بسم الله الرحمن الرحیم

جلسه ۱۲ر۸ر۱۴۰۱

$$\mathcal{H}(\{S\}) = -H\sum_r S(r) - \underbrace{\sum J(r-r')\, S(r)\, S(r')}$$

$$\begin{cases} S(r) \longrightarrow \overbrace{(S(r)-M)}^{\delta S(r)} + M \\ S(r') \longrightarrow (\underbrace{S(r')-M}_{\delta S(r')}) + M \end{cases}$$

①

$$\underbrace{\bar{J} \equiv \sum J(r-r')}_{②}$$

$$M = \tanh\left[\beta(H + 2dJM)\right] \longrightarrow \text{Non-trivial fixed Point}$$

$$t = \frac{T-T_c}{T_c}$$

$M(T,H) = ✓$

$M(t,h) = ✓$

Critical Exponents

نماهای بحرانی

$$M \propto |-t|^{\beta}$$

$$C \propto |t|^{-\alpha}$$

$$\chi \propto |t|^{-\gamma}$$

$$G(R) \propto \frac{1}{R^{d-2+\eta}}$$

$$M \propto h^{1/\delta}$$

$$\tau \propto \xi^{z} \qquad \xi \propto |t|^{-\nu}$$

$$\boxed{\begin{matrix}\beta\\ \alpha\\ \gamma\\ \eta\\ \nu\\ \delta\\ z\end{matrix}}$$

☆ در اینجا برای محاسبه این نماهای معیاری ؟ از حل تابع پارش و نهایتاً محاسبه انرژی آزاد استفاده می کنیم.

رهیافت های دیگری وجود دارند مانند

☆ نظریه موثر Landau-Ginzburg

☆ نظریه بازبهنجارش Renormalization Group RG

☆ همچنین روش‌های مبتنی بر شبیه‌سازی.

We are interested in evaluating Critical Exponents

Near critical point

$$t \to 0^{(\pm)}$$

$$h \to 0^{\pm}$$

We turn to Free Energy. $F \to f = \dfrac{F}{V_d}$

$$\underline{f^{\text{Mean-field}}(M,H,t)} = -\frac{K_B T}{N} \ln Z = \frac{1}{2} J M^2 - \frac{\ln \cosh \beta (H + JM)}{\beta}$$

$$f^{\text{Landau-Ginzburg}} = $$ با عبارت بالا متفاوت است

$$\lim_{\substack{M \to 0 \\ t \to 0 \\ H \to 0}} f^{\text{Mean-field}} = a + btM^2 + CM^4 + dHM + \mathcal{O}(M^6)$$

$$= a_0 + a_2 M^2 + a_3 M^3 + a_4 M^4 + a_1 M^1 + \mathcal{O}(M^5)$$

$$\underbrace{a,\ b,\ c,\ d}$$ $\boxed{a_1 = d}$

ضرایب فقط هستند که به صورت خیلی ضعیف به t، h و ... وابسته هستند

$a = cts$

ضرایب مربوط به M^{n_i}

$$\text{for } H = 0 \longrightarrow Z_2\text{-symmetry} \Rightarrow \underset{\substack{\downarrow \\ n=1}}{a_1} = \underset{\substack{\downarrow \\ n=2}}{a_3} = \underset{\substack{\downarrow \\ n=3}}{a_5} = \cdots$$

$a_2 \equiv bt$

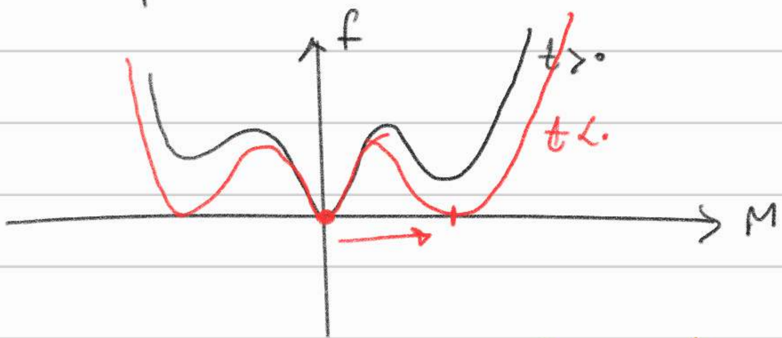
$a_4 \equiv c$

$a_1 \equiv dH$

For $t<0$, $T<T_c$, $M\neq 0$ $\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $a_2<0$, $a_4>0$ $\Rightarrow$ $M\neq 0$, $f(M\neq 0) =$ is minimized

For $t>0$, $T>T_c$, $M=0$ $\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $a_2>0$, $a_4>0$ $\Rightarrow$ $M=0$, $f(M=0) =$ is minimized

for $a_4<0 \longrightarrow M\to\infty \Rightarrow$ first order Phase transition



حول و نزدیکی نقطه بحرانی

$M\to 0$

$$f(M,h,t) = a + btM^2 + cM^4 + dHM + O(M^6)$$

① $H=0$ $\quad \frac{\partial f}{\partial M}=0 \longrightarrow 2btM + 4cM^3 = 0$

$$M \propto (-t)^{1/2} \longrightarrow \boxed{\beta = 1/2}$$

$M\to 0$

② $H\neq 0$, $t\to 0$ $\quad \frac{\partial F}{\partial M}=0 \longrightarrow 2btM + 4cM^3 + dH = 0$

$M\to 0$ صرف نظر کرده‌ایم

$$M \propto -\frac{dH}{2bt}$$

$$\chi = \left.\frac{\partial M}{\partial H}\right|_{H\to 0} = -\frac{d}{2bt} \propto t^{-1} \longrightarrow \boxed{\gamma = 1}$$

$$\left\{ \begin{array}{l} \chi \sim \xi^{2-\eta} \\ t^{-1} \sim \xi^{2-\eta} \;\text{—}\; \xi \sim t^{-\frac{1}{2-\eta}} \longrightarrow \boxed{\nu = \dfrac{1}{2-\eta}} \end{array} \right\}$$

③ $t=0 \; (T = T_c^{MF})$

$$\frac{\partial f}{\partial M} = 0 \Rightarrow 3CM^3 + dH = 0 \Rightarrow M \propto H^{1/3}$$

$$\boxed{\delta = 3}$$

④ C_H

$\rightarrow t>0 \; (T > T_c^{MF}) \longrightarrow M=0 \; (H=0)$

$f = a \;, \; H=0 \quad C_H = \dfrac{\partial^2 f}{\partial t^2} = ?$ تعریف نمی‌شود

$M \to 0$

$H \to 0$

$\rightarrow t<0 \; (T < T_c^{MF})$

$M \neq 0$

$$f = a + btM^2 + \dots$$
$$= a + bt\,(t^{1/2})^2 + \dots$$

$$\frac{\partial^2 f}{\partial t^2} = cts \;\text{—}\; C \sim t^{-\alpha} = cts$$

for $t<0$ $\quad \boxed{\alpha = 0}$

for $t>0$ $\quad \boxed{?}$

MF کار نمی‌کند $\longrightarrow$

نقص می‌گیرد

$t>0 \leftarrow$ افت و خیزها حاکم

نظریه میدان میانگین $\leftarrow$ for $t>0$

اثر میانگین

⑤ Correlation function

$$G(R) \approx \frac{1}{R^{d-2+\eta}}$$

$$\xi \sim |t|^{-\nu} \qquad \boxed{\nu = \frac{1}{2-\eta}}$$

$R = |r_i - r_j|$

$$G(r_i, r_j) = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle$$

$$\mathcal{H} = -J \sum S_i S_j - H \sum S_i$$

$$Z = \mathrm{Tr}\ \exp[\ \beta J \sum S_i S_j + \beta H \sum S_i]$$

$\langle x \rangle = \int x\, p(x)\, dx$

$$\sum_i \langle S_i \rangle = \frac{1}{Z} \mathrm{Tr}\left(\sum S_i e^{-\beta H} \right)$$

$$\sum_i \langle S_i \rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial H}$$

$$\sum_{ij} \langle S_i S_j \rangle = \frac{1}{\beta^2} \frac{1}{Z} \frac{\partial^2 Z}{\partial H^2}$$

$$\chi = \frac{\partial M}{\partial H} = \frac{1}{N\beta} \frac{\partial^2 \ln Z}{\partial H^2}$$

$$t^{-1} \sim \chi = \frac{1}{N} k_B T \left[\frac{1}{Z} \frac{\partial^2 Z}{\partial H^2} - \frac{1}{Z^2} \left(\frac{\partial Z}{\partial H} \right)^2 \right]$$

$$t^{-1} \sim \chi = \frac{\beta}{N} \left[\underbrace{\sum \langle S_i S_j \rangle - \left(\sum \langle S_i \rangle \right)^2}_{} \right]$$

$R \equiv r_i - r_j$

$$t^{-1} \sim \chi = \frac{\beta}{N} \sum_{ij} G(r_i, r_j) \sim \frac{\beta}{N} \int \frac{d^d r_i}{a^d} \int \frac{d^d r_j}{a^d}\ \tilde{G}(r_i, r_j)$$

$$t^{-1} \sim \chi \simeq \frac{\beta}{a^d} \int d^d R\ G(R)$$

Suppose that
$$G(R) \simeq \frac{e^{-R/\xi}}{R^{\frac{d-1}{2}}\ \xi^{\frac{d-3}{2}}}$$

$$t^{-1} \sim \chi \sim \frac{\beta}{a^d} \int d^dR \; G(R)$$

$$\underbrace{t^{-1} \sim \xi^2} \quad , \quad \underbrace{\xi \sim t^{-\nu}}$$

$$\boxed{\nu = 1/2}$$

$$\boxed{\underset{R \gg \xi}{G(R) \sim \frac{1}{R^{d-2+\eta}}}}$$

$$\nu = \frac{1}{2-\eta}$$

$$\nu = \frac{1}{2}$$

$$\boxed{\eta = 0}$$

☆ Second approach

$\nu = ?$

$\eta = ?$

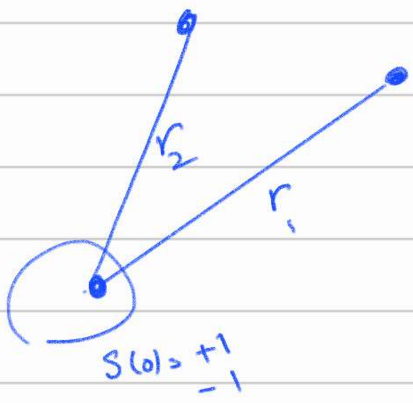
$$S(r)\,S(r') = \Big[\underbrace{M(r) + S(r) - M(r)}\Big]\Big[\underbrace{M(r') + S(r') - M(r')}\Big]$$

$\boxed{\text{Conditional}}$ Correlation function

$$\langle S(r) \,|\, S(0) \rangle \Big|_{S(0)=+1} = M(r)$$

فرض

مقدار متوسط مغناطش در r
به شرطی که در مبداء r=0 مقدار
اسپین S(0)=+1



according to mean field theory we obtained

$$M = \tanh \beta (H + JM)$$

$$\langle S(r) | S(0) \rangle \equiv \underline{M(r)} = \tanh \beta \Big(\sum_{r'} \underbrace{\underline{J(r-r')}}_{\text{Kernel (window function)}} \underline{M(r')} \Big)$$

$H=0$

for $M \to 0$
$t \to 0$

$$M(r) \sim \beta \sum_{r'} J(r-r') M(r') \equiv \beta J \otimes M$$

Convolution

Fourier Space

$$\tilde{G}(K) \equiv \tilde{M}(K) = \beta \tilde{J}(K) \tilde{M}(K) + cts$$

Suppose $J(r) \sim J_0 \, e^{-\frac{r^2}{2\ell^2}}$ = Gaussian function

$$\tilde{J}(K) \sim \tilde{J}_0 \, e^{-\frac{K^2 \ell^2}{2}} \sim \boxed{\tilde{J}_0 (1 - K^2 \ell^2 + \cdots)}$$

$$\left\{ \begin{array}{l} \ell \to 0 \\ r \to \infty \\ K \to 0 \end{array} \right\}$$

قبلاً توضیح دادیم

$$\boxed{T_c^{MF} \equiv \frac{J_0}{K_B}}$$

$$\tilde{G}(K) = \tilde{M}(K) = \frac{cts}{1 - \beta \tilde{J}_0 (1 - K^2 \ell^2 + \cdots)}$$

$$\tilde{G}(K) \simeq \frac{cts}{1 - \frac{T_c^{MF}}{T} (1 - K^2 \ell^2)}$$

$$\tilde{G}(K) \sim \frac{\ell^{-2}}{K^2 + \xi^{-2}}$$

$$\xi^{-2} \equiv \frac{T - T_c^{MF}}{\ell^2 T_c^{MF}}$$

$$\boxed{\nu = 1/2} \leftarrow \begin{cases} \xi \sim \ell (T - T_c^{MF})^{-1/2} \\ \xi \sim t^{-\nu} \end{cases}$$

Real space $\quad G(r) = \dfrac{1}{r^{d-2+\eta}} \Rightarrow \boxed{\eta = 0}$

Table 3.1 (Goldenfeld) ✓

Ginzburg Criterion

To examine the accuracy of Mean field theory we consider 3 following Questions

Ⓐ Which d is good for? $\quad d \equiv$ Dimension

Ⓑ How much is the contribution of $\underbrace{\text{correction}}_{\mathcal{H}_1}$ term in $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$?

$\leftarrow$ in MF $\quad \mathcal{H}_1 \to 0$

ⓒ What is the relation between T_c and T_c^{MF} ?

$T_c^{MF} - T_c > 0$

Recall that

① Small fluctuation $\delta S = S - \langle S \rangle \sim 0$

② Mean Value for Intraction $\bar{J} \equiv \sum_R J(R)$

$$\mathcal{H} = -H \sum_r S(r) - \sum_{rr'} J(r-r') S(r) S(r')$$

$$S(r) = \underbrace{S(r) - \langle S \rangle}_{\delta S(r)} + \langle S \rangle$$

$$S(r') = \underbrace{S(r') - \langle S \rangle}_{\delta S(r')} + \langle S \rangle$$

$$\mathcal{H} = -H \sum_r S(r) - 2 \sum_{rr'} J(r-r') \langle S \rangle S(r') + \sum_{rr'} \frac{J}{2} \langle S \rangle \langle S \rangle$$

$$\underbrace{- \sum_{rr'} \frac{J(r-r') \delta S(r) \delta S(r')}{2}}$$

$$= \bar{\mathcal{H}}_0 + \bar{\mathcal{H}}_1$$

$$\mathcal{H} = V^d \mathcal{H}_0 + V^d \mathcal{H}_1$$

$$V^d \mathcal{H}_1 \equiv - \sum J(r-r') \delta S(r') \delta S(r)$$

$$V^d \mathcal{H}_0 \equiv -H \sum S(r) - \sum J \langle S \rangle S(r) + \tfrac{1}{2} \langle S \rangle^2 \sum J$$

Ⓐ To answer first Question we evaluate

$$\frac{\delta M_V^2}{M_V^2} = ?$$

$$\delta M_V^2 = \sum_{rr'} G(r,r') = \sum_{rr'} \langle \delta S(r)\, \delta S(r') \rangle$$

$$= \int d^d r\, d^d r'\, G(r-r') = V^d \int d^d R\, G(R)$$

$$= V^d \underbrace{\tilde{G}(K=0)}_{\text{Static Response}} = \xi^d \xi^2$$

$$\boxed{[\delta M_V^2] = \xi^{d+2}}$$

$$G(R) = \frac{e^{-R/\xi}}{R^{d-2+\eta}} \qquad \eta = 0$$

$$\delta M_V^2 = V^d \int d^d R\, G(R)$$

$$[\delta M_V^2] = V^d \xi^2\, \mathcal{L} \leftarrow \quad ضرب$$

$$[\ \ ] = \xi^{d+2}$$

$$M_V^2 = ?$$

$$f(M,H,t) = a + btM^2 + cM^4 + dHM$$

$$\left.\frac{\partial f}{\partial M}\right|_{H=0} = 2btM + 4cM^3 = 0 \rightarrow M = \left(-\frac{bt}{2c}\right)^{1/2}$$

$$M_V^2 = (V^d M)^2 = -\frac{bt}{2c} V^{2d} = -\frac{b}{2c}\left(\frac{T-T_c^{MF}}{T_c^{MF}}\right) \xi^{2d}$$

$$\left[\frac{(\delta M_V)^2}{(M_V)^2}\right] = \frac{\xi^{d+2}\,\mathcal{L}}{-\frac{b}{2c}\left(\frac{T-T_c^{MF}}{T_c^{MF}}\right)\xi^{2d}} \qquad \xi \sim |t|^{-1/2}$$

$$[\ \text{''}\ ] = \frac{\mathcal{L}\left(\frac{T-T_c^{MF}}{T_c^{MF}}\right)^{-\frac{(d+2)}{2}}}{-\frac{b}{2c}\left(\frac{T-T_c^{MF}}{T_c^{MF}}\right)\left(\frac{T-T_c^{MF}}{T_c^{MF}}\right)^{-d}}$$

$$[\ \text{''}\ ] = \frac{t^{-\frac{(d+2)}{2}}}{t\, t^{-d}} = t^{\frac{d}{2}-2}$$

$$\left[\frac{\delta M_V^2}{M_V^2}\right] \sim t^{\frac{d}{2}-2}$$

for $\frac{d}{2}-2 \geqslant 0 \Longrightarrow \boxed{d \geqslant 4}$ $\lim\limits_{t\to 0} \frac{(\delta M_V)^2}{M_V^2} = 0$

تقریب خوبی است (البته نتیجه دقیق نیست) $T \to T_c^{MF}$

Exercise: According to Saddle Point approximation

$$\lim_{d\to\infty} \text{Mean-field} = \text{Exact Solution}$$

(B) To answer second Question we consider

$$V^d \mathcal{H}_1 \equiv \sum_{rr'} J(r-r') \langle \delta S(r)\, \delta S(r') \rangle$$

$$= V^d \int d^d R \; J(R)\, G(R)$$

$$V^d \mathcal{H}_1 = V^d \int \frac{d^d K}{(2\pi)^d} \underbrace{\tilde{J}(K)} \underbrace{\tilde{G}(K)}$$

$$\boxed{\begin{aligned} J(r) &= J_0\, e^{-\frac{r^2}{2\ell^2}} \\ \tilde{J}(K) &= \tilde{J}_0\, e^{-\frac{K^2 \ell^2}{2}} \end{aligned}} \qquad \boxed{\tilde{G}(K) = \frac{\ell^{-2}}{K^2 + \xi^{-2}}}$$

$J \simeq J_0$ انتخاب مقطع کارآمد

$$\left[\tilde{J}_0 \int \underbrace{\frac{d^d K}{(2\pi)^d}}_{K^d} \underbrace{\frac{\ell^{-2}}{K^2 + \xi^{-2}}}_{K^{-2}} \right] = K^{d-2} \sim \xi^{2-d}$$

Near Critical Point $\xi \longrightarrow \infty$

$K \to 0$

$$\ln V^d \mathcal{H}_1 = V^d J \int \frac{d^d K}{(2\pi)^d} \frac{\ell^{-2}}{K^2 + \xi^{-2}}$$

$\lambda \geq a \to |K| \leq 1/a$

$\xi \to \infty$

$K \to 0$

$|K| \leq 1/a \to$ unit cell

UV-cut off $\leftarrow$

$$\simeq V^d J\, l^{-2} \int_0^{|K| \leq 1/a} \frac{d^d k}{(2\pi)^d}\, \underbrace{\frac{1}{K^2 + \xi^{-2}}}$$

$$\frac{1}{K^2\left(1+ \xi^{-2}/K^2\right)} \cong \frac{1}{K^2}\left(1 - \frac{\xi^{-2}}{K^2+\xi^{-2}}\right)$$

تقریب تیلوری تا ثبت

$$\frac{1}{K^2}\left(1 - \frac{\xi^{-2}}{K^2} + \cdots\right)$$

$$V^d \mathcal{H}_1 = \frac{J_0 V^d}{l^2} \int_0^{|K|<1/a} \frac{d^d k}{(2\pi)^d} \left[\frac{1}{K^2} - \frac{\xi^{-2}}{K^2(K^2+\xi^{-2})}\right]$$

$$= \frac{J_0 V^d}{l^2} \int^{K \leq 1/a} \underbrace{\frac{d^d K}{(2\pi)^d} \frac{1}{K^2}}_{K^{d-2}} - \underbrace{\frac{J_0 V^d}{l^2} \int \overbrace{\frac{d^d K}{(2\pi)^d}}^{K^d} \frac{\xi^{-2}}{\underbrace{K^2(K^2+\xi^{-2})}_{K^4}}}_{\text{واستگی به T دارد.}}$$

$d>2 \longrightarrow K=0$ واگرایی ندارد

واستگی هم ندارد

$$V^d \mathcal{H}_1 = \frac{J_0 V^d}{l^2}\left[cts - \xi^{-2} \underbrace{K^{d-4}}_{\xi^{4-d}} \right]$$

$$[\;"\;] = \frac{J_0}{l^2} \xi^d \left[\xi^{2-d} \right]$$

$$[\mathcal{H}_1] = [\xi^{2-d}]$$

$$\mathcal{H}_0 = J \sum \langle s \rangle \langle s \rangle \simeq J M^2 \sim -Jt$$

$$[\mathcal{H}_1] \sim \xi^{2-d}$$

$$[\mathcal{H}_0] \sim [-t]$$

$$[\xi \sim l\, t^{-1/2}]$$

$$t \sim l^2/\xi^2$$

$$\mathcal{H}_1 \overset{?}{\ll} \mathcal{H}_0$$

$$\frac{\cancel{J}}{l^2} \xi^{2-d} < \cancel{J} t$$

$$\xi^{2-d} \ll l^4/\xi^2$$

$$\xi^{4-d} \ll l^4 = cts$$

one can ignore $\mathcal{H}_1$ compare to $\mathcal{H}_0$ $\quad 4-d \leq 0 \rightarrow \boxed{d \geq 4}$

© $\quad T_c, T_c^{MF} \qquad T_c^{MF} > T_c$

Continuous Ising Model.

$$\mathcal{H} = \int d^d r \left[\frac{(\nabla s)^2}{2} + t\, s^2(r) + u\, s^4(r) \right]$$

$$[s] \sim l^{\frac{2-d}{2}} \longleftarrow \left[d^d r \frac{(\nabla s)^2}{2} \right] = \left[l^d \frac{s^2}{l^2} \right] = 1$$

$\langle s^4 \rangle = 3 \langle s^2 \rangle^2$

Small fluctuation $\quad \nabla s \sim 0 \qquad$ Wick's theorem

$$\mathcal{H} = \int \underbrace{d^d r}_{[\ell^d]} \underbrace{[\, t + 3 u M^2 \,]}_{[\ell^{-2}]} \underbrace{s^2(r)}_{[\ell^{2-d}]}$$

Correlation length scale.

$$[\xi^{-2}] \equiv [t + 3 u M^2]$$

$$M^2 = \langle s^2 \rangle = G(0) = \int \frac{d^d k}{(2\pi)^d} \frac{R^{-2}}{k^2 + \xi^{-2}}$$

$$\xi^{-2} = t + 3u \left[\int \frac{d^d k}{(2\pi)^d} \frac{R^{-2}}{k^2 + \xi^{-2}} \right]$$

$$\xi^{-2} = t + 3u \left[\underbrace{\int_{|k| \leq 1/a} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2}}_{\text{ثابت}} - \int_{|k| \leq 1/a} \frac{d^d k}{(2\pi)^d} \frac{\xi^{-2} R^{-2}}{k^2 [k^2 + \xi^{-2}]} \right]$$

$$\xi^{-2} = t + A - \text{Cts}\, \mathcal{O}(\xi^{2-d})$$

$$\boxed{\xi^{-2} = T - T_c^{MF} + A - \text{Cts}\, \mathcal{O}(\xi^{2-d})}$$

For $d > 2 \implies \xi \to \infty \to 0 = \underbrace{T_c - T_c^{MF}} + A - 0$

$T \to T_c$

$\implies \boxed{T_c < T_c^{MF}}$

for $d < 2$ $\qquad 0 = \underbrace{T_c - T_c^{MF}} + A - \infty$

$T \to T_c$

$\xi \to \infty$

نمی‌توان در خصوص رابطه T_c ، T_c^{MF} صحبت کرد

Summary

① $\dfrac{(\delta M_v)^2}{M_v^2} \sim t^{\frac{d}{2} - 2}$ $\quad \boxed{d \geq 4}$ تقریب خوبی است

② $\dfrac{\mathcal{H}_1}{\mathcal{H}_0} \ll 1 \to \xi^{4-d} \ll cts$

تقریب خوبی است $\boxed{d > 4}$

③ $\xi^{-2} = T_c - T_c^{MF} + A - \mathcal{O}(\xi^{2-d})$

$d > 2 \to T_c < T_c^{MF}$

Effective Theory

Landau-Ginzburg (LG)

RG