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Answer to Exercise set 4

1. We use Cumulative Probability Distribution Function or CDF with definition:

$$p(a < X < b) = \mathcal{G}_X(b) - \mathcal{G}_X(a)$$

if a and b are very close together we have:

$$P(x) = \frac{d\mathcal{G}(x)}{dx}$$

in N dimensional we can get:

$$P(\mathcal{A}) = \frac{\partial^N \mathcal{G}(\mathcal{A})}{\partial \mathcal{A}_1 ... \partial \mathcal{A}_N} \Rightarrow \partial^N \mathcal{G}(\mathcal{A}) = P(\mathcal{A})\partial \mathcal{A}_1 ... \partial \mathcal{A}_N = P(\mathcal{A})d\mathcal{A}$$

Average of $\mathcal{F}$ calculated like this:

$$\begin{aligned} \langle \mathcal{F} \rangle &= \int d\mathcal{A}\mathcal{F}(\mathcal{A})P(\mathcal{A}) \\ &= \int \mathcal{F}(\mathcal{A})\partial^N \mathcal{G}(\mathcal{A}) \end{aligned} \tag{1}$$

Now we use integration by part recursively, we obtain:

$$\begin{aligned} \langle \mathcal{F} \rangle &= \int_{-\infty}^{\infty} \mathcal{F}(\mathcal{A})\partial^N \mathcal{G}(\mathcal{A}) \\ &= \mathcal{F}(\mathcal{A})\partial^{N-1}\mathcal{G}(\mathcal{A})\Big]_{-\infty}^{\infty} - \partial\mathcal{F}(\mathcal{A})\partial^{N-2}\mathcal{G}(\mathcal{A})\Big]_{-\infty}^{\infty} + \partial^2\mathcal{F}(\mathcal{A})\partial^{N-3}\mathcal{G}(\mathcal{A})\Big]_{-\infty}^{\infty} + ... \\ &+ (-1)^{n-1}\partial^{N-1}\mathcal{F}(\mathcal{A})\mathcal{G}(\mathcal{A})\Big]_{-\infty}^{\infty} + (-1)^n \int_{-\infty}^{\infty} \partial^N \mathcal{F}(\mathcal{A})\mathcal{G}(\mathcal{A}) \end{aligned}$$

$\mathcal{F}$ is well behave function so in infinity $\mathcal{F}$ and all it's derivative is zero, so we have:

$$\langle \mathcal{F} \rangle = (-1)^n \int_{-\infty}^{\infty} \partial^N \mathcal{F}(\mathcal{A})\mathcal{G}(\mathcal{A})$$

Using following relation:

$$\partial^N \mathcal{G}(\mathcal{A}) = P(\mathcal{A})d\mathcal{A} = \hat{\mathcal{L}}_{\mathcal{A}} P_G(\mathcal{A})d\mathcal{A} \Rightarrow \frac{\partial^N \mathcal{G}(\mathcal{A})}{\partial \mathcal{A}_1 ... \partial \mathcal{A}_N} = \hat{\mathcal{L}}_{\mathcal{A}} P_G(\mathcal{A})$$

Then we find $\mathcal{G}(\mathcal{A}) = P_G(\mathcal{A})$ and $d\mathcal{F}(\mathcal{A}) = \hat{\mathcal{L}}_{\mathcal{A}}\mathcal{F}(\mathcal{A})d\mathcal{A}$, so we can get:

$$\langle \mathcal{F} \rangle = (-1)^n \int \partial^N \mathcal{F}(\mathcal{A})\mathcal{G}(\mathcal{A}) = (-1)^n \int P_G(\mathcal{A})\hat{\mathcal{L}}_{\mathcal{A}}\mathcal{F}(\mathcal{A})d\mathcal{A}$$

$$\langle \mathcal{F} \rangle = (-1)^n \int P_G(\mathcal{A}) \hat{\mathcal{L}}_{\mathcal{A}} \mathcal{F}(\mathcal{A}) d\mathcal{A} = (-1)^n \Big\langle \hat{\mathcal{L}}_{\mathcal{A}} \mathcal{F}(\mathcal{A}) \Big\rangle_G$$

Finally:

$$\begin{aligned}
\langle \mathcal{F} \rangle &= (-1)^n \Big\langle \hat{\mathcal{L}}_{\mathcal{A}} \mathcal{F}(\mathcal{A}) \Big\rangle_G \\
&= (-1)^n \Big\langle \exp\Big[\sum_{n=3}^{\infty} \frac{(-1)^n}{n!} \Big\{ \sum_{\mu_1 \ldots, \mu_n}^{N} \mathcal{K}^{(n)}_{\mu_1 \ldots \mu_n} \frac{\partial^n}{\partial \mathcal{A}_{\mu_1} \ldots \partial \mathcal{A}_{\mu_n}} \Big\} \Big] \mathcal{F}(\mathcal{A}) \Big\rangle_G \\
&= \Big\langle \exp\Big[\sum_{n=3}^{\infty} \frac{1}{n!} \Big\{ \sum_{\mu_1 \ldots, \mu_n}^{N} \mathcal{K}^{(n)}_{\mu_1 \ldots \mu_n} \frac{\partial^n}{\partial \mathcal{A}_{\mu_1} \ldots \partial \mathcal{A}_{\mu_n}} \Big\} \Big] \mathcal{F}(\mathcal{A}) \Big\rangle_G
\end{aligned}$$

2. We should calculate this:

$$\langle \mathcal{F} \rangle = \left\langle \hat{\mathcal{L}} \mathcal{F}(\mathcal{A}) \right\rangle_G$$

$$\hat{\mathcal{L}} = \exp\Big[\sum_{n=3}^{\infty} \frac{1}{n!} \Big\{ \sum_{\mu_1 \ldots, \mu_n}^{N} \mathcal{K}^{(n)}_{\mu_1 \ldots \mu_n} \frac{\partial^n}{\partial \mathcal{A}_{\mu_1} \ldots \partial \mathcal{A}_{\mu_n}} \Big\} \Big]$$

With expansion of $\hat{\mathcal{L}}$ we have:

$$\begin{aligned}
\langle \mathcal{F} \rangle &= \Big\langle \exp\Big[\sum_{n=3}^{\infty} \frac{1}{n!} \Big\{ \sum_{\mu_1 \ldots, \mu_n}^{N} \mathcal{K}^{(n)}_{\mu_1 \ldots \mu_n} \frac{\partial^n}{\partial \mathcal{A}_{\mu_1} \ldots \partial \mathcal{A}_{\mu_n}} \Big\} \Big] \mathcal{F}(\mathcal{A}) \Big\rangle_G \\
&= \Bigg\langle \Bigg\{ 1 + \Big[\sum_{n=3}^{\infty} \frac{1}{n!} \Big\{ \sum_{\mu_1 \ldots, \mu_n}^{N} \mathcal{K}^{(n)}_{\mu_1 \ldots \mu_n} \frac{\partial^n}{\partial \mathcal{A}_{\mu_1} \ldots \partial \mathcal{A}_{\mu_n}} \Big\} \Big] \\
&+ \frac{1}{2!} \Bigg(\Big[\sum_{n=3}^{\infty} \frac{1}{n!} \Big\{ \sum_{\mu_1 \ldots, \mu_n}^{N} \mathcal{K}^{(n)}_{\mu_1 \ldots \mu_n} \frac{\partial^n}{\partial \mathcal{A}_{\mu_1} \ldots \partial \mathcal{A}_{\mu_n}} \Big\} \Big] \mathcal{F}(\mathcal{A}) \Bigg)^2 \\
&+ \frac{1}{3!} \Bigg(\Big[\sum_{n=3}^{\infty} \frac{1}{n!} \Big\{ \sum_{\mu_1 \ldots, \mu_n}^{N} \mathcal{K}^{(n)}_{\mu_1 \ldots \mu_n} \frac{\partial^n}{\partial \mathcal{A}_{\mu_1} \ldots \partial \mathcal{A}_{\mu_n}} \Big\} \Big] \Bigg)^3 + \ldots \Bigg\} \mathcal{F}(\mathcal{A}) \Bigg\rangle_G
\end{aligned} \tag{2}$$

From here we use normalized cumulant with definition:

$$\hat{\mathcal{K}}^{(n)} = \frac{\mathcal{K}^{(n)}}{\sigma_0^{n-2}}$$

So we have:

$$\begin{aligned}
\langle \mathcal{F} \rangle &= \langle \mathcal{F}(\mathcal{A}) \rangle_G + \Big(\frac{1}{3!} \sum_{\mu_1 \mu_2 \mu_3} \hat{\mathcal{K}}^{(3)}_{\mu_1 \mu_2 \mu_3} \langle \mathcal{F}_{,\mu_1 \mu_2 \mu_3} \rangle_G \Big) \sigma_0 \\
&+ \Bigg(\frac{1}{4!} \sum_{\mu_1 \mu_2 \mu_3 \mu_4} \hat{\mathcal{K}}^{(4)}_{\mu_1 \mu_2 \mu_3 \mu_4} \langle \mathcal{F}_{,\mu_1 \mu_2 \mu_3 \mu_4} \rangle_G + \frac{1}{2!(3!)^2} \sum_{\mu_1 \mu_2 \mu_3} \sum_{\nu_1 \nu_2 \nu_3} \hat{\mathcal{K}}^{(3)}_{\mu_1 \mu_2 \mu_3} \hat{\mathcal{K}}^{(3)}_{\nu_1 \nu_2 \nu_3} \langle \mathcal{F}_{,\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \rangle_G \Bigg) \sigma_0^2 \\
&+ \Bigg(\frac{1}{5!} \sum_{\mu_1 \mu_2 \mu_3} \hat{\mathcal{K}}^{(5)}_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \langle \mathcal{F}_{,\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \rangle_G \\
&+ \frac{1}{3!(3!)^3} \sum_{\mu_1 \mu_2 \mu_3} \sum_{\nu_1 \nu_2 \nu_3} \sum_{\eta_1 \eta_2 \eta_3} \hat{\mathcal{K}}^{(3)}_{\mu_1 \mu_2 \mu_3} \hat{\mathcal{K}}^{(3)}_{\nu_1 \nu_2 \nu_3} \hat{\mathcal{K}}^{(3)}_{\eta_1 \eta_2 \eta_3} \langle \mathcal{F}_{,\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3 \eta_1 \eta_2 \eta_3} \rangle_G \Bigg) \sigma_0^3 + \mathcal{O}(\sigma_0^4)
\end{aligned}$$

With use of this definition$\langle \mathcal{F}_{,\mu_1 \mu_2 \mu_3} \rangle_G = \frac{\partial^3 \mathcal{F}(\mathcal{A})}{\partial \mathcal{A}_{\mu_1} \partial \mathcal{A}_{\mu_2} \partial \mathcal{A}_{\mu_3}}$. Now insert Dirac delta function so:

$$\langle\delta_D(\alpha-\nu)\rangle = \langle\delta_D(\alpha-\nu)\rangle_G + \Big(\frac{1}{3!}\sum_{\mu_1\mu_2\mu_3}\hat{\mathcal{K}}^{(3)}_{\mu_1\mu_2\mu_3}\langle\frac{\partial^3\delta_D(\alpha-\nu)}{\partial\mathcal{A}_{\mu_1}\partial\mathcal{A}_{\mu_2}\partial\mathcal{A}_{\mu_3}}\rangle_G\Big)\sigma_0$$

$$+\Bigg(\frac{1}{4!}\sum_{\mu_1\mu_2\mu_3\mu_4}\hat{\mathcal{K}}^{(4)}_{\mu_1\mu_2\mu_3\mu_4}\langle\frac{\partial^4\delta_D(\alpha-\nu)}{\partial\mathcal{A}_{\mu_1}\partial\mathcal{A}_{\mu_2}\partial\mathcal{A}_{\mu_3}\partial\mathcal{A}_{\mu_4}}\rangle_G$$

$$+\frac{1}{2!(3!)^2}\sum_{\mu_1\mu_2\mu_3}\sum_{\nu_1\nu_2\nu_3}\hat{\mathcal{K}}^{(3)}_{\mu_1\mu_2\mu_3}\hat{\mathcal{K}}^{(3)}_{\nu_1\nu_2\nu_3}\langle\frac{\partial^6\delta_D(\alpha-\nu)}{\partial\mathcal{A}_{\mu_1}\partial\mathcal{A}_{\mu_2}\partial\mathcal{A}_{\mu_3}\partial\mathcal{A}_{\nu_1}\partial\mathcal{A}_{\nu_2}\partial\mathcal{A}_{\nu_3}}\rangle_G\Bigg)\sigma_0^2$$

$$+\Bigg(\frac{1}{5!}\sum_{\mu_1\mu_2\mu_3}\hat{\mathcal{K}}^{(5)}_{\mu_1\mu_2\mu_3\mu_4\mu_5}\langle\frac{\partial^5\delta_D(\alpha-\nu)}{\partial\mathcal{A}_{\mu_1}\partial\mathcal{A}_{\mu_2}\partial\mathcal{A}_{\mu_3}\partial\mathcal{A}_{\mu_4}\partial\mathcal{A}_{\mu_5}}\rangle_G$$

$$+\frac{1}{3!(3!)^3}\sum_{\mu_1\mu_2\mu_3}\sum_{\nu_1\nu_2\nu_3}\sum_{\eta_1\eta_2\eta_3}\hat{\mathcal{K}}^{(3)}_{\mu_1\mu_2\mu_3}\hat{\mathcal{K}}^{(3)}_{\nu_1\nu_2\nu_3}\hat{\mathcal{K}}^{(3)}_{\eta_1\eta_2\eta_3}$$

$$\langle\frac{\partial^9\delta_D(\alpha-\nu)}{\partial\mathcal{A}_{\mu_1}\partial\mathcal{A}_{\mu_2}\partial\mathcal{A}_{\mu_3}\mathcal{A}_{\nu_1}\partial\mathcal{A}_{\nu_2}\partial\mathcal{A}_{\nu_3}\partial\mathcal{A}_{\eta_1}\partial\mathcal{A}_{\eta_2}\partial\mathcal{A}_{\eta_3}}\rangle_G\Bigg)\sigma_0^3+\mathcal{O}(\sigma_0^4)$$

Now we calculate each of six term, for first term we have:

$$FirstTerm \equiv \langle\delta_D(\alpha-\nu)\rangle_G = \int d\alpha\frac{e^{-\alpha^2/2\mathcal{K}^{(2)}}}{\sqrt{2\pi Det(\mathcal{K}^{(2)})}}\delta_D(\alpha-\nu) = \frac{1}{\sqrt{2\pi\sigma_0^2}}e^{-\nu^2/2\sigma_0^2}$$

For second term:

$$Second\ Term \equiv \frac{1}{3!}\sum_{\mu_1\mu_2\mu_3}\hat{\mathcal{K}}^{(3)}_{\mu_1\mu_2\mu_3}\langle\frac{\partial^3\delta_D(\alpha-\nu)}{\partial\mathcal{A}_{\mu_1}\partial\mathcal{A}_{\mu_2}\partial\mathcal{A}_{\mu_3}}\rangle_G = \frac{1}{6}\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\int d\alpha\frac{\partial^3\delta_D(\alpha-\nu)}{\partial\alpha^3}\frac{1}{\sqrt{2\pi\sigma_0^2}}e^{-\alpha^2/2\sigma_0^2}$$

$$= \frac{1}{6}\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\int d\alpha\frac{\partial^3\delta_D(\alpha-\nu)}{\partial\alpha^3}\frac{1}{\sqrt{2\pi\sigma_0^2}}e^{-\alpha^2/2\sigma_0^2}H_0(\alpha)$$

$$= \frac{1}{6}\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\Big\langle\frac{\partial^3\delta_D(\alpha-\nu)}{\partial\alpha^3}H_0(\alpha)\Big\rangle_G$$

Hermit polynomial first term is $H_0(\alpha)=1$ and nth term is $H_n(\alpha)=e^{-\alpha^2/2}(-\frac{\partial}{\partial\alpha})^n e^{\alpha^2/2}$ with aid of helpful hint in class note that is:

$$\Big\langle\frac{\partial^k\delta_D(\alpha-\nu)}{\partial\alpha^k}H_n(\alpha)\Big\rangle_G =^{1} \frac{e^{-\nu^2/2\sigma_0^2}}{\sqrt{2\pi}}H_{n+k}(\nu)$$

We can calculate second term like this:

$$Second\ Term \equiv \frac{1}{6}\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\frac{e^{-\nu^2/2\sigma_0^2}}{\sqrt{2\pi}}H_3(\nu)$$

and so on:

$$Third\ Term \equiv \frac{1}{4!}\hat{\mathcal{K}}^{(4)}_{\alpha\alpha\alpha\alpha}\Big\langle\frac{\partial^4\delta_D(\alpha-\nu)}{\partial\alpha^4}H_0(\alpha)\Big\rangle_G = \frac{1}{4!}\hat{\mathcal{K}}^{(4)}_{\alpha\alpha\alpha\alpha}\frac{e^{-\nu^2/2\sigma_0^2}}{\sqrt{2\pi}}H_4(\nu)$$

[1] You can prove this using $\int dx f(x)\partial^n\delta_D(x)/\partial x^n = -\int dx(\partial^{n-1}\delta_D(x)/\partial x^{n-1})\partial f(x)/\partial x$.

$$Forth\,Term \equiv \frac{1}{2!(3!)^2}\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\Big\langle\frac{\partial^6\delta_D(\alpha-\nu)}{\partial\alpha^6}H_0(\alpha)\Big\rangle_G = \frac{1}{2!(3!)^2}\big[\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\big]^2\frac{e^{-\nu^2/2\sigma_0^2}}{\sqrt{2\pi}}H_6(\nu)$$

$$Fifth\ Term \equiv \frac{1}{5!}\hat{\mathcal{K}}^{(5)}_{\alpha\alpha\alpha\alpha\alpha}\langle\frac{\partial^5\delta_D(\alpha-\nu)}{\partial\alpha^5}H_0(\alpha)\rangle_G = \frac{1}{5!}\hat{\mathcal{K}}^{(5)}_{\alpha\alpha\alpha\alpha\alpha}\frac{e^{-\nu^2/2\sigma_0^2}}{\sqrt{2\pi}}H_5(\nu)$$

Finally last term:

$$\begin{aligned} Sixth\ Term &\equiv \frac{1}{3!(3!)^3}\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\langle\frac{\partial^9\delta_D(\alpha-\nu)}{\partial\alpha^9}\rangle_G \\ &= \frac{1}{3!(3!)^3}\big[\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\big]^3\frac{e^{-\nu^2/2\sigma_0^2}}{\sqrt{2\pi}}H_9(\nu) \end{aligned}$$

And result is:

$$\begin{aligned} \langle\delta_D(\alpha-\nu)\rangle = \frac{1}{\sqrt{2\pi\sigma_0^2}}e^{-\nu^2/2\sigma_0^2}\Bigg[&1+\frac{1}{6}\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\sigma_0H_3(\nu)+\frac{1}{4!}\hat{\mathcal{K}}^{(4)}_{\alpha\alpha\alpha\alpha}\sigma_0^2H_4(\nu) \\ &+\frac{1}{2!(3!)^2}\big[\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\sigma_0\big]^2H_6(\nu)+\frac{1}{5!}\hat{\mathcal{K}}^{(5)}_{\alpha\alpha\alpha\alpha\alpha}\sigma_0^3\sqrt{2\pi}H_5(\nu) \\ &+\frac{1}{3!(3!)^3}\big[\hat{\mathcal{K}}^{(3)}_{\alpha\alpha\alpha}\sigma_0\big]^3\sqrt{2\pi}H_9(\nu)+\mathcal{O}(\sigma_0^4)\Bigg] \end{aligned}$$