On Biclique Covering

Hossein Hajiabolhassan and Farokhlagha Moazami

Department of Mathematical Sciences
Shahid Beheshti University, G.C.
Tehran, Iran

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Definition

A biclique partition of a graph G is a collection of bicliques (complete bipartite subgraphs) of G such that each edge of G is in exactly one of the bicliques. The number of bicliques in a minimum biclique partition of G is called the biclique partition number of G and denoted by $bp(G)$.

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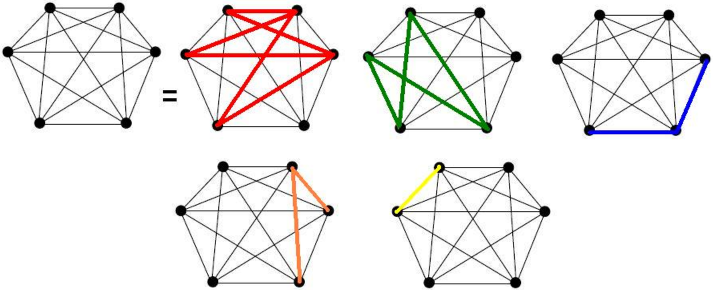

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- Assume that $H_1, \ldots, H_m$ disjointly cover all edges of K_n, and $V(H_k) = (X_k.Y_k)$.
- Assign an $n \times n$ matrix A_k to each graph H_k such that

$$a_{ij}^k = \begin{cases} 1 & \text{if } i \in X_k \text{ and } j \in Y_k, \\ 0 & \text{otherwise.} \end{cases}$$

- The nonzero rows of A_k are equal to the same vectors so A_k has rank 1. Let $A = A_1 + \ldots + A_m$ then $A + A^t = J_n - I_n$. Assume that $m \leq n-2$.
- Let $x \in kerA \cap j^{\perp}$ and $x \neq 0$. Then $x^t(J_n - I_n)x = x^t(A + A^t)x$ therefore $x^t x = 0$

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Theorem. (D. de Caen, D.A. Gregory, D. Pritikin, 1993)

Let d be a positive integer, then

- a Hadamard matrix of order $4d$ exists if and only if $bp_{2d}(K_{4d}) = 4d - 1$.
- If a conference matrix of order $2d + 2$ exists then $bp_d(K_{2d+2}) = 2d + 1$.

Definition

A biclique cover of a graph G is a collection of bicliques (complete bipartite graphs) of G such that each edge of G is in at least one of the bicliques. The number of bicliques in a minimum biclique covering of G is called the biclique covering number of G and denoted by $bc(G)$.

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Theorem. (J.-C. Bermond, 1978)

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- Encode the vertices of K_n by binary vectors of length $m = \lceil \log n \rceil$. Define, for each $i = 1, \ldots, m$, a biclique containing all edges, the codes of whose endpoints differ in the$i$th coordinate. So, $bc(K_n) \leq \lceil \log n \rceil$.

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- If we have a biclique cover of size $bc(K_n)$ then we have a vertex coloring with $2^{bc(K_n)}$ color for the graph K_n. So $n = \chi(K_n) \leq 2^{bc(K_n)}$.

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Theorem. (Fronček, Jerebic, Klavžar, and Kovář, 2007)

For any positive integer n we have $bc(K^-_{n,n}) = \min\{k : n \leq \binom{k}{\lfloor \frac{k}{2} \rfloor}\}$.

Theorem. (H. H. and F. Moazami, 2011)

Let d be a positive integer, then $bc_d(K^-_{4d-1,4d-1}) = 4d - 1$ if and only if there exists a Hadamard matrix of order $4d$.

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Theorem. (H. H. and F. Moazami, 2011)

Let d be a positive integer such that there exists a Hadamard matrix of order $4d$, then

1. $bc_{2d}(K_{8d}) = 4d$,
2. $bc_d(K^-_{8d-2,8d-2}) = 4d$.

Frameproof Codes

■ = "detectable positions"

pirate #1	1	1	1	0	1	0	1	0	0	0	0	1
#2	1	0	1	0	1	0	1	0	1	0	1	1
#3	1	0	1	0	1	0	1	0	0	0	1	1
#4	1	1	1	0	0	0	1	1	0	0	0	1
Attacked Content	1	0/1	1	0	0/1	0	1	0/1	0/1	0	0/1	1

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Attacked Content	1	0/1	1	0	0/1	0	1	0/1	0/1	0	0/1	1

Marking Assumption (D. Boneh and J. Shaw, 1998)

Pirates detect fingerprint positions by finding differences in their copies.
They make changes only in the detectable positions.

Suppose $C = \{w^{(u_1)}, w^{(u_2)}, \ldots, w^{(u_d)}\} \subseteq \Gamma$. Let $U(C)$ be the set of undetectable bit positions for C. Set

$$F(C) = \{x \in \{0,1\}^v : x|_{U(C)} = w^{(u_i)}|_{U(C)} \text{ for all } w^{(u_i)} \in C\}.$$

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Definition

Suppose that Γ is a (v,t)-code. Γ is said to be an *r-secure frameproof* code if for any $C_1, C_2 \subseteq \Gamma$ such that $|C_1| \leq r$, $|C_2| \leq r$ and $C_1 \cap C_2 = \varnothing$, we have that $F(C_1) \cap F(C_2) = \varnothing$. We will say that Γ is an $r - SFPC(v,t)$ for short.

Definition

The Kneser graph $\mathrm{KG}(t, r)$ is the graph with vertex set $\binom{[t]}{r}$, and A is adjacent to B if and only if $A \cap B = \varnothing$.

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Theorem. (H. H. and F. Moazami, 2011)

Let r, t, and v be positive integers, where $t \geq 2r$. An $r - \mathrm{SFPC}(v,t)$ exists if and only if there exists a biclique cover of size v for the Kneser graph $\mathrm{KG}(t,r)$.

Cover-Free Family

Definition

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Let w and r be positive integers, a set system $(X, \mathcal{B})$ where $|X| = n$ and $\{B_1, \ldots, B_t\}$ is called an $(r, w) - CFF(n, t)$ if for any two sets of indices $L, M \subseteq [t]$ such that $L \cap M = \varnothing$, $|L| = r$, and $|M| = w$, we have

$$\bigcap_{l \in L} B_l \nsubseteq \bigcup_{m \in M} B_m$$

where $B_i \in \mathcal{F}$.

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2004 Stinson and Wei generalized the definition of cover-free family.

Generalized Cover-Free Family

Definition

Let w and r be positive integers, a set system $(X, \mathcal{B})$ where $|X| = n$ and $\{B_1, \ldots, B_t\}$ is called an $(r, w; d) - CFF(n, t)$ if for any two sets of indices $L, M \subseteq [t]$ such that $L \cap M = \varnothing$, $|L| = r$, and $|M| = w$, we have

$$| \bigcap_{l \in L} B_l \setminus \bigcup_{m \in M} B_m | \geq d$$

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- $N((r, w; d), t)$ denotes the minimum number of points in any $(r, w; d) - CFF$ having t blocks.

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For $0 < w \leq r \leq t$, the bi-intersection graph $I_t(r,w)$ is a bipartite graph whose vertices are the w- and r-subsets of a t-element set where two vertices are adjacent if and only if their intersection is empty.

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Theorem. (H. H. and F. Moazami, 2011)

For every positive integer r, w, d and t, where $t \ge r + w$ we have

$$N((r, w; d), t) = bc_d(I_t(r, w)).$$

Theorem. (D.R. Stinson and R. Wei, 2004)

Let r, w, and t be positive integers where $t \geq r + w$. Then

$$N((r,w;d),t) \geq 2c\frac{\binom{w+r}{w}}{\log(w+r)}\log t + \frac{1}{2}c\binom{w+r}{w}(d-1)$$

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$$N((r,w;d),t) \geq 0.7c\frac{\binom{w+r}{w}(w+r)}{\log(w+r)}\log t + \frac{1}{2}c\binom{w+r}{w}(d-1)$$

Theorem. (H. H. and F. Moazami, 2011)

For every integer $0 \leq s < w \leq r$ and $t \geq r + w$,

$$N((r, w; d), t) \geq \sum_{i=0}^{s} \binom{s}{i} N((r - i, w - s + i; d), t - s).$$

Theorem. (H. H. and F. Moazami, 2011)

For every positive integer r, w, d and t, where $t \geq r + w$ we have

$$N((r, w), t) \geq \binom{r + w - 2}{r - 1} N((2, 1); t - r - w + 3).$$

Theorem. (H. H. and F. Moazami, 2011)

For any positive integers r, w, and t, where $t \geq r + w$, $r \geq w$, and $r \geq 2$, we have

$$N((r, w), t) \geq c \frac{\binom{r+w}{w+1} + \binom{r+w-1}{w+1} + 3\binom{r+w-4}{w-2}}{\log r} \log(t - w + 1),$$

where c is a constant.

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- For $1 \leq k \leq d$, let $n(k, d)$ denote the maximum possible cardinality of a k-neighborly family of standard boxes in R^d.

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Theorem. (N. Alon, 1997)

For $1 \leq k \leq d$, $n(k, d)$ is precisely the maximum number of vertices of a complete graph that admits a biclique covering of order k and size d.

Theorem. (N. Alon, 1997)

Let d be a positive integer and $1 \leq k \leq d$ then

1. $d + 1 = n(1, d) \leq n(2, d) \leq \cdots \leq n(d-1, d) \leq n(d, d) = 2^d$.
2. $(\frac{d}{k})^k \leq \prod_{i=0}^{k-1}(\lfloor \frac{d+i}{k} \rfloor + 1) \leq n(K, d) \leq \sum_{i=0}^{k} 2^i \binom{d}{i} < 2(\frac{2ed}{k})^k$.

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Definition

Let K be a set of k positive integers. We say that a biclique cover of the graph G is of type K if for every edge e of the graph G, the number of bicliques that cover e is an element of the set K.

Definition

The family $\mathcal{F} = \{(A_1, B_1), \ldots, (A_g, B_g)\}$ is called a weakly cross-intersecting set-pairs of size g on a ground set of cardinality h whenever all A_i's and B_i's are subsets of an h-set and for every i, where $1 \leq i \leq g$, $A_i \cap B_i = \varnothing$ and furthermore, for every $i \neq j$, $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \varnothing$. The family $\mathcal{F}$ is called an (r, w)-weakly cross-intersecting set-pairs if $\mathcal{F}$ weakly cross-intersecting set-pairs and $|A_i| = r$ and $|B_j| = w$.

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Definition

The biclique cover $\{G_1 = (X_1, Y_1), \ldots, G_l = (X_l, Y_l)\}$ is called an (r, w)-biclique cover whenever each vertex of G belongs to at most r sets in $\{X_1, X_2, \ldots, X_l\}$ and at most w sets in $\{Y_1, Y_2, \ldots, Y_l\}$.

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Theorem. (Tuza, 1987)

Let i and j be positive integers then

- $g(i, 1) = 2i + 1$
- $g(i, j) < \frac{(i+j)^{(}i+j)}{i^i j^j}$

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Theorem. (Z. Király, Z.L. Nagy, D. Pálvölgyi, and M. Visontai, 2010)

Let i and j be positive integers then

$$g(i, j) \geq (2 - o(1))\binom{i+j}{i},$$

where $f \in o(1)$ means that $\lim_{i+j \to \infty} f = 0$.

Theorem. (H. H. and F. Moazami, 2011)

Suppose that g, h, r, w, and t are positive integers. Also, let $\mathcal{F} = \{(A_1, B_1), \ldots, (A_g, B_g)\}$ be a weakly cross-intersecting set-pairs on a ground set of size h such that for any $1 \leq i \leq g$, $|A_i| \leq r$ and $|B_i| \leq w$. If $t \geq \max\{h, r + w\}$, then

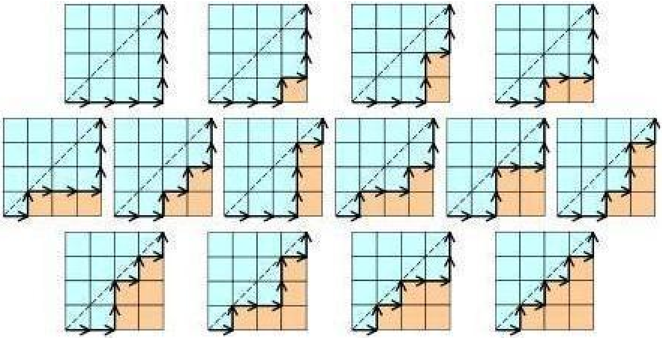
$$N((r, w; d), t) \geq \sum_{i=1}^{g} N((r - |A_i|, w - |B_i|; d), t - |A_i| - |B_i|).$$

Theorem. (K. Engel, 1996)

Let i, j, r, and w be positive integers, where $1 \leq i \leq r-1$ and $1 \leq j \leq w-1$. If there exists an (i,j)-weakly cross-intersecting set-pairs of size $g(i,j)$ on a ground set of cardinality h, then for any t, where $t \geq \max\{h, r+w\}$, we have

$$N((r,w;d),t) \geq g(i,j)N((r-i,w-j;d),t-i-j).$$

Catalan Number



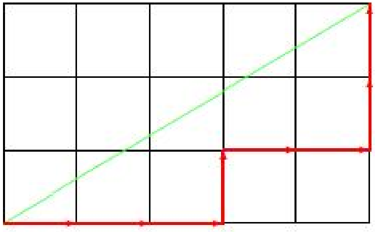
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

http://en.wikipedia.org/wiki/Catalan_number

Consider a path that

- starts from $(0,0)$ and ends at (i,j)
- each step is either an up step or a right step,
- all points visited are below the diagonal (of slope $\frac{j}{i}$).

Denote the set of all such paths by $\mathcal{L}(i,j)$.



A lattice path from $(0,0)$ to $(5,3)$.

Theorem. (Z. Király, Z.L. Nagy, D. Pálvölgyi, and M. Visontai, 2010)

There exist an (i,j)-weakly cross-intersecting set-pairs of size $(2i+2j-1)|\mathcal{L}(i,j)|$ on a ground set of size $2i+2j-1$.

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Theorem. (Bizley, 1954)

Let i and j be relatively prime numbers, then $|\mathcal{L}(i, j)| = \frac{\binom{i+j}{i}}{i+j}$.

Theorem. (Z. Király, Z.L. Nagy, D. Pálvölgyi, and M. Visontai, 2010)

There exist an (i, j)-weakly cross-intersecting set-pairs of size $(2i + 2j - 1)|\mathcal{L}(i, j)|$ on a ground set of size $2i + 2j - 1$.

- The above Theorem was proved by a charming idea and using lattice paths.

Theorem. (Bizley, 1954)

Let i and j be relatively prime numbers, then $|\mathcal{L}(i, j)| = \frac{\binom{i+j}{i}}{i+j}$.

- Unfortunately, for general (i, j), there is no explicit formula for $|\mathcal{L}(i, j)|$.

- Suppose that in an election, candidate A receives r votes and candidate B receives w votes.
- Let r_i and w_i denote the number of votes A and B have after counting the i^{th} vote where $1 \leq i \leq r + w$.
- Let k be any positive real number. A sequence is called k-good if $r > kw$ and $r_i > kw_i$ for all r.

BALLOT PROBLEM

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- Let $\mathcal{B}(r, w; k)$ be the maximum number of k-good sequences. Determining the exact value of $\mathcal{B}(r, w; k)$ is known as the generalized ballot problem.
- $\mathcal{B}(r, w - 1; \frac{r}{w}) = |\mathcal{L}(r, w)|$.

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Theorem. (Bertrand, 1887)

$\mathcal{B}(r, w; 1) = \frac{r-w}{r+w}\binom{r+w}{r}$

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