$\mathcal{H}$-theories, Fragments of HA and PA-normality

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Abstract

For a classical theory T, $\mathcal{H}(T)$ denotes the intuitionistic theory of T-normal (i.e. locally T) Kripke structures. S. Buss has asked for a characterization of the theories in the range of $\mathcal{H}$ and raised the particular question of whether HA is an $\mathcal{H}$-theory. We show that $T^i \in \text{range}(\mathcal{H})$ iff $T^i = \mathcal{H}(T)$. As a corollary, no fragment of HA extending $i\Pi_1$ belongs to the range of $\mathcal{H}$. A. Visser has already proved that HA is not in the range of $\mathcal{H}$ by different methods. We provide more examples of theories not in the range of $\mathcal{H}$. We show PA-normality of once-branching Kripke models of $HA + MP$, where it is not known whether the same holds if MP is dropped.

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0. Preliminaries

We fix the language $\{+, \cdot, <, 0, 1\}$. The principle PEM (some of whose restrictions will appear below) of Excluded Middle is $\forall \overline{x}(\varphi(\overline{x}) \vee \neg\varphi(\overline{x}))$. Heyting arithmetic HA and its fragments iPA^-, iop, lop, $i\Delta_0$, $i\Sigma_n$ and $i\Pi_n$, $n \geq 1$, are the intuitionistic counterparts of first order Peano Arithmetic PA and its fragments PA^-, Iop, Lop, $I\Delta_0$, $I\Sigma_n$ and $I\Pi_n$. We use the usual terminology about Kripke structures as in [Bus]. By MP we mean Markov's principle $\forall x(\varphi \vee \neg\varphi) \wedge \neg\neg\exists x\varphi \rightarrow \exists x\varphi$, the restrictions MP_{Δ_0} and MP_{open} being self-explanatory. For a class Γ of formulas, $\neg\Gamma$ is the class of formulas of the form $\neg\varphi$ with $\varphi \in \Gamma$. For a set T of sentences, T^i and T^c denote its intuitionistic and classical deductive closures respectively.

1. A characterization of $\mathcal{H}$-theories

In [Bus], the intuitionistic theory of the class of T-normal Kripke structures is denoted $\mathcal{H}(T)$. Buss axiomatized $\mathcal{H}(T)$ by the universal closures of all formulas of the form $(\neg\theta)^\varphi$, where θ is semipositive (i.e. each implicational subformula of θ has an atomic antecedent) and $T \vdash_c \neg\theta$. In [AM] some necessary conditions for intuitionistic theories in the range of $\mathcal{H}$ together with examples of weak fragments of HA not of that form were given. Here we give an equivalent criterion for $\mathcal{H}$-theories which fails for a variety of intuitionistic theories, including HA.

Proposition 1.1 If $T^i = \mathcal{H}(S)$, then $T \equiv_c S$.

Proof Suppose $T^i = \mathcal{H}(S)$. Every classical model of S is a one-node Kripke model of $\mathcal{H}(S)$ and so of T^i. This will be a classical model of T. By classical completeness, $S \vdash_c T$.

Now let $S \vdash_c \varphi$. By eliminating all possible occurences of $\rightarrow$'s in $\neg\varphi$ in favor of $\vee$ and $\neg$ and then pushing all $\neg$'s as much as possible inside the resulting formula, we get a semipositive formula θ classically equivalent to $\neg\varphi$. From $S \vdash_c \neg\theta$ and the Buss axioms for $\mathcal{H}(S)$, we have $\mathcal{H}(S) \vdash_i (\neg\theta)^\perp$, where $(\neg\theta)^\perp$ is Friedman's translation of $\neg\theta$ by $\perp$ (see, e.g., [TD, III.5.2] for the definition of Friedman's translation). But $(\neg\theta)^\perp$ is intuitionistically equivalent to $\neg\theta$ and so $\mathcal{H}(S) \vdash_i \neg\theta$. Therefore $T^i \vdash_i \neg\theta$ and in particular $T \vdash_c \varphi$. Hence $T \vdash_c S$. □

Corollary 1.2 We have:

(i) $T^i \in \text{range}(\mathcal{H})$ iff $T^i = \mathcal{H}(T)$.

(ii) $\mathcal{H}(T)^c = T^c$.

(iii) $\mathcal{H}(\mathcal{H}(T)) = \mathcal{H}(T)$. More generally, $\mathcal{H}(\cup_{i\in I} T_i) = \mathcal{H}(\cup_{i\in I}\mathcal{H}(T_i))$.

Corollary 1.3 No fragment of HA extending $i\Pi_1$ belongs to the range of $\mathcal{H}$.

Proof Suppose $i\Pi_1 \subseteq T^i \subseteq HA$. Then $\mathcal{H}(T) \subseteq \mathcal{H}(PA)$. But as it was shown in [Bus], $i\Pi_1 \nsubseteq \mathcal{H}(PA)$. Therefore $i\Pi_1 \nsubseteq \mathcal{H}(T)$. Hence, $T^i \nsubseteq \mathcal{H}(T)$. So, by proposition 1.1, T^i does not belong to the range of $\mathcal{H}$. □

As the referee has pointed out, the Buss argument for $i\Pi_1 \nsubseteq \mathcal{H}(PA)$ easily extends to $i\Pi_1 \nsubseteq \mathcal{H}(T)$ for any consistent recursively axiomatized theory T$\supseteq I\Sigma_1$ (in the Buss proof, replace $I\Sigma_n$ by the first n axioms of T). Therefore, corollary 1.3 could be generalized as follows: No consistent recursively axiomatized extension of $i\Pi_1$ is in the range of $\mathcal{H}$.

A. Visser [V] has proved that if $\Re$ is a class of Kripke structures with respect to which HA is sound and complete, then $\Re$ is not closed under submodels. That is, removing some of the nodes of a structure in $\Re$ will not necessarily result in a member of $\Re$. He concludes that $HA \notin \text{range}(\mathcal{H})$.

Corollary 1.4 Suppose that T is a set of sentences such that $T^i \nsubseteq \mathcal{H}(T)$ or $\mathcal{H}(T) \subseteq T^i$. Let T'^i be a proper intuitionistic extension of T^i which is classically equivalent with it. Then T'^i is not in the range of $\mathcal{H}$.

Proof This is immediate from proposition 1.1. □

Examples 1.5 In each of the following cases, no intuitionistic proper extension of T^i classically equivalent with it is in the range of $\mathcal{H}$:

(i) Let $T^i = HA, T^c = PA$ for which both disjuncts in the preceding corollary are satisfied. We observe, however, that any such extension is complete with respect to its

PA-normal Kripke models (this can be seen as in [W1, thm. 8.1]).

(ii) Let $T^i = i\Pi_1, T^c = I\Pi_1$ for which we observe that only the first disjunct is satisfied. For, on one hand, as recalled earliear from [Bus], $i\Pi_1 \not\subseteq \mathcal{H}(PA)$ and so $i\Pi_1 \not\subseteq \mathcal{H}(I\Pi_1)$. On the other hand, by another result of Buss [Bus, Thm. 7], $i\Sigma_1 \subseteq \mathcal{H}(I\Sigma_1)$ and so by the clasical equivalence $I\Sigma_1 = I\Pi_1$ and Wehmeier's result (see [We, Coro. 4]) $i\Pi_1 \nvdash i\Sigma_1$, we get the negation of the second disjunct.

(iii) Let $T^i = lop, T^c = Lop$. As observed in [AM], $\mathcal{H}(Lop) = lop$.

As it was observed in [AM] (in the proof of 2.1 (iv)), all $\mathcal{H}$-theories are closed under Friedman's translation. So the following proposition implies that for no fragment T^i of HA is $T^i + MP_{\Delta_0}$ an $\mathcal{H}$-theory.

Proposition 1.6 For any fragment T^i of HA, $T^i + MP_{\Delta_0}$ is not closed under Friedman's translation.

Proof Let $\sigma \in \Pi_1$ be Godel's sentence ($PA \nvdash \sigma$, $\mathbb{N} \models \sigma$). Assume $\tau \equiv_c \neg\sigma \in \Sigma_1$ and let M be a classical model of $PA + \tau$. Let $\mathcal{K}$ be the two-node Kripke model obtained by putting M above $\mathbb{N}$ (the result of applying Smorynski's prime operation $'$ to M). The lower node does not force τ, since $\tau \in \Sigma_1$, $\mathcal{K} \Vdash HA$ and $\mathbb{N} \not\models \tau$. On the other hand, the upper node forces τ. This shows $\mathcal{K} \nVdash \neg\neg\tau \to \tau$ and so $\mathcal{K} \nVdash MP_{\Delta_0}$. Next apply Smorynski's Σ' operation to the one node models M and $\mathbb{N}$ to get the three-node model $\mathcal{K}_1$. As observed by Smorynski in [S, 5.6.21], if each node of a Kripke model of $i\Delta_0$ has an accessible terminal one with no new elements, then the Kripke model forces MP_{Δ_0}. Note that pruning $\mathcal{K}_1$ by σ gives $\mathcal{K}$. Therefore by the first pruning lemma of [DMKV], we conclude that $T^i + MP_{\Delta_0}$ is not closed under Friedman's translation. $\square$

We end this section with certain Π_2-conservativity results for fragments of HA augmented by restrictions of MP. Before that, we collect some related facts:

Fact 1.7 (i) Iop is not $\forall_2$-conservative over iop, see [AM, example 2.5].

(ii) $I\Sigma_1$ (respectively $I\Pi_1$) is (respectively is not) Π_2-conservative over $i\Sigma_1$ (respectively over $i\Pi_1$), see [W2].

(iii) $I\Pi_2$ is Π_2-conservative over $i\Pi_2$, see [Bur, Coro. 2.6].

Proposition 1.8 (i) Iop is $\forall_2$-conservative over $iop + MP_{\text{open}}$.

(ii) $I\Pi_1 \equiv I\Sigma_1$ is Π_2-conservative over $i\Pi_1 + MP_{\Delta_0} \equiv \neg\neg i\Sigma_1 + MP_{\Delta_0} \equiv \neg\neg i\Pi_1 + MP_{\Delta_0}$.

(iii) $I\Pi_2 \equiv I\Sigma_2$ is Π_2-conservative over $i\neg\neg\Sigma_2 + MP_{\Delta_0}$ (here $i\neg\neg\Sigma_2$ is the intuitionistic theory axiomatized by $i\Delta_0 + \{I_x\varphi : \varphi \in \neg\neg\Sigma_2\}$).

Proof (i) Straightforward.

(ii) By [W2, Coro. 3.1], $i\Sigma_1 + MP_{\Delta_0} \equiv i\Pi_1 + MP_{\Delta_0}$, so fact 1.7(ii) shows Π_2-conservativity of $I\Pi_1$ over $i\Pi_1 + MP_{\Delta_0}$. To get the equivalences, use $\neg\neg I_x\psi \vdash_{MP_{\Delta_0}} I_x\psi$ for $\psi \in \Pi_1 \cup \Sigma_1$ and iPA^- being $\forall_2$-axiomatized.

(iii) Using Fact 1.7(iii), it suffices to show $i\neg\neg\Sigma_2 + MP_{\Delta_0} \equiv i\Pi_2 + MP_{\Delta_0}$. For this purpose, we can use a modified version of the proof in [W2, Thm. 4]. For, in the presence of MP_{Δ_0} and PEM_{Δ_0}, if $\varphi \in \Delta_0$, then $\neg\exists x\forall y\varphi \equiv_i \forall x\exists y\neg\varphi$ and $\overline{\exists x\forall y\varphi} \equiv_i \neg\neg\exists x\forall y\varphi$. The latter implies $i\neg\neg\Sigma_2 \vdash \overline{\psi}$ whenever $I\Sigma_2 \vdash \psi$. Also, as it is well known, in $I\Sigma_1$, the class of Π_1-formulas is classically closed under bounded quantification, see, e.g., [HP, P.64]. □

2. Once-branching Kripke models of $HA + MP$ are PA-normal

It was proved in [W1] that any finite depth or ω-framed Kripke model of HA is PA-normal. In this section, we show PA-normality of linear and certain infinite-depth nonlinear Kripke models of $HA + MP$.

Lemma 2.1 If $\mathcal{K} \Vdash PEM_{\text{atomic}} + MP$ is linear, then $\mathcal{K} \Vdash PEM$ (and so for any node α and formula φ, $\alpha \Vdash \varphi$ if and only if $\mathcal{M}_\alpha \models \varphi$).

Proof To show $\mathcal{K} \Vdash \forall\overline{x}(\varphi(\overline{x}) \vee \neg\varphi(\overline{x}))$, do induction on the complexity of φ. Using [AM, 1.1(i)], it remains to check the case of $\exists$ in the induction step, but this is an immediate consequence of $\mathcal{K} \Vdash MP$ being linear. □

Corollary 2.2 If $\mathcal{K} \Vdash T^i + PEM_{\text{atomic}} + MP$ is linear, then $\mathcal{K}$ is T-normal.

Proposition 2.3 Any Kripke model of $HA + MP$ all whose possible branchings occur at its root, is PA-normal.

Proof It is clear from 2.2 that any node other than the root is PA. If the root were not PA, then it woud be redundant (see [W1]). Therefore, it would force PEM which by the assumption on the frame and 2.1 is forced at all nodes other than it. Hence we get the contradiction that the root does not force HA. □

As observed in [AM] (in the proof of 2.3(ii)), there are two-node non Iop-normal Kripke models of *iop*. Also, an ω-framed Kripke model of *iop* is constructed in [MM] none of whose worlds satisfying Iop. Below we show that the former (resp. latter) can not happen for end-extension models (resp. in the presence of MP_{open}).

Proposition 2.4 (i) Any reversely well founded end-extension Kripke model of *iop* is Iop-normal.

(ii) Any linear Kripke model of $iop + MP_{\text{open}}$ is Iop-normal.

Proof(i) Let $\mathcal{K} \Vdash iop$ be reversely well founded and an end-extension Kripke model. By [AM], proof of 1.4, it suffices to show that $\mathcal{K} \Vdash lop$. Let α be a node of $\mathcal{K}$ and $\alpha \Vdash \exists x\varphi(x, a)$, φ open, $a \in M_\alpha$. Let $\beta \geq \alpha$ be terminal. Then $M_\beta \models Iop \equiv_c Lop$ and so $\beta \Vdash lop$. Then for some $b \in M_\beta$, $\beta \Vdash \varphi(b, a) \wedge \forall x < b\neg\varphi(x, a)$. We then have $b \in M_\alpha$, since $\alpha \Vdash \exists x\varphi(x, a)$ and $M_\alpha \subseteq_e M_\beta$.

(ii) Using a suitable variant of theorem 6.3 in [W1], the proof is routine.□

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References

[AM] M. Ardeshir and Mojtaba Moniri, Intuitionistic Open Induction and Least Number Principle and the Buss Operator, Notre Dame J. Formal Logic, 39 (1998) 212-220.

[Bur] W. Burr, Fragments of Heyting Arithmetic, to appear in J. Symbolic Logic.

[Bus] S. Buss, Intuitionistic Validity in T-normal Kripke Structures, Ann. Pure Appl. Logic, 59 (1993) 159-173.

[DMKV] D. Van Dalen et al, Finite Kripke Models of HA Are Locally PA, Notre Dame J. Formal Logic, 27 (1986) 528-532.

[HP] P. Hajek and P. Pudlak, Metamathematics of First-order Arithmetic, Springer, 1993.

[MM] Morteza Moniri and Mojtaba Moniri, Some Weak Fragments of HA and Certain Closure Properties, to appear.

[S] C. Smorynski, Applications of Kripke Models, in: Metamathematical Investigations of Intuitionistic Arithmetic and Analysis, edited by A.S. Troelstra, Springer, 1973.

[TD] A.S. Troelstra and D. van Dalen, Constructivism in Mathematics, v.I, North-Holland, 1988.

[V] A. Visser, Submodels of Kripke Models, Logic Group Preprint Series, No. 189, Dep't. of Philosophy, Utrecht university, 1998.

[W1] K.F. Wehmeier, Classical and Intuitionistic Models of Arithmetic, Notre Dame J. Formal Logic, 37 (1996), 452-461.

[W2] K.F. Wehmeier, Fragments of HA Based on Σ_1-Induction, Arch. Math. Logic, 37 (1997) 37-49. n,

[W3] K. F. Wehmeier, PhD dissertation, Muenster, 1996.