PARTITION MATROID HYPERGRAPHS ARE NICE

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1. Introduction

For a hypergraph $\mathcal{H}$, the Kneser hypergraph $\mathrm{KG}(\mathcal{H})$ is a graph with vertex set $E(\mathcal{H})$ and two vertices are adjacent if corresponding edges are disjoint. Alternation number of a hypergraph $\mathcal{H}$ is a combinatorial parameters defined by the first and second authors [1]. For each $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \{+,-,0\}^n$, an *alternating subsequence of* $\boldsymbol{x}$ is a sequence $x_{i_1}, \ldots, x_{i_k}$ of nonzero terms of $\boldsymbol{x}$ ($i_1 < \cdots < i_k$) such that $x_{i_j} \neq x_{i_{j+1}}$ for each $j \in [k-1]$. The length of the longest alternating subsequence of $\boldsymbol{x}$ is denoted by $\mathrm{alt}(\boldsymbol{x})$. Let $\mathcal{H}$ be a hypergraph with n vertices. For a bijection $\sigma[n] \longrightarrow V(\mathcal{H})$, we define

$$\mathrm{alt}_\sigma(\mathcal{H}) = \max_{\boldsymbol{x}\in\{+,-,0\}^n} \left\{\mathrm{alt}(\boldsymbol{x}) : \text{ neither } \sigma(\boldsymbol{x}^+) \text{ nor } \sigma(\boldsymbol{x}^-) \text{ contains any edge of } \mathcal{H}\right\},$$

where

$$\boldsymbol{x}^+ = \{i : x_i = +\} \quad \text{and} \quad \boldsymbol{x}^- = \{i : x_i = -\}.$$

The *alternation number* of $\mathcal{H}$, $\mathrm{alt}(\mathcal{H})$ is defined as follows.

$$\mathrm{alt}(\mathcal{H}) = \min\left\{\mathrm{alt}(\mathcal{H}) : \sigma : [n] \longrightarrow V(\mathcal{H}) \text{ is a bijection}\right\}.$$

Next theorem is proved in [1].

Theorem A. [1] Let $\mathcal{H}$ is a hypergraph with n vertices. Then

$$\chi(\mathrm{KG}(\mathcal{H})) \geq n - \mathrm{alt}(\mathcal{H}).$$

A nonempty hypergraph $\mathcal{H}$ with no singleton is called *nice,* if there is a bijection $\sigma : [n] \longrightarrow V(\mathcal{H})$ for which

- $\chi(\mathrm{KG}(\mathcal{H})) = n - \mathrm{alt}_\sigma(\mathcal{H})$.
- for each $\boldsymbol{x} \in \{+,-,0\}^n$, if $\mathrm{alt}(\boldsymbol{x}) \geq \mathrm{alt}_\sigma(\mathcal{H})$ and $|\boldsymbol{x}| > \mathrm{alt}_\sigma(\mathcal{H})$, then at least one of $\sigma(\boldsymbol{x}^+)$ and $\sigma(\boldsymbol{x}^-)$ contains some edge of $\mathcal{H}$.

The notation $|\boldsymbol{x}|$ stands for the cardinality of $\boldsymbol{x}$ which is the number of nonzero terms of $\boldsymbol{x}$.

As a main goal of this note, we are going to prove next theorem.

Theorem 1. *Let $P_1, \ldots, P_m$ be a partition of $[n]$ and let $k, s_1, \ldots, s_m$ be positive integers. Assume that $|P_i| \neq 2s_i$ for every i and that $k \geq 2$. Let $\mathcal{H}$ be the hypergraph defined by*

$$\begin{array}{rcl} V(\mathcal{H}) & = & [n] \\ E(\mathcal{H}) & = & \left\{A \in \binom{[n]}{k} : |A \cap P_i| \leq s_i \textit{ for every } i\right\}. \end{array}$$ [1]

If $\mathcal{H}$ has at least two disjoint edges, then it is nice.

[1] The edges of such a hypergraph are the bases of a truncation of a partition matroid.

2. Proof of Theorem 1

Throughout this note, k, r, m, and n are positive integers where $r \geq 2$ and $k \geq 1$. Furthermore, suppose that $\pi = (P_1, P_2, \ldots, P_m)$ is a partition of $[n]$ and $\vec{s} = (s_1, s_2, \ldots, s_m)$ is a positive integer vector. The *partition matroid Kneser graph* $\mathrm{PM}(\pi; \vec{s}; k)$ is a graph with the vertex set

$$V(\pi; \vec{s}; k) = \{A : \ A \subseteq [n],\ |A| = k, \forall\ 1 \leq i \leq m;\ |A \cap P_i| \leq s_i\}$$

and two vertices are adjacent if the corresponding subsets are disjoint, i.e., for $A, B \in V(\pi; \vec{s}; k)$, $\{A, B\}$ is an edge if $A \cap B = \varnothing$. Note that $V(\pi; \vec{s}; k)$ can be considered as the edge set of a hypergraph $\mathcal{H} = \mathcal{H}(\pi; \vec{s}; k)$ with the vertex set $[n]$. The hypergraph $\mathcal{H}(\pi; \vec{s}; k)$ is called *the partition matroid hypergraph*. Clearly, $\mathrm{PM}(\pi; \vec{s}; k) \cong \mathrm{KG}(\mathcal{H}(\pi; \vec{s}; k))$

The partition matroid graphs are introduced in [1] as a generalization of a family of graph named *the fractional multiple of the complete graphs* introduced in [4]. However, it should be mentioned that the partition matroid graphs are called the multiple Kneser graphs in [1]. For positive integers m, r, and t, the fractional multiple of the complete graph $K_m^{r,t}$ has the vertex set consisting of all r-independent sets in a disjoint union of t copies of K_m which two vertices are adjacent if their corresponding subsets are disjoint. Clearly, for $n = tm$, $\pi = (P_1, P_2, \ldots, P_m)$ where $|P_1| = \cdots = |P_m| = t$, and $\vec{s} = (1, 1, \ldots, 1)$, the graphs $\mathrm{PM}(\pi; \vec{s}; r)$ and $K_m^{r,t}$ are isomorphic. Note that for $m = 1$, $K_m^{r,t} \cong \mathrm{KG}(t, r)$. The chromatic number of the fractional multiple of the complete graph $K_m^{r,t}$ was studied in [2, 3] and was determined for even m. However, it was conjectures that for odd $m \geq 3$, $\chi(K_m^{r,t}) = m(t - r + 1)$. The chromatic number of partition matroid graphs is determined in [1] which gives an affirmative answer to the aforementioned conjecture. Moreover, there is a colorful type theorem presented in [1] which implies that the equality of chromatic number and circular chromatic number for any partition matroid graph with even chromatic number.

In what follows, we prove that the hypergraph $\mathcal{H}(\pi; \vec{s}; k)$ with at least two disjoint edges is nice provided that $|P_i| \neq 2s_i$, for each $1 \leq i \leq m$. Although, the chromatic number of $\mathrm{PM}(\pi; \vec{s}; k) \cong \mathrm{KG}(\mathcal{H}(\pi; \vec{s}; k))$ is computed in [1], but we recompute the chromatic number with a slightly different formula.

Let $\pi = (P_1, P_2, \ldots, P_m)$ be a partition of $[n]$ and $\vec{s} = (s_1, s_2, \ldots, s_m)$ be a positive integer. One can readily check the following properties of $\mathcal{H}(\pi; \vec{s}; k)$.

- The hypergraph $\mathcal{H}(\pi; \vec{s}; k)$ has nonempty edge set if and only if $\sum_{i=1}^{m} f_j \geq k$,
- $M_\pi < n$ if and only if $\sum_j f_j \geq 2k - 1$,
- there are at least two disjoint edges in $\mathcal{H}(\pi; \vec{s}; k)$ if and only if $\sum_j f_j \geq 2k$.

Define

$$M_\pi = \max_{J \subseteq [m]} \left\{ 2k - 2 + \sum_{j \in J} (|P_j| - f_j) \ : \ \sum_{j \in J} f_j \leq 2k - 2 \right\}$$

where $f_i = \min\{2s_i, |P_i|\}$.

In the next lemma, we present an upper bound for the chromatic number of $\mathrm{PM}(\pi; \vec{s}; k)$.

Lemma 1. *Let k, m, and n be positive integers and $\vec{s} = (s_1, s_2, \ldots, s_m)$ be a positive integer vector. Also, let $\pi = (P_1, P_2, \ldots, P_m)$ be a partition of $[n]$ such that $\sum_{i=1}^{m} f_j \geq k$. Then $\chi(\mathrm{PM}(\pi; \vec{s}; k)) \leq \max\{1, n - M_\pi\}$*

Proof. Since $\sum_{i=1}^{m} f_j \geq k$ the vertex set of $\mathrm{PM}(\pi; \vec{s}; k))$ is not empty. If $\mathrm{PM}(\pi; \vec{s}; k))$ has no edge, then $\chi(\mathrm{PM}(\pi; \vec{s}; k)) = 1$ which verifies the desired inequality. Now, we may assume that $\mathrm{PM}(\pi; \vec{s}; k)$

has some edges. It implies that $\sum_{j=1}^{m} f_j \geq 2k$ and consequently, $M_\pi < n$. Without loss of generality, suppose that t is the largest positive integer such that the value of M_π is attained. Moreover, suppose that M_π is obtained by $P_m, P_{m-1}, \ldots, P_{m-t+1}$. In other words, we have $M_\pi = 2k - 2 + \sum_{j=m-t+1}^{m} (|P_j| - f_j)$ and $\sum_{j=m-t+1}^{m} f_j \leq 2k-2$. If $m - t = 0$, then $\sum_{j=1}^{m} f_j \leq 2k-2$ which is impossible. Hence, suppose $m - t \geq 1$. By the definition of M_π and because of the maximality of t, for any $1 \leq i \leq m-t$, one can see that $2k - 1 - \sum_{j=m-t+1}^{m} f_j \leq f_j$. Let L be a subset of P_{m-t} of size $2k - 1 - \sum_{j=m-t+1}^{m} f_j$. Note that $|P_{m-t}| \geq f_{m-t} \geq 2k - 1 - \sum_{j=m-t+1}^{m} f_j$. Set $T = L \cup \left(\bigcup_{j=m-t+1}^{m} P_j \right)$. Now we present a proper coloring for $\mathrm{PM}(\pi; \vec{s}; k)$ using $n - M_\pi$ colors. To this end, it will be shown that all the vertices of $\mathrm{PM}(\pi; \vec{s}; k)$ which are subsets of T form an independent set. On the contrary, suppose that $A_1, A_2 \in V(\mathrm{PM}(\pi; \vec{s}; k))$ form an edge in $\mathrm{PM}(\pi; \vec{s}; k)$ where $A_1, A_2 \subseteq T$. According to the definition of $\mathrm{PM}(\pi; \vec{s}; k)$, we have $|A_1 \cap P_j| + |A_2 \cap P_j| \leq f_j$ for any $m - t + 1 \leq j \leq m$ and also $|A_1 \cap L| + |A_2 \cap L| \leq |L|$. Thus

$$
\begin{aligned}
2k = |\bigcup_{i=1}^{r} A_i| &= (\sum_{i=1}^{2} |A_i \cap L|) + \sum_{j=m-t+1}^{m} \sum_{i=1}^{2} |A_i \cap P_j| \\
&\leq |L| + \sum_{j=m-t+1}^{m} f_\pi(P_j) \\
&= (2k-1) - \sum_{j=m-t+1}^{m} f_j + \sum_{j=m-t+1}^{m} f_j
\end{aligned}
$$

which is a contradiction.

Note that the size of $D = (\bigcup_{j=1}^{m-t} P_j) \setminus L$ is $n - M_\pi - 1$ and $T \cup D = [n]$. As mentioned, all the vertices of $\mathrm{PM}(\pi; \vec{s}; k)$ which are subsets of T form an independent set; and therefore, we can assign a color to all of them. Since every other vertex A has a nonempty intersection with D, we define the color of this vertex to be the minimum integer j such that $A \cap D \neq \varnothing$. One can see that this coloring is a proper coloring using at most $|D| + 1 = n - M_\pi$ colors. $\square$

Lemma 2. *Let k, m, and n be positive integers and $\vec{s} = (s_1, s_2, \ldots, s_m)$ be a positive integer vector. Also, assume that $\pi = (P_1, P_2, \ldots, P_m)$ is a partition of $[n]$, where each P_i is a subset of $|P_i|$ consecutive numbers. Let $I : [n] \longrightarrow [n] = \bigcup_i P_i$ be the identity bijection. If $\sum_{j=1}^{m} f_j \geq 2k$, then, for any $\boldsymbol{z} \in \{+1, 0, -1\}^n$ with $alt(\boldsymbol{z}) \geq M_\pi$ and $|\boldsymbol{z}| > M_\pi$, at least one of $I(\boldsymbol{z}^+) = \boldsymbol{z}^+$ or $I(\boldsymbol{z}^-) = \boldsymbol{z}^-$ contains an edge of $\mathcal{H}(\pi; \vec{s}; k)$.*

Proof. Note that since $\sum_{j=1}^{m} f_j \geq 2k$, we have $M_\pi \leq n - 1$. Clearly, it is sufficient to show that for a $\boldsymbol{z} \in \{+1, 0, -1\}^n \setminus \{(0, 0, \ldots, 0)\}$ with $\mathrm{alt}(\boldsymbol{z}) \geq M_\pi$ and $|\boldsymbol{z}| = M_\pi + 1$ at least one of $\boldsymbol{z}^+$ or $\boldsymbol{z}^-$ contains some edge. Let $\boldsymbol{x} \in \{+1, 0, -1\}^n$ such that $\boldsymbol{x} \subseteq \boldsymbol{z}$ and $\mathrm{alt}(\boldsymbol{x}) = |\boldsymbol{x}| = M_\pi$. If $\boldsymbol{x}^+$ or $\boldsymbol{x}^-$ contains some edge of $\mathcal{H}(\pi; \vec{s}; k)$, then there is nothing to prove. Therefore, we may assume that neither $\boldsymbol{x}^+$ nor $\boldsymbol{x}^-$ contains any edge of $\mathcal{H}(\pi; \vec{s}; k)$. Also, let $l_0 \in \{1, 2, \ldots, n\}$ be the unique integer that $z_{l_0} \neq 0$ and $x_{l_0} = 0$. Note that since
$alt(\boldsymbol{x}) = |\boldsymbol{x}|$, any two consecutive nonzero terms of $\boldsymbol{x}$ have different signs. Therefore, the sequence of nonzero terms of $\boldsymbol{x}$ is the longest alternating subsequence of $\boldsymbol{x}$. Define $\mathrm{alt}(\boldsymbol{x}, P_j)$ to be

the length of the longest alternating subsequence of $\boldsymbol{x}$ lying in P_j, i.e., the number of nonzero terms of $\boldsymbol{x}$ in P_j. Also, let $\mathrm{alt}^+(\boldsymbol{x}, P_j)$ (resp. $\mathrm{alt}^-(\boldsymbol{x}, P_j)$) be the number of positive (resp. negative) entries of this alternating subsequence of $\boldsymbol{x}$ in P_j. Set $K^+ = \sum_{j=1}^{m} \min\{s_j, \mathrm{alt}^+(\boldsymbol{x}, P_j)\}$ and $K^- = \sum_{j=1}^{m} \min\{s_j, \mathrm{alt}^-(\boldsymbol{x}, P_j)\}$. Clearly, we have $\mathrm{alt}(\boldsymbol{x}, P_j) = \mathrm{alt}^+(\boldsymbol{x}, P_j) + \mathrm{alt}^-(\boldsymbol{x}, P_j)$ and $\max\{K^+, K^-\} \leq k - 1$. So

$$I(\boldsymbol{x}) = \sum_{j=1}^{m} \min\{s_j, \mathrm{alt}^+(\boldsymbol{x}, P_j)\} + \sum_{j=1}^{m} \min\{s_j, \mathrm{alt}^-(\boldsymbol{x}, P_j)\} \leq 2k - 2.$$

Also one can see that $I(\boldsymbol{x}) = \sum_{j=1}^{m} \min\{f_j,\ \mathrm{alt}(\boldsymbol{x}, P_j)\}$. For convenience, set

$$J_1 = \{j :\ \mathrm{alt}_I(\boldsymbol{x}, P_j) \geq f_j\}$$

and

$$J_2 = \{j :\ 0 < \mathrm{alt}(\boldsymbol{x}, P_j) < f_j\}.$$

Therefore, we have

$$I(\boldsymbol{x}) = \sum_{j=1}^{m} \min\{f_j,\ \mathrm{alt}(\boldsymbol{x}, P_j)\} = \sum_{j \in J_1} f_j + \sum_{j \in J_2} \mathrm{alt}(\boldsymbol{x}, P_j) \leq 2k - 2 \tag{1}$$

This means that $\sum_{j \in J_1} f_j \leq 2k - 2$, and according to the definition of M_π, we have

$$2k - 2 + \sum_{j \in J_1} (|P_j| - f_j) \leq M_\pi. \tag{2}$$

Hence,

$$2k - 2 - M_\pi + \sum_{J \in J_1} |P_j| \leq \sum_{J \in J_1} f_j. \tag{3}$$

Now we have

$$2k - 2 - M_\pi + \sum_{J \in J_1} |P_j| + \sum_{J \in J_2} \mathrm{alt}(\boldsymbol{x}, P_j) \leq I(\boldsymbol{x}) \leq 2k - 2. \tag{4}$$

Thus

$$M = \mathrm{alt}(\boldsymbol{x}) \leq \sum_{j \in J_1} |P_j| + \sum_{j \in J_2} \mathrm{alt}(\boldsymbol{x}, P_j) \leq M_\pi. \tag{5}$$

Since $\mathrm{alt}(\boldsymbol{x}) = M_\pi$, we have $K^+ = k - 1$ and $K^- = k - 1$ and moreover we have the equality in the inequalities 1, 2, 3, 4, and 5. Having equality in inequalities 2 and 5 implies that

$$\sum_{j \in J_2} \mathrm{alt}(\boldsymbol{x}, P_j) = 2k - 2 - \sum_{j \in J_1} f_j. \tag{6}$$

For a contradiction, suppose that there is some $l \in J_2$ such that $f_l < |P_l|$ and $\sum_{i\in J_1} f_\pi(P_i) + f_l \leq 2k-2$. It implies that

$$\begin{array}{rcl} M_\pi & \geq & 2k-2+\sum_{j\in J_1\cup\{l\}}(|P_j|-f_j) \\ & = & M_\pi + (|P_l|-f_l) \\ & > & M_\pi \end{array}$$

which is impossible. Therefore, for any $l \in J_1$ with $f_l < |P_l|$, we have

$$2k-2-\sum_{i\in J_1} f_i < f_l. \tag{7}$$

Claim. For any $l \in J_2$ such that $f_l < |P_l|$, we have $2k-2-\sum_{j\in J_1} f_j < f_l - 1$.

Suppose, contrary to the claim, that there is an $l \in J_2$ such that $f_l < |P_l|$ and $f_l - 1 \leq 2k-2-\sum_{j\in J_1} f_j$. Therefore, in view of Inequality 7, we have $2k-2-\sum_{j\in J_1} f_j = f_l - 1$. Note that $f_l < |P_l|$ implies that $f_l = 2s_l$ and consequently, $2k-2-\sum_{j\in J_1} f_j = f_l - 1 = 2s_l - 1$. Therefore, there should be some $j_0 \in J_1$ such that f_{j_0} is an odd integer which implies that $f_{j_0} = |P_{j_0}| \leq 2s_{j_0} - 1$. Define $J' = (J_1 \setminus \{j_0\}) \cup \{l\}$. Clearly, we have $\sum_{j\in J'} f_j \leq 2k-2$. Therefore,

$$\begin{array}{rcl} M_\pi & \geq & 2k-2+\sum_{j\in J'}(|P_j|-f_j) \\ & = & M_\pi - (|P_{j_0}|-f_{j_0}) + (|P_l|-f_l) \\ & > & M_\pi, \end{array}$$

a contradiction. In view of prior discussion, for any $l \in J_2$, we have one of the following condotions

I) $f_l = |P_l| \leq 2s_l - 1$,
II) $f_j < |P_j|$ and $2k-2-\sum_{j\in J_1} f_j \leq f_l - 2 \leq 2s_l - 2$.

Therefore, in view if inequality 6, for any $l \in [m] \setminus J_1$, we have

$$\mathrm{alt}(\boldsymbol{x}, P_l) \leq \min\left\{|P_l|-1, 2k-2-\sum_{j\in J_1} f_j,\right\} \leq 2s_l - 2.$$

Consequently, we have $\mathrm{alt}^+(\boldsymbol{x}, P_l) < s_l$ and $\mathrm{alt}^-(\boldsymbol{x}, P_l) < s_l$. Having equality in 5 implies that $\mathrm{alt}(\boldsymbol{x}, P_j) = |P_j|$ for each $j \in J_1$. Thus $\boldsymbol{z}$ and $\boldsymbol{x}$ have the same entriy in the i-th coordinate for any $i \in \bigcup_{j\in J_1} P_j$, i.e., $l_0 \in [n] \setminus \bigcup_{j\in J_1} P_j$. Without loss of generality, assume that $z_{l_0} = +1$ and $l_0 \in P_{j_0}$. Since $l_0 \in [m] \setminus J_1$ we have $\mathrm{alt}(\boldsymbol{x}, P_{l_0}) \leq 2s_{l_0} - 2$. This clearly implies that $\mathrm{alt}^+(\boldsymbol{x}, P_{l_0}) \leq s_{l_0} - 1$ and $\mathrm{alt}^-(\boldsymbol{x}, P_{l_0}) \leq s_{l_0} - 1$. Consequently,

$$\sum_{j=1}^{m} \min\{s_j, \mathrm{alt}^+(\boldsymbol{y}, P_j)\} = \sum_{j=1}^{m} \min\{s_j, \mathrm{alt}^+(\boldsymbol{x}, P_j)\} + 1 = k.$$

Therefore, $\boldsymbol{y}^+$ has some edges of $\mathcal{H}(\pi; \vec{s}; k)$ which completes the proof. □

2.1. **Proof of Theorem 1.** First note that since $k \geq 2$, the hypergraph $\mathcal{H}(\pi; \vec{s}; k)$ has no singleton. Also, since $\mathcal{H}(\pi; \vec{s}; k)$ has at least two disjoint edges, we have $\sum_{j=1}^{m} f_j \geq 2k$. This implies that $M_\pi < n$ and consequently, by Lemma 1, we have $\chi(\mathrm{PM}(\pi; \vec{s}; k)) \leq n - M_\pi$. On the other hand, Lemma 2 immediately yields $\mathrm{alt}_I(\mathcal{H}(\pi; \vec{s}; k)) \leq M_\pi$. Therefore, in view of Theorem A and Lemma 1, we have $n - M_\pi \leq n - \mathrm{alt}_I(\mathcal{H}(\pi; \vec{s}; k)) \leq \chi(\mathrm{PM}(\pi; \vec{s}; k)) \leq n - M_\pi$. Consequently, $\mathrm{alt}_I(\mathcal{H}(\pi; \vec{s}; k)) = M_\pi$. Now, by Lemma 2, it is clear that $\mathcal{H}(\pi; \vec{s}; k)$ is a nice hypergraph. □

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