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Answer to Exercise set 6

1&3. We answer question 1 and 3 together. We have:

$$P(x,t|x',t') = e^{\int L_{KM}(x,t)dt}\delta(x-x')$$

for $\tau = t - t' \ll 1$ we can expand above so:

$$P(x,t+\tau|x',t) = [1 + L_{KM}(x,t)\tau + \mathcal{O}(\tau^2)]\delta(x-x') \tag{1}$$

We know Keramers-Moyal coefficint bigger than 3 is zero so:

$$P(x,t+\tau|x',t) = [1 + L_{FP}(x,t)\tau + \mathcal{O}(\tau^2)]\delta(x-x')$$

and:

$$L_{FP} = -\frac{\partial}{\partial x}D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2}D^{(2)}(x,t)$$

Plugin above expression in 1 we obtain:

$$\begin{aligned}
P(x,t+\tau|x',t) &= [1 - \frac{\partial}{\partial x}D^{(1)}(x,t)\tau + \frac{\partial^2}{\partial x^2}D^{(2)}(x,t)\tau]\delta(x-x') \\
&= \exp\Big[-\frac{\partial}{\partial x}D^{(1)}(x,t)\tau + \frac{\partial^2}{\partial x^2}D^{(2)}(x,t)\tau\Big]\delta(x-x') \qquad (2)\\
{}^{1} &= \exp\Big[-\frac{\partial}{\partial x}D^{(1)}(x',t)\tau + \frac{\partial^2}{\partial x^2}D^{(2)}(x',t)\tau\Big]\delta(x-x')
\end{aligned}$$

We use the Fourier transform of the delta function, so:

$$\begin{aligned}
P(x,t+\tau|x',t) &= \exp\Big[-\frac{\partial}{\partial x}D^{(1)}(x',t)\tau + \frac{\partial^2}{\partial x^2}D^{(2)}(x',t)\tau\Big]\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iu(x-x')}du \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\Big[-iuD^{(1)}(x',t) - u^2D^{(2)}(x',t)\tau + iu(x-x')\Big]du \qquad (3)\\
&= \frac{1}{2\sqrt{\pi D^{(2)}(x',t)\tau}}\exp\Big(-\frac{[x-x'-D^{(1)}(x',t)\tau]^2}{4D^{(2)}(x',t)\tau}\Big)
\end{aligned}$$

We find the answer of the Ex#3 part a.

Conditional moments are defined as below:

$$M_n(x',t,\tau) = \int (x-x')^n P(x,t+\tau|x',t)dx$$

[1] We can replace x, x' because of $\delta(x-x')f(x') = \delta(x-x')f(x)$

We put 3 and chage of variable like $\alpha = \frac{x-x'}{2\sqrt{D^{(2)}(x',t)\tau}}$ we can calculate integral and we get:

$$
\begin{aligned}
M_n(x',t,\tau) &= \frac{\left[2\sqrt{D^{(2)}(x',t)\tau}\right]^{n+1}}{2\sqrt{\pi D^{(2)}(x',t)\tau}} \int \alpha^n \exp\Big(- [\alpha - \frac{D^{(1)}(x',t)\tau}{2\sqrt{D^{(2)}(x',t)\tau}}]^2\Big) d\alpha \\
&= \left[2\sqrt{D^{(2)}(x',t)\tau}\right]^n \frac{1}{\sqrt{\pi}} \int \alpha^n \exp\Big(- [\alpha - \frac{D^{(1)}(x',t)\tau}{2\sqrt{D^{(2)}(x',t)\tau}}]^2\Big) d\alpha \\
{}^2 &= \left[2\sqrt{D^{(2)}(x',t)\tau}\right]^n \frac{1}{\sqrt{\pi}} (2i)^{-n} \sqrt{\pi} H_n\Big\{(i\frac{D^{(1)}(x',t)\tau}{2\sqrt{D^{(2)}(x',t)\tau}})\Big\} \\
&= \left[-i\sqrt{D^{(2)}(x',t)\tau}\right]^n H_n\Big\{\frac{1}{2} i D^{(1)}(x',t)\sqrt{\tau/D^{(2)}(x',t)}\Big\}
\end{aligned}
$$

Above is answer to Ex#1. We can check our result with below calculation:

$$
\begin{aligned}
D^{(1)}(x',t) = \lim_{\tau\to 0}\frac{M_1(x',t,\tau)}{\tau} &= \lim_{\tau\to 0}\frac{1}{\tau}\left[-i\sqrt{D^{(2)}(x',t)\tau}\right] H_1\Big\{\frac{1}{2} i D^{(1)}(x',t)\sqrt{\tau/D^{(2)}(x',t)}\Big\} \\
&= \lim_{\tau\to 0}\frac{1}{\tau}\left[-i\sqrt{D^{(2)}(x',t)\tau}\right] 2\Big(\frac{1}{2} i D^{(1)}(x',t)\sqrt{\tau/D^{(2)}(x',t)}\Big) \\
&= \lim_{\tau\to 0}\frac{\tau}{\tau} D^{(1)}(x',t) = D^{(1)}(x',t) \quad \checkmark
\end{aligned}
$$

$$
\begin{aligned}
D^{(2)}(x',t) = \lim_{\tau\to 0}\frac{M_2(x',t,\tau)}{2!\tau} &= \lim_{\tau\to 0}\frac{1}{2\tau}\left[-i\sqrt{D^{(2)}(x',t)\tau}\right]^2 H_2\Big\{\frac{1}{2} i D^{(1)}(x',t)\sqrt{\tau/D^{(2)}(x',t)}\Big\} \\
&= \lim_{\tau\to 0}\frac{1}{2\tau}\left[-i\sqrt{D^{(2)}(x',t)\tau}\right]^2 \Big(- \left[D^{(1)}(x',t)\sqrt{\tau/D^{(2)}(x',t)}\right]^2 - 2\Big) \\
&= \lim_{\tau\to 0}\frac{1}{2\tau}\left[-D^{(2)}(x',t)\tau\right] \Big(- \left[D^{(1)}(x',t)\right]^2 \tau/D^{(2)}(x',t) - 2\Big) \\
&= \lim_{\tau\to 0}\frac{1}{2\tau}\left[-D^{(2)}(x',t)\tau\right] (-2 + \mathcal{O}(\tau^2)) \\
&= D^{(2)}(x',t) \quad \checkmark \\
D^{(3)}(x',t) = \lim_{\tau\to 0}\frac{M_3(x',t,\tau)}{3!\tau} &= \lim_{\tau\to 0}\frac{1}{6\tau}\left[-i\sqrt{D^{(2)}(x',t)\tau}\right]^3 H_3\Big\{\frac{1}{2} i D^{(1)}(x',t)\sqrt{\tau/D^{(2)}(x',t)}\Big\} \\
&= \lim_{\tau\to 0}\frac{1}{6\tau}\left[-i\sqrt{D^{(2)}(x',t)\tau}\right]^3 \\
&\times \Bigg(8\Big(\frac{1}{2} i D^{(1)}(x',t)\sqrt{\tau/D^{(2)}(x',t)}\Big)^3 - 12\Big(\frac{1}{2} i D^{(1)}(x',t)\sqrt{\tau/D^{(2)}(x',t)}\Big)\Bigg) \\
&= \lim_{\tau\to 0}\frac{1}{\tau}\mathcal{O}(\tau^{3/2})\left[\mathcal{O}(\tau^{3/2}) - \mathcal{O}(\tau^{1/2})\right] \\
&= \lim_{\tau\to 0}\left[\mathcal{O}(\tau^2) - \mathcal{O}(\tau)\right] \\
&= 0 \quad \checkmark
\end{aligned}
$$

[2] Here, we use helpful hint in notebook:

$$\int_{-\infty}^{\infty} x^n e^{-(x-\beta)^2} dx = (2i)^{-n}\sqrt{\pi} H_n(i\beta)$$

We can see that $D^{(3)}(x', t) = 0$, we know from Pawula theorem that all Keramers-Moyal coeffs bigger than 3 are zero and this is completely consistence with Fokker-Planck, Keramers-Moyal coeffs. But this answer are not unique, we can write differential form of Fokker-Planck operator as below:

$$\begin{aligned}
L_{FP}(x,t)\mathcal{F} &= -\frac{\partial}{\partial x}\Big(D^{(1)}(x,t)\mathcal{F}\Big) + \frac{\partial^2}{\partial x^2}\Big(D^{(2)}(x,t)\mathcal{F}\Big) \\
&= -\mathcal{F}\frac{\partial D^{(1)}(x,t)}{\partial x} - D^{(1)}(x,t)\frac{\partial \mathcal{F}}{\partial x} + \frac{\partial}{\partial x}\Big(\mathcal{F}\frac{\partial D^{(2)}(x,t)}{\partial x} + D^{(2)}(x,t)\frac{\partial \mathcal{F}}{\partial x}\Big) \\
&= -\mathcal{F}\frac{\partial D^{(1)}(x,t)}{\partial x} - D^{(1)}(x,t)\frac{\partial \mathcal{F}}{\partial x} + \frac{\partial \mathcal{F}}{\partial x}\frac{\partial D^{(2)}(x,t)}{\partial x} + \mathcal{F}\frac{\partial^2 D^{(2)}(x,t)}{\partial x^2} \\
&+ \frac{\partial D^{(2)}(x,t)}{\partial x}\frac{\partial \mathcal{F}}{\partial x} + D^{(2)}(x,t)\frac{\partial^2 \mathcal{F}}{\partial x^2}
\end{aligned}$$

So:

$$\begin{aligned}
L_{FP}(x,t) &= -\frac{\partial D^{(1)}(x,t)}{\partial x} + \frac{\partial^2 D^{(2)}(x,t)}{\partial x^2} \\
&- \Big[D^{(1)}(x,t) - 2\frac{\partial D^{(2)}(x,t)}{\partial x}\Big]\frac{\partial}{\partial x} + D^{(2)}(x,t)\frac{\partial^2}{\partial x^2}
\end{aligned}$$

Put it in 1 we can get:

$$\begin{aligned}
P(x,t+\tau|x',t) &= [1 + L_{KM}(x,t)\tau + \mathcal{O}(\tau^2)]\delta(x-x') \\
&= \Big[1 - \tau\frac{\partial D^{(1)}(x,t)}{\partial x} + \tau\frac{\partial^2 D^{(2)}(x,t)}{\partial x^2} \\
&- \tau\Big(D^{(1)}(x,t) - 2\frac{\partial D^{(2)}(x,t)}{\partial x}\Big)\frac{\partial}{\partial x} + \tau D^{(2)}(x,t)\frac{\partial^2}{\partial x^2} + \mathcal{O}(\tau^2)\Big]\delta(x-x') \\
&= \exp\Big[-\tau\frac{\partial D^{(1)}(x,t)}{\partial x} + \tau\frac{\partial^2 D^{(2)}(x,t)}{\partial x^2} \\
&- \tau\Big(D^{(1)}(x,t) - 2\frac{\partial D^{(2)}(x,t)}{\partial x}\Big)\frac{\partial}{\partial x} + \tau D^{(2)}(x,t)\frac{\partial^2}{\partial x^2} + \mathcal{O}(\tau^2)\Big]\delta(x-x') \\
&= \exp\Big[-\tau\frac{\partial D^{(1)}(x,t)}{\partial x} + \tau\frac{\partial^2 D^{(2)}(x,t)}{\partial x^2} \\
&- \tau\Big(D^{(1)}(x,t) - 2\frac{\partial D^{(2)}(x,t)}{\partial x}\Big)\frac{\partial}{\partial x} + \tau D^{(2)}(x,t)\frac{\partial^2}{\partial x^2} + \mathcal{O}(\tau^2)\Big]\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iu(x-x')}du \\
&= \frac{1}{2\pi}\exp\Big[-\tau\frac{\partial D^{(1)}(x,t)}{\partial x} + \tau\frac{\partial^2 D^{(2)}(x,t)}{\partial x^2}\Big] \\
&\times \int_{-\infty}^{\infty}\exp\Big[iu\Big(-\tau D^{(1)}(x,t) + 2\tau\frac{\partial D^{(2)}(x,t)}{\partial x} + (x-x')\Big) - u^2\tau D^{(2)}(x,t)\Big]du
\end{aligned}$$

Final result is the answer of part b of Ex#3:

$$\begin{aligned}
P(x,t+\tau|x',t) &= \frac{1}{2}\sqrt{\frac{1}{\pi D^{(2)}(x,t)\tau}} \\
\times \exp&\Bigg(-\tau\frac{\partial D^{(1)}(x,t)}{\partial x} + \tau\frac{\partial^2 D^{(2)}(x,t)}{\partial x^2} - \frac{\Big[(x-x') - \tau D^{(1)}(x,t) + 2\tau\partial D^{(2)}(x,t)/\partial x\Big]^2}{4\tau D^{(2)}(x,t)}\Bigg)
\end{aligned}$$

2. Equation of Ornstein-Uhlenbeck is as below:

$$\dot{\xi}_i + \gamma_{ij}\xi_j = \Gamma_i(t); \qquad i = 1, ..., N$$

We use Einstein sum rule on indices's and noise part have this properties $\langle \Gamma_i(t) \rangle = 0$ and $\langle \Gamma_i(t)\Gamma_j(t') \rangle = q_{ij}\delta(t - t')$. We have homogeneous and inhomogeneous solution with initial condition $\xi_i(0) = x_i$. For homogeneous part we have:

$$\dot{\xi}_i + \gamma_{ij}\xi_j = 0$$

Solution of above is $\xi_i^h(t) = G_{ij}(t)x_j$, with help of boundary condition we have:

$$\xi_i^h(0) = G_{ij}(0)x_j \Rightarrow G_{ij}(0) = \delta_{ij}$$

Homogeneous solution should satisfy on the equation so:

$$\begin{aligned}
&\frac{d}{dt}(G_{ij}x_j) + \gamma_{ik}G_{kj}xj = 0 \\
&\dot{G}_{ij}x_j + G_{ij}\dot{x_j} + \gamma_{ik}G_{kj}x_j = 0 \\
&\big(\dot{G}_{ij} + \gamma_{ik}G_{kj}\big)x_j + G_{ij}\dot{x_j} = 0 \quad \Rightarrow \text{We know that} \quad \dot{x_j} = 0 \\
&\dot{G}_{ij} + \gamma_{ik}G_{kj} = 0 \quad \checkmark
\end{aligned}$$

3. This problem is like path integral so we can write $p(x_0, t_0)$ with infinitesimal time difference $\tau = (t - t_0)/N$ when $N \to \infty$ so we have:

$$p(x,t) = \int dx_{N-1} \int dx_{N-1} ... \int dx_0 p(x,t|x_{N-1},t_{N-1}) p(x_{N-1},t_{N-1}|x_{N-2},t_{N-2}) ... \\ \times p(x_1,t_1|x_0,t_0) p(x_0,t_0)$$

By replacing 3 in above we have:

$$\begin{aligned} p(x,t) = &\lim_{N\to\infty} \int dx_{N-1} \int dx_{N-2} ... \int dx_0 \\ &\Big[4\pi D^{(2)}(x,t)\Big]^{-1/2} \exp\Big(- \frac{[x_1 - x_0 - D^{(1)}(x_0,t_0)\tau]^2}{4D^{(2)}(x_0,t_0)\tau}\Big) \\ &\times \Big[4\pi D^{(2)}(x,t)\Big]^{-1/2} \exp\Big(- \frac{[x_2 - x_1 - D^{(1)}(x_1,t_1)\tau]^2}{4D^{(2)}(x_1,t_1)\tau}\Big) \\ &\times ... \Big[4\pi D^{(2)}(x,t)\Big]^{-1/2} \exp\Big(- \frac{[x_N - x_{N-1} - D^{(1)}(x_{N-1},t_{N-1})\tau]^2}{4D^{(2)}(x_{N-1},t_{N-1})\tau}\Big) \\ = &\lim_{N\to\infty} \int dx_{N-1} \int dx_{N-2} ... \int dx_0 \prod_{i=0}^{N-1} \Big[4\pi D^{(2)}(x_i,t_i)\Big]^{-1/2} \\ &\times \exp\Big(- \sum_{i=0}^{N-1} \frac{[x_{i+1} - x_i - D^{(1)}(x_i,t_i)\tau]^2}{4D^{(2)}(x_i,t_i)\tau}\Big) p(x_0,t_0) \end{aligned}$$

If evolution equation be like $x_{i+1} - x_i = \dot{x}(t_i)\tau$ we can write:

$$\sum_{i=0}^{N-1} \frac{[\dot{x}(t_i) - D^{(1)}(x_i,t_i)]^2}{4D^{(2)}(x_i,t_i)} \tau = \int_{t_0}^{t} \frac{[\dot{x}(t') - D^{(1)}(x(t'),t')]^2}{4D^{(2)}(x(t'),t')} dt'$$

This called generalized Onsager-Machlup equation.