Commutative Graph Algebras

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Abstract

Let G be a graph and K be an arbitrary field. Consider the Stanley-Reisner ring $R(G)$ as K-algebra. In this note, we prove that, for every two graphs G_1 and G_2 if $R(G_2)$ is an epimorphic image of $R(G_1)$ as K-algebra, then G_2 is isomorphic to a subgraph of G_1. As a consequence of this result we have, $R(G_1)$ and $R(G_2)$ are isomorphic if and only if G_1 and G_2 are isomorphic.

Key words: K-Algebra, Stanley-Reisner Ring, Graph Isomorphism.

1 Isomorphism Theorems

The study of combinatorial objects using algebraic tools is one of the important branches of algebra. In this direction, one of the natural way is to assign an algebraic structure like a group, a ring and etc., to a given combinatorial object. In particular, graphs can be investigated by this approach. In 1980, Hang Kim, L. Makar-Limanov, J. Neggers and F. W. Roush [2], introduced a non-commutative algebra $K(G)$, for any graph G, and proved that $K(G_1) \simeq K(G_2)$ if and only if $G_1 \simeq G_2$. In 1987, Droms [1], assigned a group to a given graph by introducing the generators and relations of that group and proved a similar isomorphism theorem. In this paper we consider the Stanley-Reisner ring $R(G)$ as K-algebra and we prove a similar isomorphism theorem for this association.

Throughout this paper we consider simple graphs, which are finite, undirected, with no loops or multiple edges. A *matching* in a graph G is a set of edges, no two of which have a vertex in common. We say that a matching *saturates* a given subset S of $V(G)$, if each element of S is an end point of an edge of that matching. For any subset S of $V(G)$ the set of neighbors of S is denoted by $N(S)$ and defined to be $\bigcup_{a\in S} N(a)$; where $N(a)$ is the set of neighbors of a. Next, we give a formulation of the well-known matching theorem of P. Hall.

Theorem A. *Let G be a bipartite graph with vertex set $X \cup Y$. Graph G has a matching which saturates X if and only if for every $S \subseteq X$, $|\,N(S)\,| \geq |\,S\,|$.*

Suppose G is a graph. Denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively, where $V(G) = \{v_1, \ldots, v_n\}$. Throughout the paper, let K be a fixed field and set ideal $I(G)$ in $K[x_1, \ldots, x_n]$ as follows,

$$I(G) = \langle x_i x_j \,|\, i \neq j,\ v_i v_j \notin E(G)\rangle.$$

Now, set,

$$R(G) = K[x_1, \ldots, x_n]/I(G),$$

where $R(G)$ is called the *Stanley-Reisner ring* of the graph G (see [3] and [4]).

In order to prove the main theorem we need some additional definitions and lemmas.

Definition. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$ and assume that $R(G) = K[x_1, \ldots, x_n]/I(G)$, where $I(G)$ is the ideal defined as above. Let $f \in R(G)$, and set $f = k + z + s + I(G)$; where $k \in K$, z is a linear combination of x_i's and s consists of terms of degree greater than 1. We call z the *linear part* of f. Define the set $V(f)$ to be the set of those v_i's for which the coefficient of x_i in z is not zero.

In what follows, we consider two graphs G_1 and G_2 and denote their vertex sets by $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ respectively. Let $R(G_1) = K[x_1, \ldots, x_n]/I(G_1)$ and $R(G_2) = K[y_1, \ldots, y_m]/I(G_2)$ and denote by X_i and Y_j the images of x_i and y_j under the natural maps $K[x_1, \ldots, x_n] \longrightarrow R(G_1)$ and $K[y_1, \ldots, y_m] \longrightarrow R(G_2)$, respectively.

Definition. Let $\varphi : R(G_1) \longrightarrow R(G_2)$ be a K-algebra homomorphism. We define a bipartite graph $\mathcal{G}(\varphi)$ by regarding $V(G_2)$ as one of its parts and the set $\{V(\varphi(X_1)), \ldots, V(\varphi(X_n))\}$ as the other part. We take w_i to be adjacent to $V(\varphi(X_j))$ if $w_i \in V(\varphi(X_j))$.

Lemma 1. *Let G_1 and G_2 be two graphs. Also, let $\varphi : R(G_1) \longrightarrow R(G_2)$ be a K-algebra epimorphism, then $|\,V(G_1)\,| \geq |\,V(G_2)\,|$ and $\mathcal{G}(\varphi)$ has a matching which* *saturates* $V(G_2)$.

Proof. Since φ is surjective, for every $i, 1 \leq i \leq m$, there exist λ_i and $\lambda_{i_1}, \dots, \lambda_{i_n}$ in K, such that $y_i + I(G_2) = \lambda_i + \sum_{j=1}^{n} \lambda_{i_j} z_j +$ (terms with degree greater than 1)$+I(G_2)$; where z_j is the linear part of $\varphi(X_j)$. Now, we know that the elements of $I(G_2)$ have degree greater than 1, so

$$y_i = \sum_{j=1}^{n} \lambda_{i_j} z_j, \ 1 \leq i \leq m.$$

This implies that the m-dimensional K-vector space generated by $\{y_1, \dots, y_m\}$ is exactly equal to the K-vector space generated by $\{z_1, \dots, z_n\}$. Hence, $m \leq n$. Next, we prove that $\mathcal{G}(\varphi)$ has a matching. By renumbering w_i's and $V(\varphi(X_j))$'s we may assume that $w_1, \dots, w_r$ is an arbitrary r-subset of $V(G_2)$ and $V(\varphi(X_1)), \dots, V(\varphi(X_t))$, $t < r$ are the only elements of $\mathcal{G}(\varphi)$ which contain at least one of the w_i's, $1 \leq i \leq r$. In this case, $z_{t+1}, \dots, z_n$ can be written as a linear combination of $y_{r+1}, \dots, y_m$. Thus, the vector space generated by $\{z_1, \dots, z_n\}$, which we know has dimension m, is equal to the vector space generated by $\{z_1, \dots, z_t, y_{r+1}, \dots, y_m\}$, which clearly has dimension at most $(m-r)+t$. Obviously this is a contradiction. Thus by Theorem A, $\mathcal{G}(\varphi)$ has a matching which saturates $V(G_2)$. □

Lemma 2. *Let G_1 and G_2 be two graphs and $\varphi : R(G_1) \longrightarrow R(G_2)$ be a K-algebra epimorphism. If $v_i v_j \notin E(G_1)$, and $V(\varphi(X_i))$ and $V(\varphi(X_j))$ are non- empty, then for every $w_r \in V(\varphi(X_i))$ and $w_s \in V(\varphi(X_j)), w_r w_s \notin E(G_2)$. Equivalently, if $w_r w_s \in E(G_2)$ for some $w_r \in V(\varphi(X_i))$ and $w_s \in V(\varphi(X_j))$, then $v_i v_j \in E(G_1)$.*

Proof. Let $v_i v_j \notin E(G_1)$, then $X_i X_j = 0$ and therefore $\varphi(X_i)\varphi(X_j) = 0$. So we have

$$(k_i + z_i + s_i + I(G_2))(k_j + z_j + s_j + I(G_2)) = I(G_2);$$

where $k_i, k_j \in K$, z_i and z_j are the linear parts of $\varphi(X_i)$ and $\varphi(X_j)$, respectively, and s_i, s_j consist of terms with degree greater than 1. Since the elements of $I(G_2)$ have degree greater than 1, by the above equation we obtain $k_i k_j = k_i z_j = k_j z_i = 0$. Now, by our assumptions we must have $k_i = k_j = 0$, and so $z_i z_j$ can be written as a linear combination of generators of $I(G_2)$. Next, let $z_i = \lambda_{i_1} y_{i_1} + \cdots + \lambda_{i_k} y_{i_k}$ and $z_j = \mu_{j_1} y_{j_1} + \cdots + \mu_{j_l} y_{j_l}$, where the λ_{i_t} and μ_{j_s} are all non-zero coefficients. We claim the set $\{y_{i_1}, \dots, y_{i_k}\} \cap \{y_{j_1}, \dots, y_{j_l}\}$ is empty. For, if y belongs to this set, then y^2 will appear in the product $z_i z_j = \sum_{t=1}^{k} \sum_{s=1}^{l} \lambda_{i_t} \mu_{j_s} y_{i_t} y_{j_s}$ and it can not be canceled by other terms. This contradicts the assumption that $X_i X_j = 0$. Hence, in the product $z_i z_j$ no term can be canceled by other terms, and since this product is a linear combination of the generators of $I(G_2)$, we have $y_{i_t} y_{j_s} \in I(G_2)$, which means that $w_{i_t} w_{j_s} \notin E(G_2)$, as desired. □

Theorem 1. *Let G_1 and G_2 be two graphs, then $R(G_2)$ is an epimorphic image of $R(G_1)$, as K-algebras, if and only if G_2 is isomorphic to a subgraph of G_1.*

Proof. By Lemma 1, there is a one-one map σ from $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$ such that for every $1 \leq i \leq m$, $w_i \in V(\varphi(X_{\sigma(i)}))$. By Lemma 2 if $w_i w_j \in E(G_2)$ then $v_{\sigma(i)} v_{\sigma(j)} \in E(G_1)$. □

Now, the following theorem is clear.

Theorem 2. *For any two graphs G_1 and G_2, $R(G_1)$ is isomorphic to $R(G_2)$, as K-algebras, if and only if G_1 is isomorphic to G_2.*

Acknowledgement. The authors are indebted to the Research Council of the Sharif University of Technology and Institute for Theoretical Physics and Mathematics for their support. In addition they wish to thank Professor E. S. Mahmoodian for his helpful comments.

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