بسم الله الرحمن الرحیم

جلسه ۸ / ۸ / ۱۴۰۱

Mean field Theory — نظریه میدان میانگین

① Simple to apply

② Gives a Correct qualitative Picture of Phase diagram and Phase Transition

③ For small fluctuations [e.g. $S_i - \langle S_i \rangle = \delta S_i \sim 0$] is also quantitative correct and accurate

④ For large No. of spatial Dimension is accurate

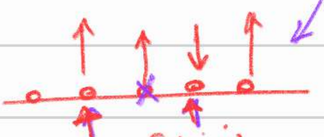
D = Dimension $\quad \lim_{D\to\infty}$ Mean field theory ≡ accurate

$\boxed{\text{Ginzburg Criterion} \quad \text{محک گینزبرگ}}$

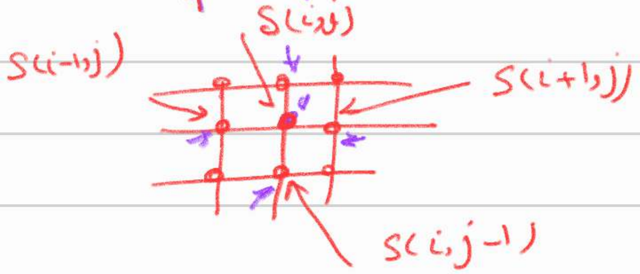
دقیق

$\frac{(\delta M)^2}{(M)^2} \ll 1$?

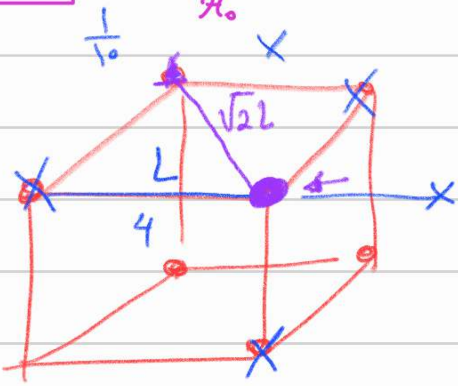
$\frac{\Delta \mathcal{H}}{\mathcal{H}_0} \ll 1$



→ 1 Dimension



2D



$$J = cts \longleftarrow \begin{cases} J(r_i, r_j) = J(R) \\ R = |r_i - r_j| \end{cases}$$

$$\begin{cases} J(r_i) = \sum_{r_j} J(r_i, r_j) \longleftarrow \text{هر چقدر تعداد همسایه‌ها زیادتر شود و J گستره بزرگتر باشد} \\ = J = cts \end{cases}$$

Ex 1. 3.7: Mean field Theory

Goldenfeld

$\left\{\begin{array}{l} \text{* Weiss' mean field Theory} \\ \text{* Bragg – Williams mean field Theory} \\ \text{* Bethe mean field Theory.} \end{array}\right.$

Nearest Neighbor Ising model

نزدیک‌ترین همسایه

Intraction

External field

$$\mathcal{H}_\Omega(\{S\}) = -J\sum_{\langle ij\rangle} S_i S_j \underbrace{- H\sum S_i}$$

{Intraction only with External field}

A $J = 0$

$$\mathcal{H}_\Omega(\{S\}) = -H\sum_{i=1}^{N} S_i$$

$$\underbrace{\mathcal{H}}_{+\beta H \sum_{i=1}^{N} S_i}$$

مدل پارامغناطیس

$$\to Z_\Omega(J=0, H) = \sum_{\{S\} = \pm 1} e$$

$$\{S_1, S_2, \dots, S_N\}^{\pm 1}$$

$S_1 \to \pm 1$, $S_2 \to \pm 1$

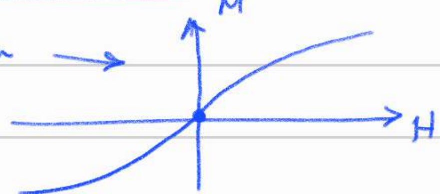
$$Z_\Omega(J=0,H) = \prod_{i=1}^{N}\left[\sum_{S_i=-1}^{+1} e^{+\beta H S_i}\right]$$

$$Z_\Omega(J=0,H) = \left[e^{-\beta H} + e^{+\beta H}\right]^N$$

$$F_\Omega = -K_B T \ln Z_\Omega(J=0,H) \longrightarrow f = \frac{F}{N}$$

$$M = \langle S\rangle = -\frac{1}{N}\frac{\partial F}{\partial H} \Rightarrow \boxed{M = \tanh(\beta H)}$$

Paramagnetism System



(B) For $J \neq 0$

$$\mathcal{H}_\Omega(J=0, H) = -H\sum_{i=1}^{N} s_i \Longrightarrow \mathcal{H}_\Omega(J, H) = -H\sum_{i=1}^{N} s_i - J\underbrace{\sum_{\langle ij\rangle} s_i s_j}_{\text{Internal Intract.}}$$

at $\underline{\underline{s_i}}$ location we have $\underline{\underline{H}}$ for $J=0$

for $J \neq 0$ $\quad H_i = H \longrightarrow \boxed{H_i = H + J\sum_j s_j}$

$\sum_j \to 2d$ Coordinate No. Regular Network

$$\underline{-H s_i} \longrightarrow \underbrace{\left(-H - J\sum_{j=1} s_j\right) s_i}_{\bar{H} s_i}$$

$\bar{H}$: Effective magnetic field at ith spin

We are interested in calculating $\underline{\text{Partition function}}$

$$\mathcal{Z}_\Omega(J, H) = ?$$

$$\mathcal{Z}_\Omega(J, H) = \overline{\sum_{\{s\}}} e^{-\beta\mathcal{H}(J, H, \{s\})}$$

$\delta s_j \equiv$ fluctuation

$$H_i = H + \sum_j J_{ij} s_j = H + \sum_j J_{ij}\left[\overbrace{s_j - \langle s_j\rangle}^{\delta s_j} + \langle s_j\rangle\right]$$

$$H_i = \underbrace{H}_{\text{External field}} + \underbrace{\sum_j J_{ij}\langle s_j\rangle}_{\text{Mean field}} + \underbrace{\sum_j J_{ij}\,\delta s_j}_{\text{fluctuation term}}$$

for $\delta s_j \longrightarrow 0$ (Small fluctuation Regime)

$$H_i \simeq H + \sum_j J_{ij}\langle s_j\rangle \simeq H + \sum_j J_{ij} M = H + 2dJM$$

→ Remember that for $J=0$ $\quad M=\tanh(\beta H)$

for $J\neq 0$ (Mean field theory)

$$H \rightarrow \underbrace{H+2dJM}$$

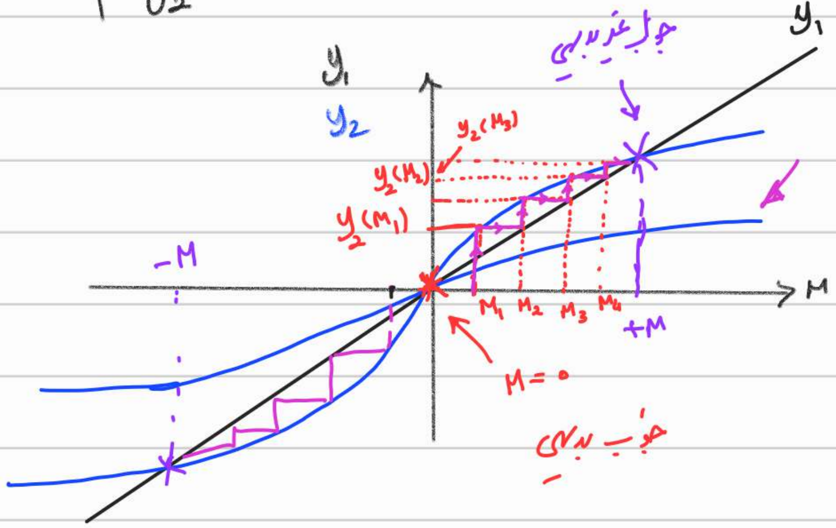
$$\boxed{M=\tanh\left[\beta(H+2dJM)\right]}$$

trivial Solution

For $H=0$ $\quad$ $M=\tanh(\beta 2dJM) \rightarrow \begin{matrix} M=0 \\ M\neq 0 \end{matrix}$

$$\begin{cases} y_1 \equiv M \\ y_2 \equiv \tanh(\beta 2dJM) \end{cases}$$

M_1



$$M=\tanh(\beta(H+2dJM)) \qquad H=0$$

$$M=\tanh(2d\beta JM)$$

$$\frac{2dJ}{K_B T}<1$$

$$\frac{2dJ}{K_B}<T$$

$$\underbrace{\frac{2dJ}{K_B}}_{T_c}$$

دمای گذار فاز در مدل نظریه میدان متوسط (میانگین)

J_{effect}

For $T>T_c$ $\quad$ $\boxed{T_c \equiv \frac{2dJ}{K_B}}$ $\quad$ $M=0$

For $T<T_c$ $\quad$ $M\neq 0 \leftarrow$ Non-Trivial fixed point

Ex 2: Mean field Theory (other approach)

chapter 2: (Cardy _ Book)

$$\mathcal{H}_\Omega(\{S\}) = -H\sum S(r) - \sum_{rr'} \frac{J(r-r')}{2}\, \widetilde{S(r)S(r')}$$

$Z_\Omega = ?$, $f = ?$, M , χ , C

ξ , $G(R)$ $\longrightarrow$ Critical Exponents

(A) small fluctuations

(B) Mean Value for Intraction at each site

$$\begin{cases} S(r) \longrightarrow \widetilde{S(r)-M} + M \equiv \delta S(r) + M \\ S(r') \text{ — } \widetilde{S(r')-M} + M \equiv \delta S(r') + M \end{cases}$$

$$S(r)\,S(r') = [M + \delta S(r)][M + \delta S(r')]$$

صرف نظر

$$= -M^2 + M S(r) + M S(r') + \widetilde{\mathcal{O}(\delta S^2)}$$

$$Z = \mathrm{Tr}\, e^{-\beta\mathcal{H}} = \mathrm{Tr}\left\{ e^{\underbrace{\beta H \sum_r S(r) + \frac{\beta}{2}\sum_{rr'} J(r-r')\, S(r)\, S(r')}_{A}} \right\}$$

$$A \equiv \beta H \sum_r S(r) + \frac{\beta}{2}\sum_{rr'} J(r-r')\, \overbrace{S(r)}^{M+\delta S(r)}\, \underbrace{S(r')}_{M+\delta S(r')}$$

صرف نظر می کنیم

$$= \beta H \sum S(r) - \frac{\beta M^2}{2}\sum_{rr'} J(r-r') + \beta M \sum_{rr'} J(r-r')\, S(r) + \widetilde{\mathcal{O}(\delta S^2)}$$

$$\sum_{rr'} J(\overbrace{r-r'}^{R}) = N\sum_R J(R) \equiv N\bar{J} \qquad \bar{J} \equiv \sum_R J(R)$$

$$= N\bar{J}$$

$= N\rho J \to$ effective intraction $\equiv J_{eff}$

No. sites ← ↑ No. of Neighbor

$$A = -\beta \frac{M^2}{2} \overset{J_{eff}}{\widetilde{NJ}} + (\beta MJ + \beta H) \sum_r s(r) + \mathcal{O}(\delta s^2)$$

$$\mathcal{Z}_\Omega(J,H) = \sum_{\{s\}} e^{-\frac{\beta M^2 NJ}{2}} e^{+(\beta MJ + \beta H)\sum s(r) + \mathcal{O}(\delta s^2)}$$

$$= e^{-\frac{\beta M^2 NJ}{2}} \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} \cdots e^{(\beta MJ + \beta H)(S_1 + S_2 + \cdots S_N) + \mathcal{O}(\delta s^2)}$$

$$\simeq e^{-\frac{\beta M^2 NJ}{2}} \left(\sum_{S=\pm 1} e^{(\beta MJ + \beta H) S} \right)^N$$

$$\simeq e^{\frac{\beta M^2 NJ}{2}} \Big(\underbrace{e^{-\beta(MJ+H)} + e^{+\beta(MJ+H)}}_{2\cosh \beta(MJ+H)} \Big)^N$$

$$F = -K_B T \ln Z \qquad f = \frac{F}{N}$$

$$\boxed{M = -\frac{\partial f}{\partial H} = \tanh(\beta(MJ + H))} \qquad \boxed{H \to H + M\overset{J_{eff.}}{J}}$$

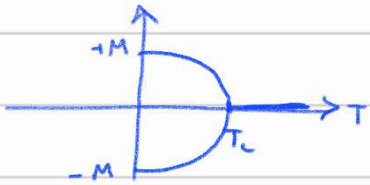
Exercise

$$F^{\text{Mean-field}} \geqslant F$$

$$T_c^{\text{Mean-field}} \geqslant T_c \leftarrow$$

Ex3: Evaluation f versus M

$M(T) = ?$ $\longleftrightarrow$



※ Self-Conssistent   خودسازگار

※ $f = -\frac{K_B T}{N} \ln Z$

mean-field

$$\rightarrow f = \frac{1}{2} J M^2 - \frac{\ln \cosh(\beta(JM+H))}{\beta} \longleftarrow f$$

Landau-Ginzburg

روش نظریه موثر

Mean field Theory   Effective Theory

If $H=0$ Tylor Expansion of f around $M \simeq 0$ and set $H=0$

$$f \simeq -M\beta H + \frac{M^2 J}{2}(1-\beta J) + \frac{2M^4\beta^3 J^4}{4!} + \dots$$

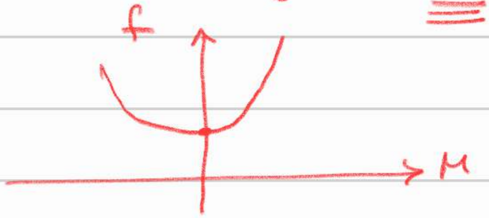
$$f(H=0) = \frac{M^2 J}{2}(1-\beta J) + M^4 \frac{2\beta^3 J^4}{4!} + \dots$$

$$\frac{df}{dM} = J(1-\beta J)M + O(M^3) \longrightarrow 0$$

$$\frac{d^2 f}{dM^2} = J(1-\beta J) + O(M^2) \longrightarrow \begin{cases} > 0 \\ < 0 \end{cases}$$

For $(1-\beta J) > 0$ $1 - \frac{J}{K_B T} > 0 \longrightarrow 1 > \frac{J}{K_B T} \longrightarrow T > \frac{J}{K_B} \equiv T_c^{mf}$

$$\left.\frac{d^2 f}{dM^2}\right|_{M=0} > 0$$

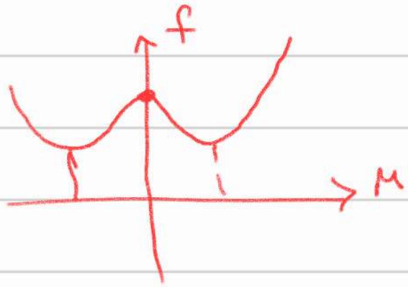


for $(1-\beta J) < 0$

$T < T_c^{MF}$

$\left.\frac{d^2 f}{dM^2}\right|_{M=0} < 0$

$M \neq 0$



For $H > 0$

$T > T_c$

$T < T_c$