سیگنال و سیستم

(تجزیه و تحلیل سیستم‌ها)

۱۳-۱۱-۱۸۰

بخش پنجم

تبدیل فوریه زمان گسسته

دانشگاه شهید بهشتی

دانشکده‌ی مهندسی برق و کامپیوتر

پاییز ۱۳۹۴

احمد محمودی ازناوه

http://faculties.sbu.ac.ir/~a_mahmoudi/

فهرست مطالب

- تبدیل فوریه گسسته
- خواص تبدیل فوریه گسسته
- تبدیل فوریه‌ی سیگنال پریودیک
- خواص تبدیل فوریه گسسته

تبدیل فوریه‌ی سیگنال‌های زمان‌گسسته

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}$$

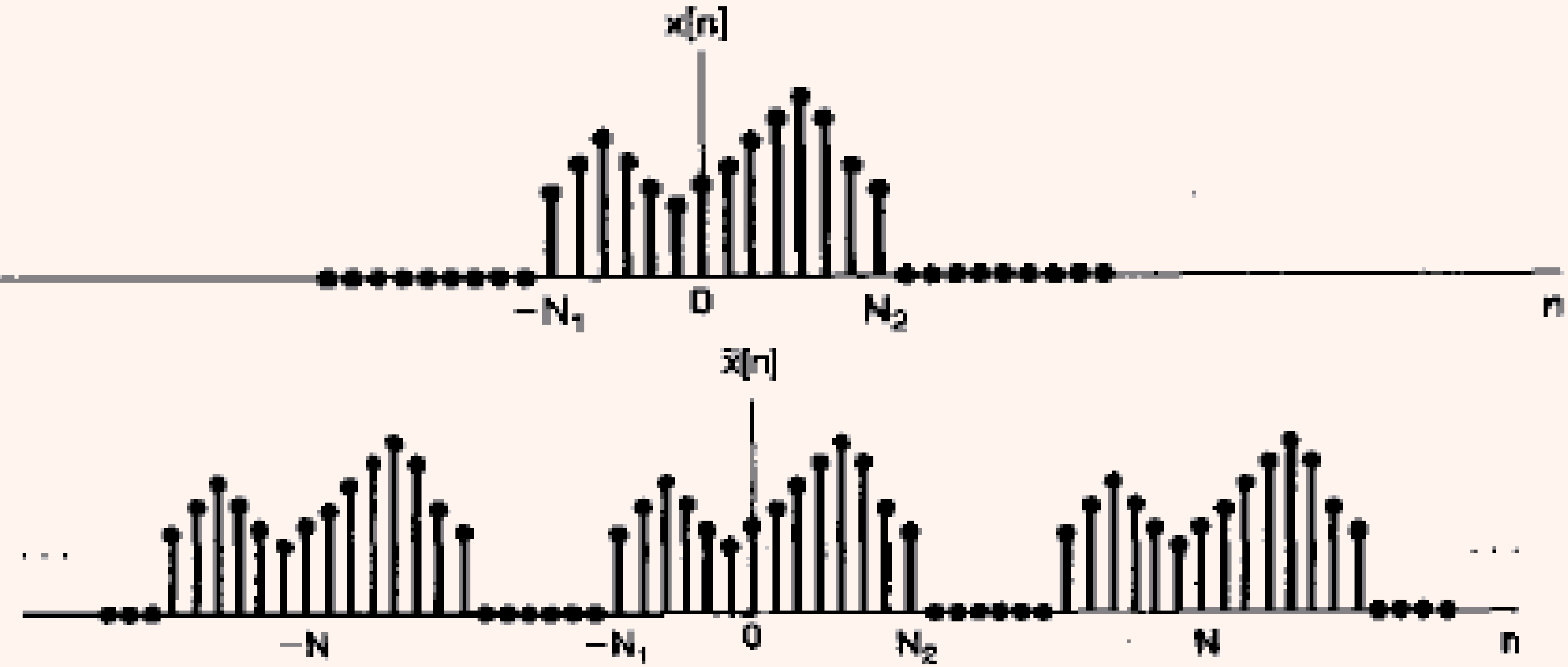
Synthesis equation

$$\boldsymbol{a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n}}$$

Analysis equation

سری فوریه سیگنال‌های پریودیک زمان گسسته، برای تحلیل فرکانسی مورد استفاده قرار می‌گرفتند، به طریقی مشابه با سیگنال‌های زمان‌پیوسته؛ تبدیل فوریه‌ی زمان گسسته مطرح می‌شود.

تبدیل فوریه‌ی سیگنال‌های زمان‌گسسته



$$\tilde{x}[n]=\sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}$$

$$a_k=\frac{1}{N}\sum_{k=<N>}\tilde{x}[n]e^{-jk\frac{2\pi}{N}n}=\frac{1}{N}\sum_{n=-N_1}^{N_2}x[n]e^{-jk\frac{2\pi}{N}n}=\frac{1}{N}\sum_{n=-\infty}^{\infty}x[n]e^{-jk\frac{2\pi}{N}n}$$

تبدیل فوریه‌ی سیگنال‌های زمان‌گسسته

$$a_k = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n] e^{-jk\frac{2\pi}{N}n}$$

تعریف می‌کنیم:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

متناوب با دوره‌ی تناوب 2π

خواهیم داشت:

$$a_k = \frac{1}{N} X(e^{jk\omega_0})$$

بنابراین

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} = \frac{1}{2\pi}\sum_{k=<N>} X(e^{jk\omega_0}) e^{jk\omega_0 n}\omega_0$$

تبدیل فوریه‌ی سیگنال‌های زمان‌گسسته

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

As N→∞,

$$\tilde{x}[n] \to x[n] \qquad \omega_0 \to 0, \quad \sum \omega_0 \to \int d\omega$$

$$\Rightarrow \quad x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

شرایط همگرایی مانند همگرایی تبدیل پیوسته است

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \qquad \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

factor

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Synthesis equation
DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Analysis equation
Inverse DTFT

1. **A linear combination of complex exponentials.**
2. $X(e^{j\omega})$ —— **Spectrum** $x[n]$

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega)$$

DTFS

a_k is samples of $X(e^{j\omega})$

DTFT

$$x[n]=\sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$x[n]=\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$a_k=\frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\omega_0 n}$$

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n]=\tilde{x}[n]$$

در یک دوره

CTFT **DTFT**

x[n] has a finite interval of integration

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega \qquad x[n]=\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(j\omega)=\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \qquad X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$X(e^{j\omega})$ **is periodic**

مثال

$$x[n]=\delta[n]$$

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}=\sum_{n=-\infty}^{\infty}\delta[n]e^{-j\omega n}=1 \qquad \delta[n]\xleftrightarrow{FT}1$$

$$x[n]=\delta[n-n_0]$$

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x[n-n_0]e^{-j\omega n}=\sum_{n=-\infty}^{\infty}\delta[n-n_0]e^{-j\omega n}=e^{-j\omega n_0}$$

$$\delta[n-n_0]\xleftrightarrow{FT}e^{-j\omega n_0}$$

مثال

$$x[n] = a^n u[n] \quad |a| < 1$$

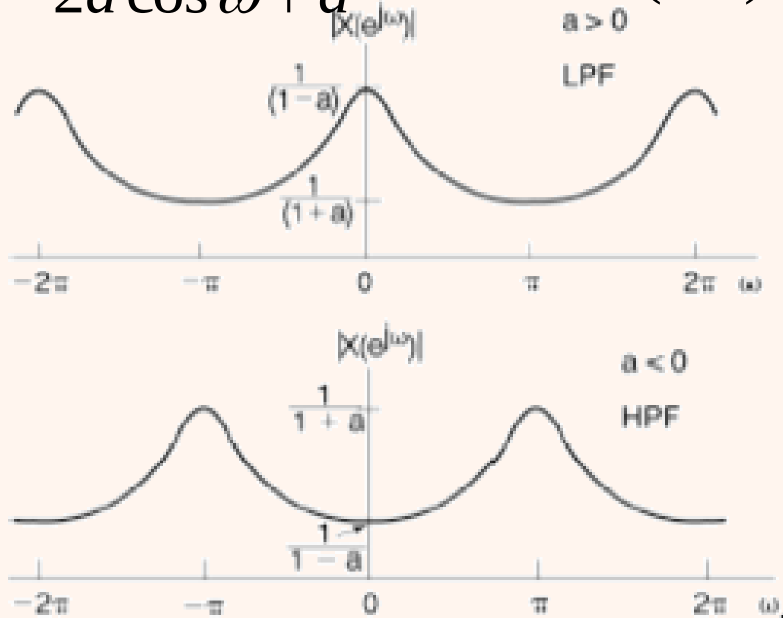
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

$$a^n u[n] \overset{FT}{\longleftrightarrow} \frac{1}{1 - ae^{-j\omega}}$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

$$\angle X(e^{j\omega}) = -\arctan\left(\frac{a\sin\omega}{1 - a\cos\omega}\right)$$



مثال

$$x[n] = a^{|n|} \qquad |a| < 1$$

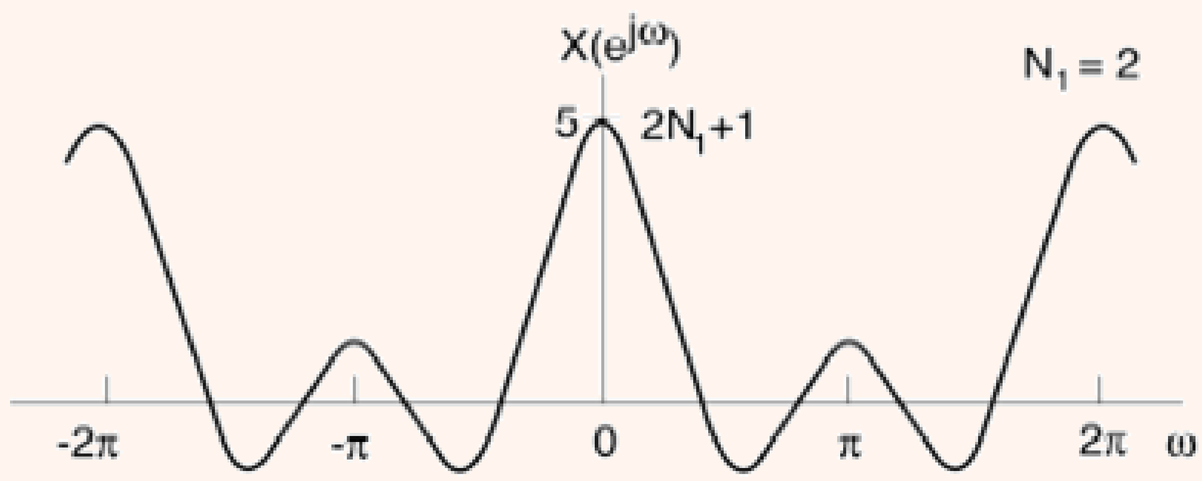
$$a^{|n|} \quad (|a| < 1) \overset{FT}{\longleftrightarrow} \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

مثال

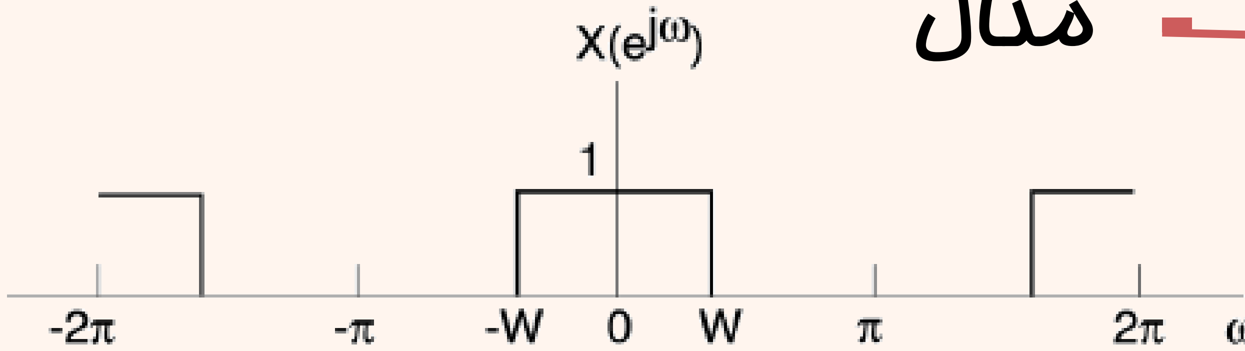
$$x[n]=u[n+N_1]-u[n-N_1-1]=\begin{cases}1 & |n|\le N_1\\ 0 & |n|>N_1\end{cases}$$

$$X(e^{j\omega})=\sum_{n=-N_1}^{N_1}e^{-j\omega n}=e^{j\omega N_1}\sum_{m=0}^{2N_1}e^{-j\omega m}=\frac{e^{j\omega(N_1+1/2)}}{e^{j\omega(1/2)}}\frac{1-e^{-j\omega(2N_1+1)}}{1-e^{-j\omega}}$$

$$=\frac{e^{j\omega(N_1+1/2)}-e^{-j\omega(N_1+1/2)}}{e^{j\omega/2}-e^{-j\omega/2}}=\frac{\sin(\omega(N_1+1/2))}{\sin(\omega/2)}$$



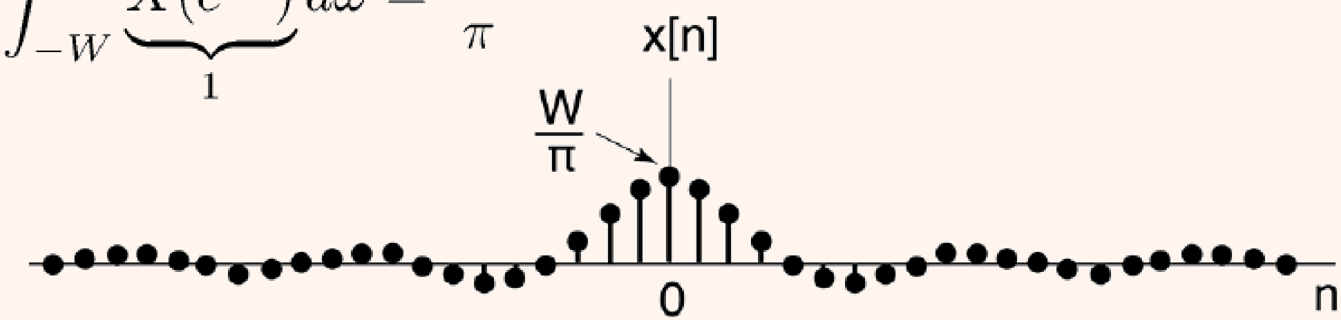
مثال



$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}d\omega = \frac{1}{2\pi}\times\frac{e^{jn\omega}}{jn}\Bigg|_{-W}^{W} = \frac{\sin Wn}{\pi n}$$

$$x[0] = \frac{1}{2\pi}\int_{-W}^{W}\underbrace{X(e^{j\omega})}_{1}d\omega = \frac{W}{\pi}$$



تبدیل فوریه‌ی سیگنال پریودیک

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l)$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{2\pi \sum_{m=-\infty}^{\infty} \delta(\omega-\omega_0-2\pi m)}_{X(e^{j\omega})} e^{j\omega n} d\omega = e^{j\omega_0 n}$$

$$x[n] = e^{j\omega_0 n} \xleftrightarrow{FT} X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l)$$

تبدیل فوریه‌ی سیگنال پریودیک (ادامه...)

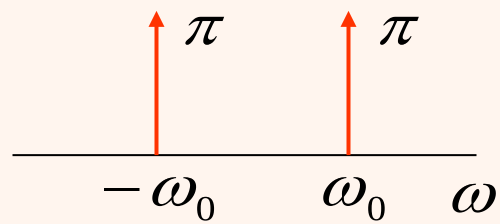
$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N})$$

مثال

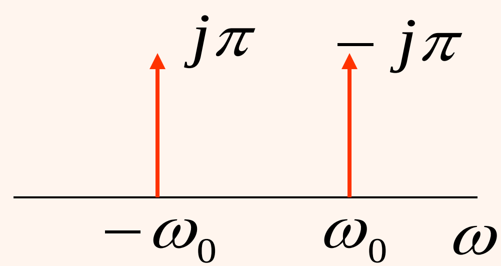
$$\cos\omega_0 n \xleftrightarrow{FT} \sum_{l=-\infty}^{\infty}\pi\delta(\omega+\omega_0-2\pi l)+\sum_{l=-\infty}^{\infty}\pi\delta(\omega-\omega_0-2\pi l)$$

$$\cos\omega_0 n=\frac{1}{2}e^{j\omega_0 n}-\frac{1}{2}e^{-j\omega_0 n}$$



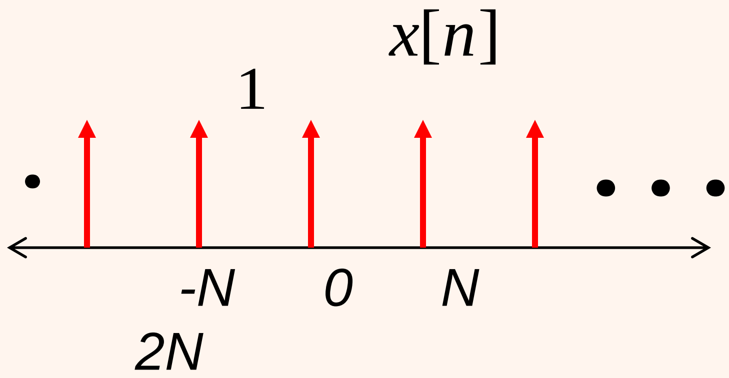
$$\sin\omega_0 n \xleftrightarrow{FT} \sum_{l=-\infty}^{\infty}j\pi\delta(\omega+\omega_0-2\pi l)-j\sum_{l=-\infty}^{\infty}\pi\delta(\omega-\omega_0-2\pi l)$$

$$\sin\omega_0 n=\frac{1}{2j}e^{j\omega_0 n}-\frac{1}{2j}e^{-j\omega_0 n}$$



مثال

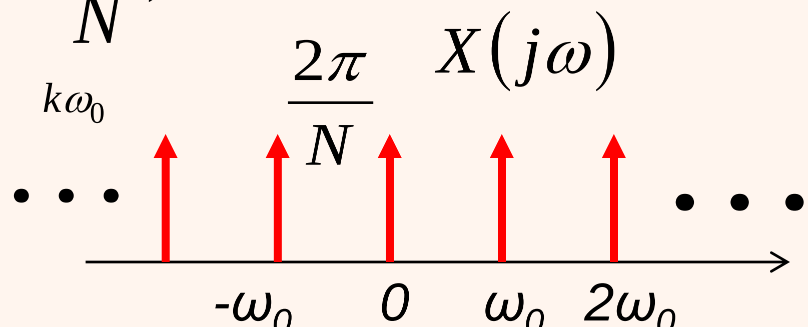
$$x[n]=\sum_{k=-\infty}^{+\infty}\delta[n-kN]$$



$$x[n]\leftrightarrow a_k=\frac{1}{N}\sum_{n=<N>}x[n]e^{-j\omega_0 n}=\frac{1}{N}$$

$$\Downarrow\ x[n]=\sum_{k=<N>}a_k e^{jk\omega_0 n}=\sum_{k=<N>}\frac{1}{N}e^{jk\omega_0 n}$$

$$X(e^{j\omega})=\sum_{k=-\infty}^{+\infty}\frac{2\pi}{N}\delta(\omega-k\frac{2\pi}{N})$$



خواص تبدیل فوریه گسسته

Periodicity of DTFT

$$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$$

Linearity

$$x[n] \xleftrightarrow{\text{FT}} X\left(e^{j\omega}\right) \quad y[n] \xleftrightarrow{\text{FT}} Y\left(e^{j\omega}\right)$$

$$ax[n]+by[n] \xleftrightarrow{\text{F}} aX\left(e^{j\omega}\right)+bY\left(e^{j\omega}\right)$$

Time Reversal

$$x[-n] \xleftrightarrow{\text{FT}} X\left(e^{-j\omega}\right)$$

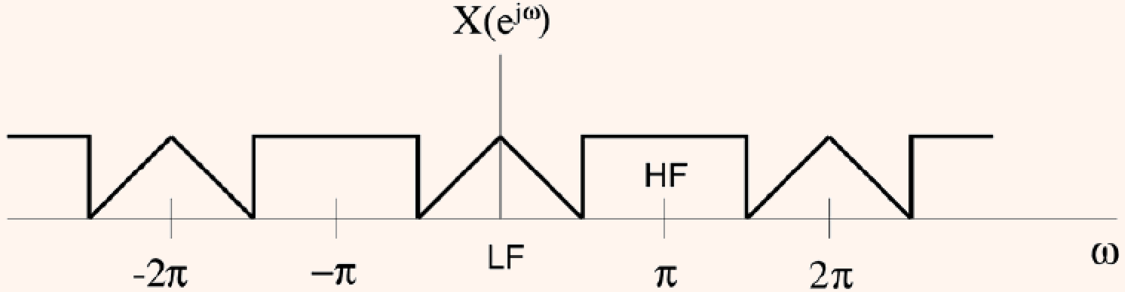
خواص تبدیل فوریه گسسته (ادامه...)

Time Shifting and Frequency Shifting

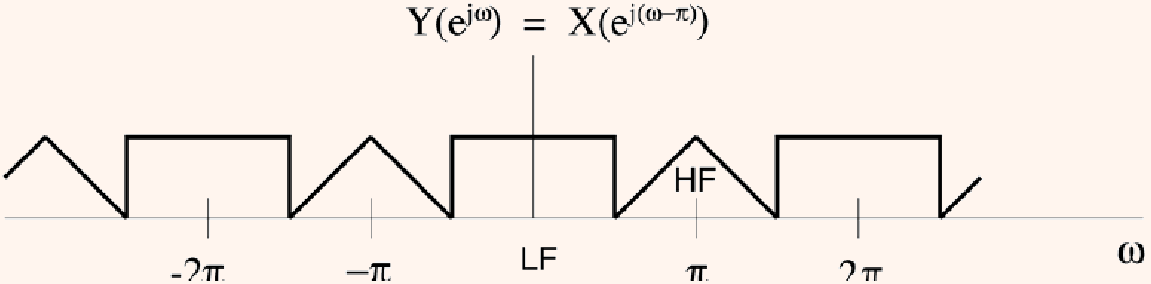
$$x[n] \xleftrightarrow{FT} X\left(e^{j\omega}\right)$$

$$x[n-n_0] \xleftrightarrow{FT} X\left(e^{j\omega}\right)e^{-j\omega n_0}$$

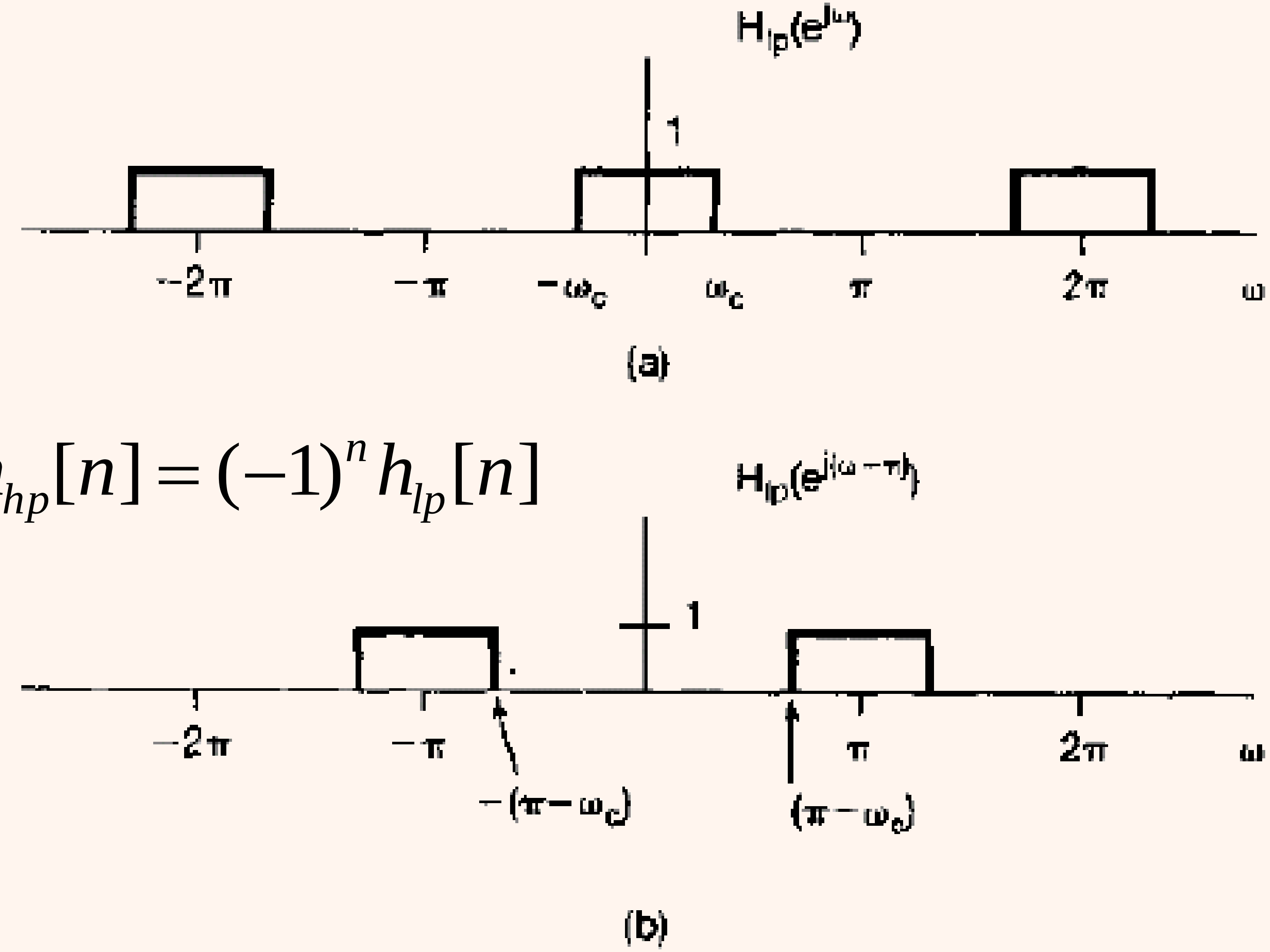
$$e^{jn\omega_0}x[n] \xleftrightarrow{FT} X\left(e^{j(\omega-\omega_0)}\right)$$



$$\omega_0 = \pi\ ,\ y[n] = e^{j\pi n}x[n] = (-1)^n x[n]$$



مثال



$$h_{hp}[n] = (-1)^n h_{lp}[n]$$

Conjugation and Conjugate Symmetry

$$x^*[n] \overset{F}{\longleftrightarrow} X^*\left(e^{-j\omega}\right)$$

$$x[n] = x^*[n] \Rightarrow X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right)$$

$$x[n] = x^*[n] \Rightarrow \begin{cases} \operatorname{Re}\left\{X\left(e^{-j\omega}\right)\right\} = \operatorname{Re}\left\{X\left(e^{j\omega}\right)\right\} \\ \operatorname{Im}\left\{X\left(e^{-j\omega}\right)\right\} = -\operatorname{Im}\left\{X\left(e^{j\omega}\right)\right\} \end{cases}$$

$$x[n] = x^*[n] \Rightarrow \begin{cases} \left|X\left(e^{-j\omega}\right)\right| = \left|X\left(e^{j\omega}\right)\right| \\ \angle X\left(e^{-j\omega}\right) = -\angle X\left(e^{j\omega}\right) \end{cases}$$

خواص تبدیل فوریه گسسته (ادامه...)

$x[n]$ **real even** $\Rightarrow$ $X(e^{j\omega})$ **real even**

$x[n]$ **real odd** $\Rightarrow$ $X(e^{j\omega})$ **Purely imaginary odd**

$$Ev\{x[n]\} \xleftrightarrow{F} \mathrm{Re}\{X(e^{j\omega})\} \qquad Od\{x[n]\} \xleftrightarrow{F} j\,\mathrm{Im}\{X(e^{j\omega})\}$$

مثال

$$a^{|n|} = a^n u[n] + a^{-n}u[-n] - \delta[n]$$

$$a^n u[n] + a^{-n}u[-n] = 2Ev\{a^n u[n]\}$$

$$a^n u[n] \xleftrightarrow{FT} \frac{1}{1-ae^{-j\omega}}$$

$$a^{|n|} \xleftrightarrow{FT} 2\,\mathrm{Re}\left\{\frac{1}{1-ae^{-j\omega}}\right\} - 1 = \frac{1-a^2}{1-2a\cos\omega + a^2}$$

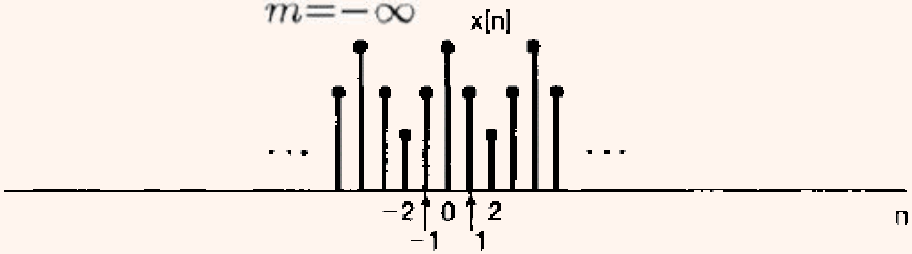
خواص تبدیل فوریه گسسته (ادامه...)

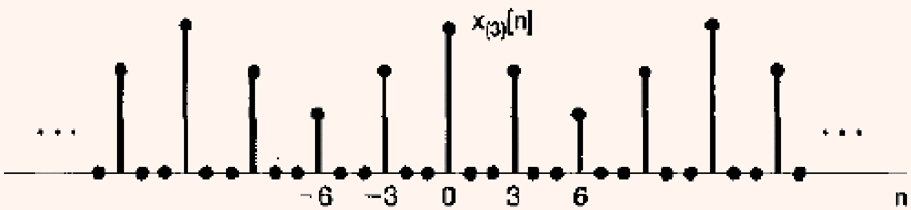
$$x_{(k)}[n] = \begin{cases} x[n/k] & n \text{ is a multiple of } k \\ 0 & n \text{ is not a multiple of } k \end{cases}$$

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n]e^{-j\omega n} \overset{n=mk}{=} \sum_{m=-\infty}^{\infty} x_{(k)}[mk]e^{-j\omega mk}$$

$$= \sum_{m=-\infty}^{\infty} x[m]e^{-j(k\omega)m} = X(e^{jk\omega})$$

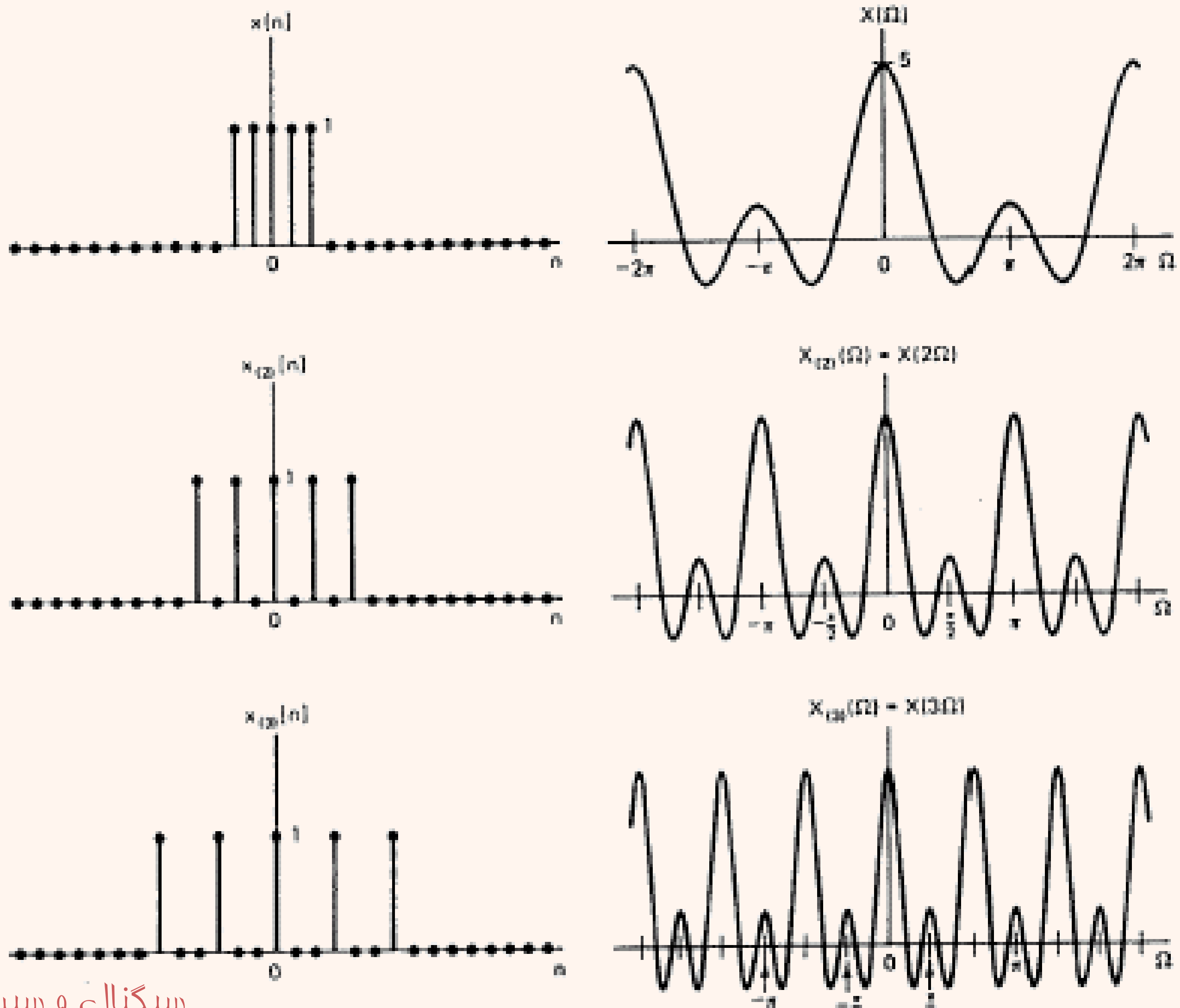
-compressed by a factor of k in frequency domain





$$x_{(k)}[n] \xleftrightarrow{FT} X(e^{jk\omega})$$

خواص تبدیل فوریه گسسته (ادامه...)



مثال

- تبدیل فوریه‌ی سیگنال زیر را بیابید:

$$y[n]=x[1-n]+x[-1-n]\overset{\text{FT}}{\longleftrightarrow}?$$

$$x[-n]\overset{\text{FT}}{\longleftrightarrow}X\left(e^{-j\omega}\right)$$

$$x[1-n]\overset{\text{FT}}{\longleftrightarrow}e^{-jw}X\left(e^{-j\omega}\right)$$

$$x[-1-n]\overset{\text{FT}}{\longleftrightarrow}e^{jw}X\left(e^{-j\omega}\right)$$

$$y[n]\overset{\text{FT}}{\longleftrightarrow}2\cos\omega X\left(e^{-jw}\right)$$

خواص تبدیل فوریه گسسته (ادامه...)

Difference

$$x[n]-x[n-1] \overset{FT}{\longleftrightarrow} (1-e^{-j\omega})X\left(e^{j\omega}\right)$$

Summation

$$\sum_{m=-\infty}^{n} x[m] \overset{\text{FT}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

مثال

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \overset{FT}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

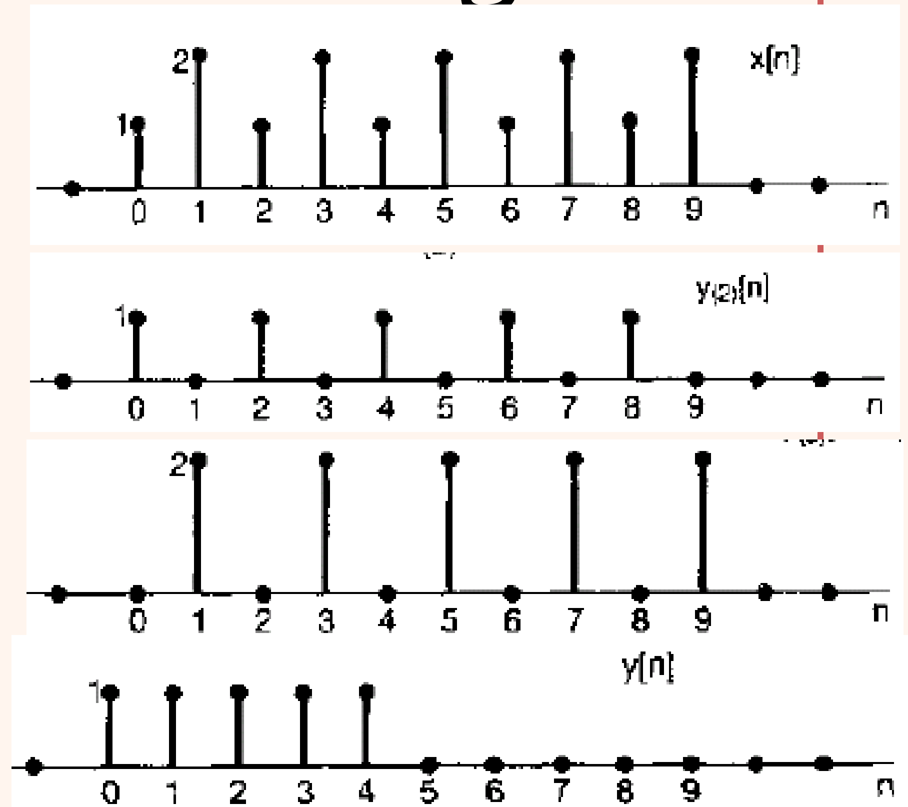
مثال

$$x[n] = y_{(2)}[n] + y_{(2)}[n-1]$$

$$y[n] = u[n] - u[n-5]$$

$$Y(e^{j\omega}) = e^{-j2\omega} \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

$$Y_{(2)}(e^{j\omega}) = Y(e^{j2\omega}) = e^{-j4\omega} \frac{\sin(5\omega)}{\sin(\omega)}$$



$$X(e^{j\omega}) = \frac{\sin(\omega(N_1 + 1/2))}{\sin(\omega/2)}$$

$$X(e^{j\omega}) = e^{-j4\omega}(1 + 2e^{-j\omega}) \frac{\sin(5\omega)}{\sin\omega}$$

خواص سری فوریه‌ی زمان گسسته

Differentiation in Frequency Domain

$$x[n] \overset{FT}{\longleftrightarrow} X\left(e^{j\omega}\right) \qquad nx[n] \overset{FT}{\longleftrightarrow} j\frac{dX\left(e^{j\omega}\right)}{d\omega}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\frac{d}{d\omega}X(e^{j\omega}) = -j\sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n}$$

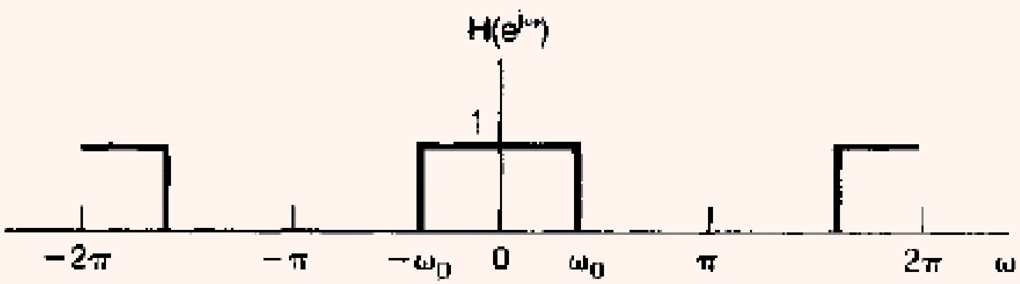
Parseval's Relation

$$\sum_{n=-\infty}^{\infty}\left|x[n]\right|^2 = \frac{1}{2\pi}\int_{2\pi}\left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

کانولوشن

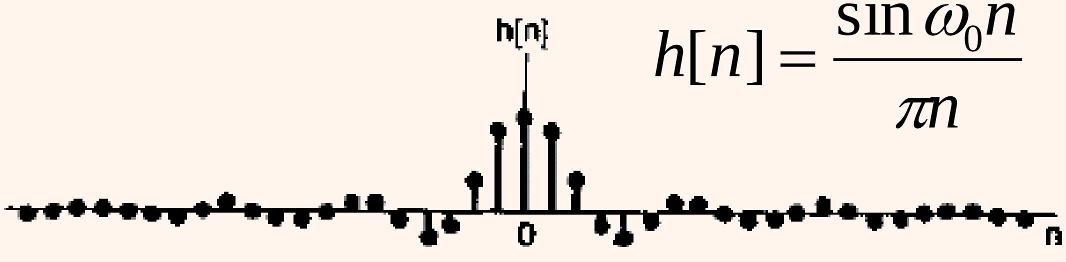
$$y[n] = x[n] * h[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$



(a)

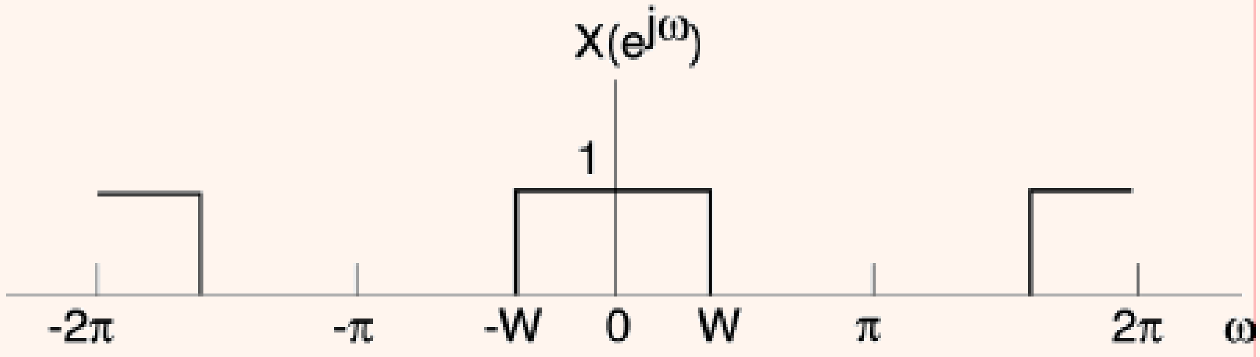
$$h[n] = \frac{\sin \omega_0 n}{\pi n}$$



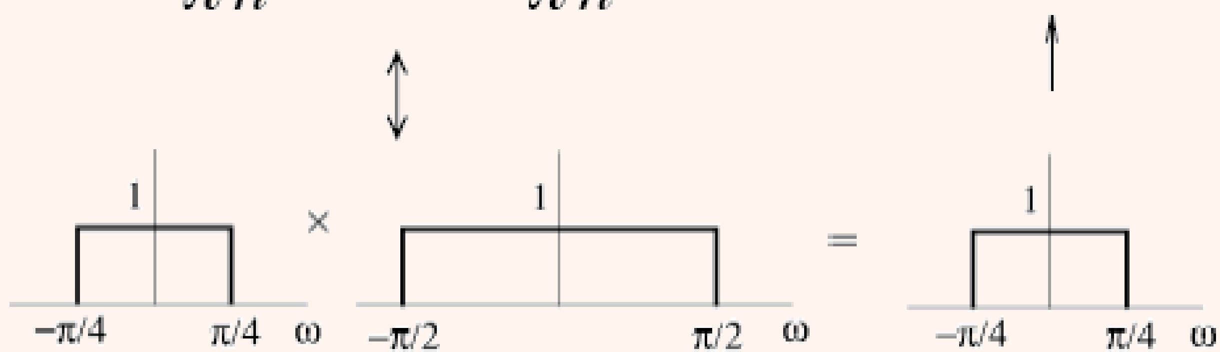
(b)

مثال

$$x[n]=\frac{\sin Wn}{\pi n}$$



$$\frac{\sin(\pi n/4)}{\pi n} * \frac{\sin(\pi n/2)}{\pi n} = ?$$



مثال

$$h[n] = \alpha^n u[n], \quad x[n] = \beta^n u[n] \qquad |\alpha|, |\beta| < 1$$

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}, \quad X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \frac{1}{1 - \beta e^{-j\omega}}$$

α≠β,

$$Y(e^{j\omega}) = \frac{A}{1 - \alpha e^{-j\omega}} + \frac{B}{1 - \beta e^{-j\omega}}$$

$$y[n] = A\alpha^n u[n] + B\beta^n u[n]$$

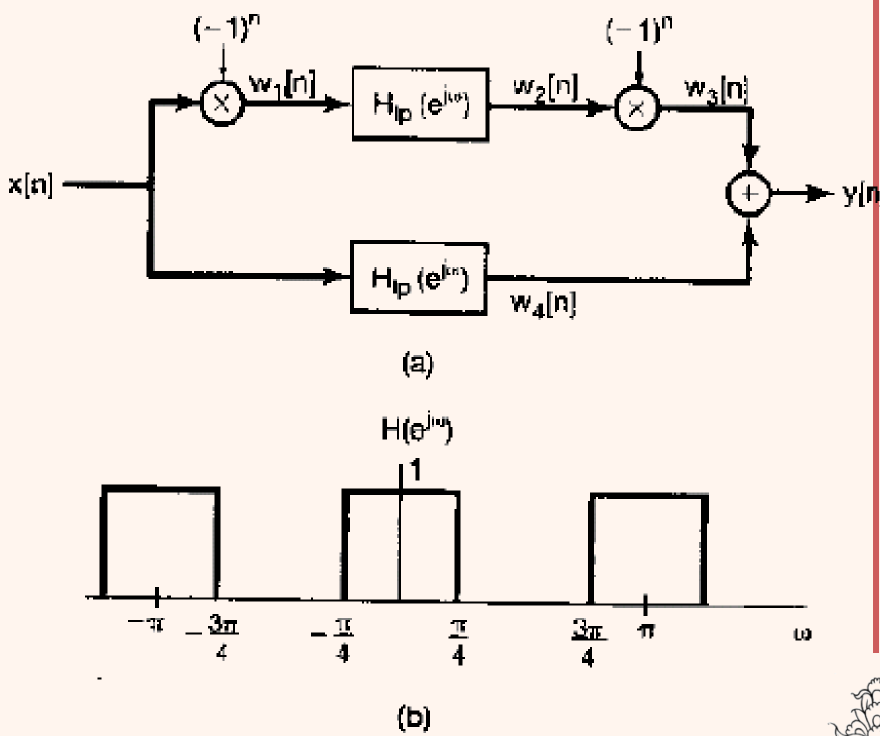
α=β,

$$Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^2}$$

$$y[n] = (n+1)\alpha^n u[n]$$

مثال

Bandstop filter



$$W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)})$$

$$W_3(e^{j\omega}) = W_3(e^{j(\omega-\pi)}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega})$$

$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j\omega})$$

معادلات تفاضلی

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(j\omega) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(j\omega)$$

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

مثال

$$y[n] - ay[n-1] = x[n]$$

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$h[n] = a^n u[n]$$

مثال

مطلوبست پاسخ فرکانسی

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$$

$$H(e^{j\omega})=\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}}=\frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$A_1=H(e^{j\omega})\left(1-\frac{1}{4}e^{-j\omega}\right)\Big|_{e^{j\omega}=\frac{1}{4}}=-2$$

$$A_2=H(e^{j\omega})\left(1-\frac{1}{2}e^{-j\omega}\right)\Big|_{e^{j\omega}=\frac{1}{2}}=4$$

$$H(e^{j\omega})=\frac{4}{1-\frac{1}{2}e^{-j\omega}}-\frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

$$h[n]=4\left(\frac{1}{2}\right)^n u[n]-2\left(\frac{1}{4}\right)^n u[n]$$

ادامه‌ی مثال

خروجی را به دست آورید:

$$x[n]=\left(\frac{1}{4}\right)^n u[n]$$

$$X(e^{j\omega})=\frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$Y(e^{j\omega})=X(e^{j\omega})H(e^{j\omega})=\frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$A_{11}=\left(-e^{-j\omega}\right)^1\frac{d}{de^{-j\omega}}\left\{Y(e^{j\omega})\left(1-\frac{1}{4}e^{-j\omega}\right)^2\right\}\Bigg|_{e^{-j\omega}=4}$$

$$=\left(-e^{-j\omega}\right)\frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)^2}\Bigg|_{e^{-j\omega}=4}=-4$$

ادامه‌ی مثال

$$A_{12} = Y(e^{j\omega})\left(1-\frac{1}{4}e^{-j\omega}\right)^2\Bigg|_{e^{j\omega}=\frac{1}{4}} = -2 \quad A_2 = Y(e^{j\omega})\left(1-\frac{1}{2}e^{-j\omega}\right)\Bigg|_{e^{j\omega}=\frac{1}{2}} = 8$$

$$Y(e^{j\omega}) = \frac{4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{8}{1-\frac{1}{2}e^{-j\omega}}$$

$$y[n] = \left\{-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\}u[n]$$

$$= \left\{-2(n+3)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\}u[n]$$

مثال

مطلوبست پاسخ فرکانسی

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$$

$$H(e^{j\omega})=\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}}=\frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$A_1=H(e^{j\omega})\left(1-\frac{1}{4}e^{-j\omega}\right)\bigg|_{e^{j\omega}=\frac{1}{4}}=-2$$

$$A_2=H(e^{j\omega})\left(1-\frac{1}{2}e^{-j\omega}\right)\bigg|_{e^{j\omega}=\frac{1}{2}}=4$$

$$H(e^{j\omega})=\frac{4}{1-\frac{1}{2}e^{-j\omega}}-\frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

$$h[n]=4\left(\frac{1}{2}\right)^n u[n]-2\left(\frac{1}{4}\right)^n u[n]$$

ادامه‌ی مثال

خروجی را به دست آورید:

$$x[n]=\left(\frac{1}{4}\right)^n u[n]$$

$$X(e^{j\omega})=\frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$Y(e^{j\omega})=X(e^{j\omega})H(e^{j\omega})=\frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$A_{11}=\left(-e^{-j\omega}\right)^1\frac{d}{de^{-j\omega}}\left\{Y(e^{j\omega})\left(1-\frac{1}{4}e^{-j\omega}\right)^2\right\}\Bigg|_{e^{-j\omega}=4}$$

$$=\left(-e^{-j\omega}\right)\frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)^2}\Bigg|_{e^{-j\omega}=4}\qquad=-4$$

ادامه‌ی مثال

$$A_{12} = Y(e^{j\omega})\left(1-\frac{1}{4}e^{-j\omega}\right)^2\Bigg|_{e^{j\omega}=\frac{1}{4}} = -2 \quad A_2 = Y(e^{j\omega})\left(1-\frac{1}{2}e^{-j\omega}\right)\Bigg|_{e^{j\omega}=\frac{1}{2}} = 8$$

$$Y(e^{j\omega}) = \frac{-4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{8}{1-\frac{1}{2}e^{-j\omega}}$$

$$y[n] = \left\{-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\}u[n]$$

$$= \left\{-2(n+3)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\}u[n]$$

خواص تبدیل فوریه‌ی گسسته

ضرب

$$y[n] = x_1[n]x_2[n]$$

$$Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

$$\begin{aligned}
Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_1[n]\cdot x_2[n]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\right)x_2[n]e^{-j\omega n} \\
&= \frac{1}{2\pi}\int_{2\pi}(X_1(e^{j\theta})\underbrace{\sum_{n=-\infty}^{\infty} x_2[n]e^{-j(\omega-\theta)n})}_{X_2(e^{j(\omega-\theta)})}d\theta \\
&= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta
\end{aligned}$$

کانولوشن پریودیک

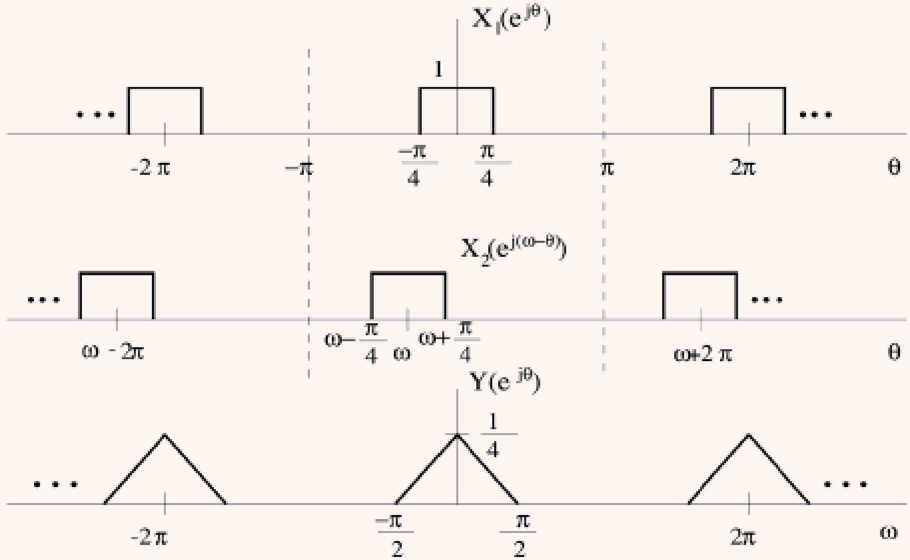
$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

$$\hat{X}_1(e^{j\theta}) = \begin{cases} X_1(e^{j\theta}), & |\theta| \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

مثال

$$y[n] = \left(\frac{\sin(\pi n/4)}{\pi n}\right)^2 = x_1[n] \cdot x_2[n], \quad x_1[n] = x_2[n] = \frac{\sin(\pi n/4)}{\pi n}$$

$$Y(e^{j\omega}) = \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega})$$



$$\frac{\sin Wn}{\pi n}$$