Intuitionistic Weak Arithmetic

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Abstract

We construct ω-framed Kripke models of $i\forall_1$ and $i\Pi_1$ non of whose worlds satisfies $\forall x \exists y(x = 2y \vee x = 2y+1)$ and $\forall x, y \exists z Exp(x, y, z)$ respectively. This will enable us to show that $i\forall_1$ does not prove $\neg\neg\forall x \exists y(x = 2y \vee x = 2y + 1)$ and $i\Pi_1$ does not prove $\neg\neg\forall x, y \exists z Exp(x, y, z)$. Therefore, $i\forall_1 \nvdash \neg\neg lop$ and $i\Pi_1 \nvdash \neg\neg i\Sigma_1$. We also prove that $HA \nvdash l\Sigma_1$ and present some remarks about $i\Pi_2$.

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0. Preliminaries

Following [W1], [AM], [MM], [M1] and [M2] this paper continues the study of some weak fragments of Heyting arithmetic and Kripke models of them.

We fix the language $L = \{+, \cdot, <, 0, 1\}$ of arithmetic throughout the paper.

By *open* formulas we mean quantifier-free formulas. $(\exists x \leq t)\varphi$ is an abbreviation for $\exists x(x \leq t \wedge \varphi)$ and $(\forall x \leq t)\varphi$ is an abbreviation for $\forall x(x \leq t \rightarrow \varphi)$, where t is a term not involving x. A formula is bounded if all quantifiers occurring in it are bounded, i.e., occur in a context as above. Σ_0, Π_0 or Δ_0-formulas are bounded formulas. For $n \geq 0, \Sigma_{n+1}$-formulas have the form $(\exists \overline{x})\varphi$ where φ in Π_n, Π_{n+1}-formulas have the form $(\forall \overline{x})\varphi$ where φ in Σ_n.

The hierarchy of $\forall_n$-formulas and of $\exists_n$-formulas are defined similarly by changing bounded formulas to open formulas.

Heything arithmetic HA and its fragments $(PA^-)^i$, $iop(= iopen)$, $lop(= lopen)$ and $i\Delta_0$ are the intuitionistic counterparts of first order Peano Arithmetic PA and its fragments PA^-, $Iop(= Iopen)$, $Lop(= Lopen)$ and $I\Delta_0$. More generally for any set Γ of formulas we will use notations such as $i\Gamma$ and $l\Gamma$ in the same manner.

We use the usual terminology about Kripke structures as in [TD]. A formula $\varphi(\overline{x})$ is decidable in a Kripke model $\mathcal{K}$ whenever $\mathcal{K} \Vdash \forall \overline{x}(\varphi(\overline{x}) \vee \neg\varphi(\overline{x}))$.

For a set T of sentences, T^i and T^c denote its intuitionistical and classical deductive closures.

Let $\neg\neg iop$ denote the intuitionistic theory axiomatized by $(PA^-)^i + \{\neg\neg I_x\varphi : \varphi \text{ is open}\}$. The theories $\neg\neg i\forall_1$ and $\neg\neg lop$ are defined similarly, by either replacing the class of open formulas by $\forall_1$-formulas or the induction scheme by LNP. Also, $\neg\neg i\Pi_1$ will stand for the intuitionistic theory axiomatized by $i\Delta_0 + \{\neg\neg I_x\varphi : \varphi \in \Pi_1\}$.

Below we give three facts which we will use throughout the paper. The proofs are straightforward.

Fact 1 A $\forall_1$ (resp. Π_1)-formula is forced at a node α of a Kripke model of $(PA^-)^i$ (resp. $i\Delta_0$) if and only if it is satisfied in (the world attached to) α and any node above α if and only if it is satisfied in the union of the worlds in any (complete) path above α.

Fact 2 Suppose that $\mathcal{K} \Vdash (PA^-)^i$ (resp. $\mathcal{K} \Vdash i\Delta_0$) and $\varphi \in \exists_1$ (resp. $\varphi \in \Sigma_1$). Then for each $\alpha \in K$, we have:

$$\alpha \Vdash \varphi \Leftrightarrow M_\alpha \vDash \varphi.$$

If $\psi \in \forall_2$ (resp. $\psi \in \Pi_2$) then:

$$\alpha \Vdash \psi \Leftrightarrow \forall\beta \geq \alpha \; M_\beta \vDash \psi.$$

Fact 3 For a linear Kripke model deciding atomic (resp. bounded)-formulas to force $i\forall_1$ (resp. $i\Pi_1$), it is necessary and sufficient that the union of the worlds in any (complete) path in it satisfies $I\forall_1$ (resp. $I\Pi_1$).

Proof It was proved in [M2], using induction on formulas, that if α is a node in a linear Kripke model deciding atomic formulas and φ is an $\exists$-free formula, then $\alpha \Vdash \varphi$ if and only if the union of the worlds above α satisfies φ. Using this the proof is straightforward.□

1. Constructing Kripke models of $i\forall_1 + \neg AEO$ and $i\Pi_1 + \neg exp$

In this section we prove two independence results for $i\forall_1$ and $i\Pi_1$.

Let AEO be the sentence $\forall x\exists y(x = 2y \vee x = 2y + 1)$. It was proved in [MM, 3.1] that, iop does not prove $\neg\neg AEO$. Here, using the same method, we show that even $i\forall_1$ does not prove $\neg\neg AEO$.

Proposition 1.1 There is an ω-framed Kripke model of $i\forall_1$ which forces $\neg AEO$.

Proof: Method 1 We use a modified version of the proof of [MM, 3.1]. Indeed we prove that for any nonstandard model M of $I\forall_1$ including an element t infinitely many times divisible by 2, there is an ω-framed Kripke model of $i\forall_1$ with no worlds satisfying AEO such that the union of its worlds is a countable submodel of M satisfying $I\forall_1$.

Let $(\psi_n)_{n\in\omega}$ be an enumeration of all universal L-formulas with a distinguished free variable. Each universal formula $\varphi(x_1,\cdots,x_k)$, $k\geq 1$, occurs k-times in this enumeration.

Let $M\vDash I\forall_1$ and $t\in M$ has the above mentioned property. Put $M_0=\mathbb{Z}[t]^{\geq 0}$ and let $\overline{p}_{0,0},\overline{p}_{0,1},\cdots$ be a list of all tuples of parameters from M_0 (an enumeration of $M_0^{<\omega}$).

Fix any $k\geq 0$. Assume that for each $i\leq k$ a subsemiring M_i of M together with an enumeration $(\overline{p}_{i,j})_{j\in\omega}$ of $M_i^{<\omega}$ is given. For each $0\leq i,j,m\leq k$ with $i+j\leq k$, if $\overline{p}_{i,j}$ does not have the same arity as the non-distinguished free variables in ψ_m or if $M_i\models\neg\psi_m(0,\overline{p}_{i,j})$ or $M\models\forall x\psi_m(x,\overline{p}_{i,j})$, where x is the distinguished free variable in ψ_m, then let $s_{i,j,m}=0$. Otherwise, let $s_{i,j,m}$ be the least element in M for which $M\models\neg\psi_m(s_{i,j,m}+1,\overline{p}_{i,j})$ (note that $I\forall_1\vdash L\exists_1$). Suppose $\psi_m(s_{i,j,m}+1,\overline{p}_{i,j})$ is $\forall\overline{y}\varphi_m(s_{i,j,m}+1,\overline{p}_{i,j},\overline{y})$, where φ_m is open. Let $\overline{t}_{i,j,m}$ be any tuple of elements of M such that $M\models\neg\varphi_m(s_{i,j,m}+1,\overline{p}_{i,j},\overline{t}_{i,j,m})$. Let $M_{k+1}=M_k[s_{i,j,m},\overline{t}_{i,j,m}:\ 0\leq i,j,m\leq k,\ i+j\leq k]^{\geq 0}$.

Consider the Kripke structure on frame ω with M_k attached to node k. We want to show that for any m, $0\Vdash I_x\psi_m(x,\overline{y})$. Fix $i\geq 0$ and let $\overline{p}_{i,j}\in M_i$, of the same arity as the number of non-distinguished free variables in ψ_m, be arbitrary. We need to show $i\Vdash I_x\psi_m(x,\overline{p}_{i,j})$. It is easy to see that $\neg\neg I_x\psi_m(x,\overline{p}_{i,j})\vdash_i I_x\psi_m(x,\overline{p}_{i,j})$ and so it suffices to prove the following claim:

Claim We have $i+j+m+1\Vdash I_x\psi_m(x,\overline{p}_{i,j})$.

Proof of the Claim In constructing $M_{i+j+m+1}$ from M_{i+j+m}, the formula $\psi_m(x,\overline{p}_{i,j})$ receives attention. Using Fact 1, one can show that if $M_i\models\neg\psi_m(0,\overline{p}_{i,j})$ or $M\models\forall x\psi_m(x,\overline{p}_{i,j})$, then $i+j+m+1\Vdash I_x\psi_m(x,\overline{p}_{i,j})$. Otherwise, by construction and Fact 1 again, $i+j+m+1$ does not force the second conjunct of the antecedent of $I_x\psi_m(x,\overline{p}_{i,j})$ and so forces $I_x\psi_m(x,\overline{p}_{i,j})$. This establishes the claim.

As any finitely generated ring is Noetherian, one can show that each of the worlds in the Kripke model is a model of $\neg AEO$. Let us prove this. Assume for the purpose of a contradiction that some world models AEO. Put $t_0=t$ and $t_{l+1}=\frac{t_l}{2}$. The ascending chain of ideals $(t_0)\subseteq(t_1)\subseteq(t_2)\subseteq\cdots$ in the ring generated by that model must stop as, by Hilbert's basis theorem, every finitely generated ring is Noetherian. So, for some $n\in\mathbb{N}$ and some g in that world, $0=(2g-1)t$. But this is impossible as $2g-1\neq 0$ and t is infinitely large. This contradiction shows that for some i, t_{i+1} does not exist, i.e., t_i is not divisible by 2. Since our world is supposed to be a model of AEO it would follow that t_i is odd, which is impossible because this world is a subring of M in which t_i is divisible by 2.

Now since the sentence AEO is $\forall_2$, the Kripke model will force $\neg AEO$ (Fact 2) and we will be done with the proposition.

Method 2 Let $M=\{p_0,p_1,p_2,\dots\}$ be a countable nonstandard model of $I\forall_1$ with $t=p_0\in M$ as above. For each $i\geq 0$, put $M_i=\mathbb{Z}[p_0,\cdots,p_i]^{\geq 0}$. Let $\mathcal{K}$ be the obvious ω-framed Kripke model. We have $\bigcup M_i=M\models I\forall_1$ and therefore by Fact 3, $\mathcal{K}\Vdash i\forall_1$.

Again, each node of $\mathcal{K}$ is finitely generated and so $\mathcal{K} \Vdash \neg AEO$.□

An intuitionistic theory T^i is said to be closed under the rule Double Negation Shift DNS if whenever $T^i \vdash \forall \overline{x} \neg\neg\varphi$, then $T^i \vdash \neg\neg\forall\overline{x}\varphi$ for any formula φ.

Theorem 1.2 (i) The theory $i\forall_1$ is not closed under the rule $DNS(\exists_1)$ (the rule DNS restricted to $\exists_1$-formulas).

(ii) $i\forall_1 \nvdash \neg\neg lop$.

Proof (i) By $Iop \vdash AEO$ and closure of iop under the negative translation we have $iop \vdash \forall x \neg\neg\exists y(x = 2y \vee x = 2y + 1)$, while the above proposition shows $i\forall_1 \nvdash \neg\neg AEO$.

(ii) By the proof of [AM, Th. 1.4], Kripke models of lop are exactly Iop-normal Kripke structures and so $lop \vdash AEO$.□

Now we consider the theory $i\Pi_1$. Recall Wehmeier's result, $i\Pi_1 \nvdash exp$, where exp is the Π_2 sentence which says the exponentiation function is total. His proof is based on constructing a two-node Kripke model of $i\Pi_1$ such that its root is not a model of exp, see [W1, Lemma 10]. Here we prove a stronger independence result.

Proposition 1.3 There is an ω-framed Kripke model of $i\Pi_1$ which forces $\neg exp$.

Proof Let M be a countable nonstandard model of $I\Pi_1$. Suppose that $a_0, a_1, a_2, \cdots$ is a cofinal sequence of the nonstandard elements of M such that $a_i^{a_i} < a_{i+1}$ for each $i \geq 0$. For each $a \in M$, $a^{\mathbb{N}}$ denotes the set $\{x \in M : x < a^n$ for some non negative integer $n\}$. Consider the Kripke Model ${a_0}^{\mathbb{N}} \subseteq {a_1}^{\mathbb{N}} \subseteq {a_2}^{\mathbb{N}} \subseteq \cdots$. By [K, P. 69], each node of this Kripke model is a Δ_o-elementary substructure of M (therefore models Π_1-theory $I\Delta_0$) and non of them satisfies exp. Therefore, it forces the negation of $exp \in \Pi_2$. Also, since the union of the worlds in this Kripke model is equal to M by Fact 3, it forces $i\Pi_1$.□

Theorem 1.4 (i) The theory $i\Pi_1$ is not closed under the rule $DNS(\Sigma_1)$ (the rule DNS restricted to Σ_1-formulas).

(ii) $i\Pi_1 \nvdash \neg\neg i\Sigma_1$.

Proof (i)The theory $i\Pi_1$ is closed under the negative translation and $I\Pi_1$ proves exp. Therefore $i\Pi_1 \vdash \forall x, y \neg\neg\exists z Exp(x, y, z)$ while the above proposition shows $i\Pi_1 \nvdash \neg\neg exp$.

(ii) By [W1, Fact 8], $I\Sigma_1$ is Π_2-conservative over $i\Sigma_1$ and so $i\Sigma_1 \vdash exp$.□

For any theory T^i containing $i\Delta_0$, we denote the intuitionistic closure of $i\Delta_0 + \{\neg\neg\varphi : \varphi \in T^i\}$ by $\neg\neg T^i$.

Proposition 1.5 If T^i contains $i\Delta_0 + exp$, then $\neg\neg T^i \nvdash T^i$.

Proof Suppose $\neg\neg T^i \vdash T^i$. Then any two-node Kripke model consisting of a model $M \models T^c$ over a Δ_0-elementary substructure of M will force T^i, and so Whehmeier's argument about the limitation of the Π_2-consequences of $i\Pi_1$ works in this situation, contradiction.□

2. Some remarks about $i\Pi_2$

What can we say about $i\Pi_2$? First, $I\Pi_2$ is Π_2-conservative over $i\Pi_2$ [Bur, Coro. 2.6]. Also, by Proposition 1.5, $\neg\neg i\Pi_2 \nvdash i\Pi_2$. This shows that, unlike $i\Pi_1$, it is not true that satisfying $I\Pi_2$ in the union of each cofinal path of a Kripke model $\mathcal{K} \Vdash i\Delta_0$ implies $\mathcal{K} \Vdash i\Pi_2$. Therefore, we should not expect to construct Kripke models of the form Proposition 1.3 for $i\Pi_2$. However, the converse remains open:

Question 1 Is it true that the union of the worlds in any cofinal path of a Kripke model of $i\Pi_2$ satisfies $I\Pi_2$?

Wehmeier [W2, Th. 5.1] proved that any reversely well founded $I\Pi_2$-normal Kripke structure forces $i\Pi_2$ (note that by [Bus, P. 72-73], there exists an ω-framed PA-normal Kripke structure which does not force even $i\Pi_1$). Also one can construct a non $I\Pi_2$-normal Kripke model of $i\Pi_2$ by putting a model M of $I\Pi_2$ above a Σ_2-elementary subsructure of M which is not a model of $I\Pi_2$. Furthermore, it is easy to see that any Σ_2-elementary $I\Pi_2$-normal Kripke structure forces $i\Pi_2$.

Question 2 Is there an ω-framed Kripke model of $i\Pi_2$ non of whose worlds satisfies $I\Pi_2$?

Here we prove a generalization of [W2, Th. 5.1].

Proposition 2.1 Any $I\Pi_2$-normal Kripke model of $\neg\neg i\Pi_2$ (with a tree as its frame) forces $i\Pi_2$.

Proof Let $\mathcal{K}$ be an $I\Pi_2$-normal Kripke model of $\neg\neg i\Pi_2$ and $\alpha \in \mathcal{K}$. Suppose that $\varphi(x, \overline{y})$ is any Π_2-formula. If $\alpha \nVdash I_x\varphi(x, \overline{y})$, then there exists a node $\beta \geq \alpha$ and $\overline{b} \in M_\beta$ such that $\beta \Vdash \varphi(0, \overline{b})$ and $\beta \Vdash \forall x(\varphi(x, \overline{b}) \rightarrow \varphi(x+1, \overline{b}))$, but $\beta \nVdash \forall x\varphi(x, \overline{b})$. By $\beta \Vdash \neg\neg i\Pi_2$ in each path above β, there exists a node which forces $I_x\varphi(x, \overline{b})$ and so does $\forall x\varphi(x, \overline{b})$. Now we can consider the nodes below these nodes and proceed by bar induction as the proof of [W2, Th. 5.1].□

We end this section by providing a proof for a stronger version of the fact $HA \nvdash LNP$, see e.g. [TD, P. 130-131] or [D, P. 117].

Proposition 2.2 $HA \nvdash l\Sigma_1$.

Proof Let $\tau \in \Pi_1$ be a Godel sentence ($PA \nvdash \tau$, $\mathbb{N} \models \tau$). Assume $\sigma \equiv_c \neg\tau \in \Sigma_1$ and let M be a classical model of $PA + \sigma$. Let $\mathcal{K}$ be the two-node Kripke model obtained by putting M above $\mathbb{N}$ (the result of applying Smorynski's prime operation $'$ to M [S]). Note that the least solution of the formula $x = 1 \vee \sigma$ in $\mathbb{N}$ is 1 and in M is 0. Hence using fact 2, one can see that $\mathcal{K} \nVdash L_x(x = 1 \vee \sigma)$. □

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