Sparse H-colourable graphs of large girth

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(Joint work with Professor Xuding Zhu)

Definition 1. A graph G is called k-colorable if there is a function $c : V(G) \to \{1, \ldots, k\}$, such that $c(u) \neq c(v)$ whenever u and v are adjacent in G. The minimum number k such that G is k-colorable is called the chromatic number of G and it is denoted by $\chi(G)$. ♠

Definition 2 The *girth* $g(G)$ of a graph G is the length of a shortest cycle in G. ♠

Problem 1. (1947) *Are there graphs of arbitrary large girth that have arbitrary large chromatic number?*

Kneser graphs:

Suppose $m \geq 2n$ are positive integers. We denote by $[m]$ the set $\{1, 2, \cdots, m\}$, and denote by $\binom{[m]}{n}$ the collection of all n-subsets of $[m]$. The *Kneser graph* $\mathrm{KG}(m, n)$ has vertex set $\binom{[m]}{n}$, in which $A \sim B$ if and only if $A \cap B = \emptyset$.

Theorem A. (Lovász 1978) *Suppose $m \geq 2n$ are positive integers. Then,*

$$\chi(\mathrm{KG}(m, n)) = m - 2n + 2.$$

$G(n,p)$:

Theorem B (Erdős 1959) *For all k, l there exists a graph G with $g(G) > l$ and $\chi(G) > k$.*

Proof. Let $G \sim G(n,p)$ and X be the number of cycles of sizes at most l. Then,

$$E(X) = \sum_{i=3}^{l} \frac{(n)_i}{2i} p^i \leq o(n),$$

Consequently,

$$P(X \geq \frac{n}{2}) = o(1). \tag{1}$$

Also,

$$P(\alpha(G) \geq \lceil \frac{3ln(n)}{p} \rceil) = o(1). \tag{2}$$

On the other hand,

$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \tag{3}$$

■

Definition 3 A graph G is said to be *uniquely* k-colorable if G is k-colorable and every k-coloring of G induces the same k-partition of $V(G)$. ♠

Problem 2 *Are there uniquely colorable graphs of arbitrary large girth that have arbitrary large chromatic number?*

Problem 3 *Are there uniquely colorable graphs of arbitrary large girth and bounded maximum degree that have arbitrary large chromatic number?*

$G(n, K_k, p)$**:**

Lemma 1. *If n is sufficiently large, then there exists a graph G in $G(n, K_k, p)$ with the following properties:*

1. *For any edge ij of K_k and any $U \subset V_i, W \subset V_j$, of size $|U| = \lceil \frac{k^{-3}n}{2} \rceil$ and $|W| = \lceil \frac{(k-1)n}{k} \rceil$, there are at least $\frac{k^7 n}{4}$ edges between U and W in G.*
2. *For any edge ij of K_k and any $U \subset V_i, W \subset V_j$, of size $|U| = |W| = \lceil \frac{n}{40k} \rceil$, there is at least one edge between U and W in G.*
3. *For any edge ij of K_k and any $U \subset V_i, W \subset V_j$, of size $k \leq |W| = k|U| \leq \frac{n}{40}$, there are less than $\frac{|U|k^{10}}{2}$ edges between U and W in G.*
4. *Let $g := \lfloor \frac{1}{11} \frac{logn}{logk} \rfloor$ and define $C := \{v : v$ is a vertex contained in a cycle in G of length at most $g - 1\}$. Then $|C| \leq \frac{n}{4k}$.*
5. *Let $Y := \{v : v$ has degree in G larger than $5k^{13}\}$. Then $|Y| \leq \frac{n}{4k} - 1$.*

Definition 4. Let G and H be two graphs. A *homomorphism* from a graph G to a graph H is a map $f : V(G) \longrightarrow V(H)$, such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. ♠

Observation: Let G and H be two graphs. If there exists a homomorphism $f : G \longrightarrow H$ then

- $\omega(G) \leq \omega(H)$
- $\chi(G) \leq \chi(H)$
- $oddg(G) \geq oddg(H)$

Theorem 1. (Hajiabolhassan and Zhu, 2004) *Let k and l be positive integers. For every graph F on at most k vertices there exists a graph G together with a surjective homomorphism $c : G \to F$ with the following properties:*

1. $g(G) > l$ *and* $\Delta(G) \leq 5k^{13}$*;*
2. *For every graph H with at most k vertices, there exists a homomorphism $g : G \to H$ if and only if there exists a homomorphism $f : F \to H$.*

$G(n, F, p)$**:**

Problem 4 *Is true that any cubic graphs with sufficiently large girth is homomorphic to C_5?*

Thanks for your attention!