{h}\cdot\vec{m} \right]}$$

Critical Point $\begin{pmatrix} t = 0 \\ h = 0 \end{pmatrix}$

$$t \equiv \frac{T - T_c}{T_c}$$

$$h = \frac{h - h_c}{h_c}$$

In Fourier Space $\quad \vec{\nabla} \equiv -i\vec{q}$

$$\vec{m}(\vec{x}) \longrightarrow \tilde{\vec{m}}(\vec{q}) = \int \frac{d^d x}{(2\pi)^d}\, e^{-i\vec{q}\cdot\vec{x}}\, \vec{m}(\vec{x})$$

$\frac{1}{\sqrt{V}}\int \dots$

$$\vec{m}(\vec{x}) = \frac{1}{\sqrt{V^{(d)}}} \int d^d q\, e^{+i\vec{q}\cdot\vec{x}}\, \tilde{\vec{m}}(\vec{q})$$

$$\beta\mathcal{H} \longrightarrow \beta\tilde{\mathcal{H}} = \left( \sum_{q\neq 0} \left[ \frac{(t + Kq^2 + Lq^4 + \dots)}{2V} |\tilde{m}(\vec{q})|^2 \right] \right) - \tilde{h}\cdot\tilde{\vec{m}}(0)$$

cts

$\vec{m}$

$$Z = \prod_q V^{-n/2} \int D\tilde{m}(\vec{q})\, e^{-\left[ \frac{(t + Kq^2 + Lq^4 + \dots)}{2V} |\tilde{m}(\vec{q})|^2 - \tilde{h}\cdot\tilde{m}(0) \right]}$$

$q = 0$

$q \neq 0$

$$Z_{q=0} = V^{-n/2} \int_{-\infty}^{+\infty} d\tilde{m}(q=0)\, \exp\left[ -\frac{t}{2V} |\tilde{m}(q=0)|^2 - \bar{h}\cdot\tilde{m}(q=0) \right]$$

$$= \left(\frac{2\pi}{t}\right)^{n/2} e^{\frac{Vh^2}{2t}}$$

$$\boxed{Z = e^{\frac{Vh^2}{2t}} \prod_q \left( \frac{2\pi}{t + Kq^2 + Lq^4 + \dots} \right)^{n/2}} \qquad (4.55)$$

$$f = -\frac{\ln Z}{V} = \frac{n}{2} \int \frac{d^d q}{(2\pi)^d} \ln(t + Kq^2 + Lq^4 + \dots) - \frac{h^2}{2t}$$

$a < x < L$

$$\frac{1}{L} < q < \frac{1}{a} \longrightarrow \boxed{-\frac{\pi}{a} < q < +\frac{\pi}{a}}$$

$$\boxed{q = \frac{2\pi}{\lambda}}$$

$$f = \frac{n}{2} \int_0^{\Lambda = 1/a} \frac{d^d q}{(2\pi)^d} \ln\left(t + Kq^2 + Lq^4 + \cdots\right) - \frac{h^2}{2t}$$

$$= \frac{n}{2} K_q \int_0^{\Lambda} dq\, q^{d-1}\, q \ln\left(t + Kq^2 + Lq^4 + \cdots\right) - \frac{h^2}{2t}$$

$$\frac{S_d}{(2\pi)^d}$$ — d-dimensional solid angle.

$$\boxed{q = \left(\frac{t}{K}\right)^{1/2} x} \qquad \boxed{dq = \left(t/K\right) dx}$$

$$f_{sing}(t,h) = \frac{n}{2} K_d \left(\frac{t}{k}\right)^{d/2} \int_0^{\Lambda/(t/k)^{1/2}} dx\, x^{d-1} \left[\ln t + \ln\left(1 + x^2 + \frac{Ltx^4}{k^2} + \cdots\right)\right] - \frac{h^2}{2t}$$

$$\boxed{f_{sing}(t,h) = -t^{d/2}\left[A + \frac{h^2}{2t^{1+d/2}}\right]}$$

$$C_v = \frac{\partial^2 f}{\partial t^2} \sim t^{-\alpha}$$

$$f_{sing}(t,h) \equiv t^{2-\alpha} g\left(\frac{h}{t^{\Delta}}\right)$$

$$2-\alpha = d/2 \rightarrow \boxed{\alpha = 2 - d/2}$$

$$\chi = \frac{\partial^2 f}{\partial h^2} \sim \frac{1}{t} \rightarrow \boxed{\gamma = 1}$$

## 4.7 The Gaussian Model (renormalization group)

$$\mathcal{H} \longrightarrow \mathcal{H}_\ell$$

$$Z \longrightarrow Z_\ell$$

$$Z = Z_\ell \Big|_{\vec{K} = \vec{K}^*} \rightarrow$$ نقطه ثابت

$$t^{d/2} \int_0^{\Lambda/(t/k)^{1/2}} dx \, x^{d-1} \frac{h^2}{2t} = \frac{h^2}{2t} t^{d/2} \left(\frac{1}{d}\right) x^d \Big|_0^{\Lambda/(t/k)^{1/2}}$$

$$= \frac{h^2}{\cancel{d}2t} t^{d/2} \cancel{d} \left(\frac{k}{t}\right)^{d/2}$$

$$= \frac{h^2}{2t} A'$$